Gravitational radiation from cosmic strings

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Gravitational radiation from oscillating loops of string is studied both analytically and numerically. The total radiated power is found to be $P=\gamma G\mu^2$, where μ is the mass density of the string and γ is a numerical coefficient ~ 100. The intensity and the spectrum of the stochastic gravitational-wave background produced by the loops are calculated. Gravitational radiation from asymmetric loops carries not only energy, but also momentum; the loop recoils and accelerates like a rocket. The momentum radiation rate from loops is calculated and it is shown that cosmological loops formed with sufficiently small initial velocities are slowed down by dynamical friction and do not rocket away.

I. INTRODUCTION AND SUMMARY

In recent years there has been considerable interest in cosmological effects of strings which could be produced at a phase transition in the early universe. Superheavy strings can generate cosmologically interesting density fluctuations and can also produce a number of distinctive observational effects. For a review of strings and their cosmological implications see Refs. 1–3.

In the string scenario of galaxy formation, galaxies and clusters condense around oscillating closed loops, while the loops gradually decay by gravitational radiation.[4] Loops oscillate at relativistic speeds, the typical frequency being $f\sim L^{-1}$, where L is the length of the loop. The gravitational radiation rate can be estimated from the quadrupole formula:

$$\frac{dE}{dt}\sim GM^2L^4f^6\sim G\mu^2 , \quad (1.1)$$

where $\mu\sim M/L$ is the mass per unit length of string and we use the system of units in which $\hbar=c=1$. The lifetime of the loop is $\tau\sim M/G\mu^2\sim L/G\mu$.

The quadrupole gravitational radiation formula applies only to slowly moving sources, and thus its validity for the radiation from loops is dubious. The motion of a loop of mass M is periodic with a period $T=L/2=M/2\mu$, and it can be shown[5,6] that at one moment during the period certain points of the loop actually reach the velocity of light. This gives rise to substantial radiation power at frequencies much greater than T^{-1}, and one could worry that the radiation rate from the loops may even be divergent.[5]

The main purpose of this paper is to calculate the gravitational radiation rate from oscillating loops using the exact loop trajectories found by Kibble and Turok.[7,5] Although the asymptotic behavior at high frequencies can be studied analytically, the total radiation rate can only be obtained by a computer calculation. Our result can be written as

$$dE/dt=\gamma G\mu^2 , \quad (1.2)$$

where γ is a numerical coefficient which takes different values for different loop trajectories, but is typically ~ 100. The large value of γ is partly due to the contribution of high frequencies. We find that the angular distribution of the radiation diverges in the directions corresponding to the superluminal motion of the loop. However, the divergence is integrable and the total power is finite.

Gravitational waves emitted by individual loops add up to a stochastic gravitational radiation background.[8] This prediction of the string scenario is particularly interesting in view of the recent observations of the millisecond pulsar.[9] A convenient measure of the gravitational radiation intensity is

$$\Omega_g(f)=\frac{f}{\rho_c}\frac{d\rho_g}{df} . \quad (1.3)$$

Here, ρ_g is the energy density of the gravitational waves, $\rho_c=2\times10^{-29}h^2$ g cm^{-3} is the critical density and h is the Hubble constant in units of 100 km sec^{-1} Mpc^{-1}. $\Omega_g(f)$ gives the energy density of gravitational waves in units of ρ_c per logarithmic frequency interval. Long-wavelength gravitational radiation would create noise in pulsar timing, and the present observational bound on the energy density in waves of period ~ 1 yr is

$$\Omega_g(1\ \text{yr}^{-1})\lesssim 10^{-5} . \quad (1.4)$$

The accuracy of the results grows rapidly with the time of observation, and $\Omega_g\sim10^{-7}$ will probably become detectable within several years.[9]

An order-of-magnitude estimate of the gravitational background from strings has been found in Refs. 8, 9, and 3. Here, we will do a more careful calculation, taking proper account of the facts that each loop contributes to the gravitational radiation in a wide range of frequencies and that these frequencies change with time as the loop shrinks. Our final result is very simple:

$$\Omega_g(f)\sim 45\alpha^{3/2}\beta(G\mu/\gamma)^{1/2}\Omega_\gamma\ , \tag{1.5}$$

where $\Omega_\gamma=2\times10^{-5}h^{-2}$ is the microwave radiation density in units of ρ_c and $\alpha^{3/2}\beta$ is a numerical coefficient ~ 1 defined in Sec. III. The precise value of this coefficient is presently unknown, but it can be found by a numerical simulation of the evolution of strings. γ in Eq. (1.5) is the average value of γ for various loop configurations. Interestingly enough, the result is insensitive to the details of the spectrum emitted by individual loops and depends only on the overall power given by γ. The string scenario of galaxy formation requires[3,4] $G\mu\sim10^{-6}$, and with $\gamma\sim100$ Eq. (1.5) gives

$$\Omega_g(f)\sim 10^{-7}\ . \tag{1.6}$$

Equation (1.5) applies for $f\gg 10^{-7}(G\mu)^{-1}$ yr^{-1}, which includes the frequency range of interest ($f\sim1$ yr^{-1}). The result (1.6) is consistent with the observational upper bound (1.4) and should be within the experimental capabilities of detection in a not too distant future.

Asymmetric loop configurations radiate not only energy, but also momentum. As a result, a loop initially at rest will accelerate like a rocket. We have found the momentum radiation rate for a few loop trajectories; the result is

$$|\,d\mathbf{p}/dt\,|=\gamma_P G\mu^2\ , \tag{1.7}$$

with $\gamma_P\sim10$. In the cosmological context, the loops are formed with mildly relativistic velocities. The "gravitational rocket" effect becomes important when the loop is sufficiently slowed down by the expansion. We will show that loops formed with sufficiently small initial velocities are decelerated to subsonic speeds by dynamical friction and do not rocket away (a similar conclusion has been reached by Hogan and Rees[9]).

II. GRAVITATIONAL RADIATION FROM LOOPS

The string trajectory in space-time can be described by a vector function $\mathbf{x}(\sigma,t)$, where σ is a parameter along the string. The equations of motion for a string can be written as (see, e.g., Refs. 7 and 3)

$$\ddot{\mathbf{x}}-\mathbf{x}''=0\ , \tag{2.1}$$

$$\dot{\mathbf{x}}\cdot\mathbf{x}'=0\ ,\quad \dot{\mathbf{x}}^2+\mathbf{x}'^2=1\ , \tag{2.2}$$

where dots and primes stand for derivatives with respect to t and σ, respectively. The energy-momentum tensor of a string can be found by varying the string action with respect to the metric or by considering the energy-momentum tensor for a straight string and performing Lorentz boosts. The result is[5]

$$T^{\mu\nu}(\mathbf{x},t)=\mu\int d\sigma(\dot{x}^\mu\dot{x}^\nu-x'^\mu x'^\nu)\delta^{(3)}(\mathbf{x}-\mathbf{x}(\sigma,t))\ , \tag{2.3}$$

where we take $x^0=t$. The total energy of a closed loop is

$$M=\mu\int d\sigma\ , \tag{2.4}$$

and thus the parameter σ varies from 0 to $L=M/\mu$ around the loop. We shall call L the invariant length of the loop. (The actual length varies with time, but usually remains of the order of L.)

The general solution of Eq. (2.1) is

$$\mathbf{x}(\sigma,t)=\tfrac{1}{2}[\mathbf{a}(\sigma-t)+\mathbf{b}(\sigma+t)]\ , \tag{2.5}$$

and Eqs. (2.2) give the following constraints for the otherwise arbitrary functions $\mathbf{a}$ and $\mathbf{b}$:

$$\mathbf{a}'^2=\mathbf{b}'^2=1\ . \tag{2.6}$$

For a closed loop these functions should be periodic:

$$\mathbf{a}(\sigma+L)=\mathbf{a}(\sigma)\ ,\quad \mathbf{b}(\sigma+L)=\mathbf{b}(\sigma)\ . \tag{2.7}$$

It is clear from Eq. (2.5) that the motion of the loop must also be periodic in time with the same period. In fact, the actual period is twice shorter, $T=L/2$, since it is easily seen that

$$\mathbf{x}(\sigma+L/2\ ,t+L/2)=\mathbf{x}(\sigma,t)\ . \tag{2.8}$$

The periodic functions $\mathbf{a}$ and $\mathbf{b}$ can be expanded in Fourier series; then the constraints (2.6) give nonlinear algebraic equations for the coefficients. Exact solutions can easily be obtained in which only a few lowest frequencies are present. A simple family of solutions involving only two frequencies has been found by Kibble and Turok:[7,5]

$$\begin{aligned}\mathbf{x}=\frac{L}{4\pi}\big(&\hat{\mathbf{e}}_1[(1-\alpha)\sin\sigma_-+\tfrac{1}{3}\alpha\sin3\sigma_-+\sin\sigma_+]\\ &-\hat{\mathbf{e}}_2[(1-\alpha)\cos\sigma_-+\tfrac{1}{3}\alpha\cos3\sigma_-+\cos\phi\cos\sigma_+]\\ &-\hat{\mathbf{e}}_3\{2[\alpha(1-\alpha)]^{1/2}\cos\sigma_-+\sin\phi\cos\sigma_+\}\big)\ .\end{aligned} \tag{2.9}$$

Here, $\sigma_\pm=(2\pi/L)(\sigma\pm t)$, $\hat{\mathbf{e}}_i$ are unit vectors in the directions of the Cartesian axes, α and ϕ are constant parameters, $0\le\alpha\le1$, $-\pi\le\phi\le\pi$. In these solutions, the points of the loop at $\sigma=L/4$, $3L/4$ reach the velocity of light at $t=(n+\frac{1}{2})L/2$, where n is an integer. We shall use the loop trajectories (2.9) for the calculation of gravitational radiation from loops.

The choice of the two-parameter family of solutions (2.9) may seem arbitrary. However, it can be argued[5] that it gives a fair representation of the loop configurations produced in the early universe. The reason is that high-frequency waves on strings are damped in the course of expansion, and so truncating the Fourier series after the first few terms is justified. Turok[5] has shown that a large fraction of the parameter space in the family (2.9) is occupied by loops which never self-intersect and thus cannot decay by breaking into smaller and smaller loops. The dominant decay mechanism for such loops is the gravitational radiation.

The gravitational radiation rate can be found using the following general equation:[10]

$$\frac{dP_n}{d\Omega}=\frac{G\omega_n{}^2}{\pi}[T^*_{\mu\nu}(\omega_n,\mathbf{k})T^{\mu\nu}(\omega_n,\mathbf{k})-\tfrac{1}{2}\,|\,T^\nu_\nu(\omega_n,\mathbf{k})\,|^2]\ . \tag{2.10}$$

Here, $dP_n/d\Omega$ is the intensity of radiation at angular frequency $\omega_n=4\pi n/L$ per unit solid angle in the direction of $\mathbf{k}$, $|\mathbf{k}|=\omega_n$, and

$$T_{\mu\nu}(\omega_n,\mathbf{k})=\frac{2}{L}\int_0^{L/2}dt\,e^{i\omega_n t}\int d^3x\,e^{-i\mathbf{k}\cdot\mathbf{x}}T_{\mu\nu}(\mathbf{x},t) \tag{2.11}$$

is the Fourier transform of the string energy-momentum tensor (2.3).[11] The total power of the radiation is

$$dE/dt=\sum_n P_n\ . \tag{2.12}$$

Let us first consider the loop trajectories (2.9) with $\alpha=0$. Rotating the coordinate frame by an angle $\phi/2$ around the direction of $\hat{\mathbf{e}}_1$ and performing simple trigonometric transformations we bring (2.9) to the form

$$\mathbf{x}=\frac{L}{2\pi}\left[\hat{\mathbf{e}}_1\sin\frac{2\pi\sigma}{L}\cos\frac{2\pi t}{L}+\hat{\mathbf{e}}_2\cos\frac{\phi}{2}\cos\frac{2\pi\sigma}{L}\cos\frac{2\pi t}{L}+\hat{\mathbf{e}}_3\sin\frac{\phi}{2}\sin\frac{2\pi\sigma}{L}\sin\frac{2\pi t}{L}\right]\ . \tag{2.13}$$

The motion of the loop described by this equation can be easily visualized. At $t=0$, the loop has the shape of an ellipse in the xy plane. It rotates and turns into a double line along the z axis at $t=L/4$. At this moment the ends of the double line are moving in the x direction with the velocity of light. At $t=L/2$ the loop returns to the elliptical shape and goes through the cycle again. The degenerate cases $\phi=0$ and $\phi=\pi$ correspond to a circular loop and to a rotating double line, respectively.

We can now substitute Eqs. (2.13) and (2.3) in Eq. (2.11) for $T_{\mu\nu}(\omega_n,\mathbf{k})$. The t and σ integrations can be done analytically if we restrict $\mathbf{k}$ to be in xy or xz plane. Although the calculation for other values of $\mathbf{k}$ still has to be done numerically, this analytic exercise is very useful for the understanding of the high-frequency behavior of $dP_n/d\Omega$. For $\mathbf{k}$ in the xy plane we use the relations

$$\begin{aligned}&\int_0^\pi dt\,\cos 2nt\,\exp(-iz\cos t)=\pi(-1)^nJ_{2n}(z)\ ,\\&\int_0^\pi J_{2n}(2na\,\sin\sigma)\cos 2\sigma\,d\sigma=-\pi J_{n-1}(na)J_{n+1}(na)\\&\qquad\equiv-2(-1)^nI\ ,\\&\int_0^\pi J_{2(n\pm 1)}(2na\,\sin\sigma)\,d\sigma=\pi J^2_{n\pm1}(na)\equiv 2(-1)^nI_\pm\ ,\end{aligned} \tag{2.14}$$

where $J_n(z)$ are Bessel functions. The resulting expressions for $T_{\mu\nu}(\omega_n,\mathbf{k})$ are

$$\begin{aligned}&T_{11}=\mu(I_++I_-+2I\cos 2\beta)\ ,\\&T_{22}=\mu\cos^2\frac{\phi}{2}(I_++I_--2I\cos 2\beta)\ ,\\&T_{33}=-\tan^2\frac{\phi}{2}T_{22}\ ,\\&T_{12}=2\mu\,\sin 2\beta\cos\frac{\phi}{2}I\ ,\\&T_{13}=T_{23}=0\ ,\end{aligned} \tag{2.15}$$

where I and $I_\pm$ are defined in Eqs. (2.14),

$$a=\left[1-\sin^2\frac{\phi}{2}\sin^2\theta\right]^{1/2}\ ,$$

$$\cos\beta=a^{-1}\cos\theta\ ,$$

and $\cos\theta=\hat{\mathbf{k}}\cdot\hat{\mathbf{e}}_1$ (θ is the angle between $\mathbf{k}$ and the x axis). The components T_{0i} and T_{00} can be found using the conservation laws, $\omega_nT_{0i}=-k^jT_{ij}$, $\omega_nT_{00}=k^ik^jT_{ij}$.

Using the asymptotic expansions for the Bessel functions, it is easily seen that in the high-frequency limit ($n\gg1$) the components of $T_{\mu\nu}(\omega_n,\mathbf{k})$ are peaked in the directions of $\theta=0$ and $\theta=\pi$. This is not surprising if we note that the high-frequency contributions come from the vicinity of the points on the loop that reach the velocity of light and that the luminal velocity at those points is along the x axis. For $n\gg1$ and $\sin^2\theta\ll1$, Eqs. (2.10) and (2.15) give

$$dP_n/d\Omega=8\pi G\mu^2n^2[J'_n(na)]^4\ , \tag{2.16}$$

where we have used the relation $J_{n+1}(z)-J_{n-1}(z)=2J'_n(z)$. The asymptotic behavior of $J'_n(na)$ at large n is

$$J'_n(na)\approx 0.41n^{-2/3} \tag{2.17}$$

for $1\ll n\ll n_*$, where

$$n_*=3[2(1-a)]^{-3/2}\approx 3\left|\sin\theta\sin\frac{\phi}{2}\right|^{-3}\ . \tag{2.18}$$

For $n\gg n_*$, $J'_n(na)$ decreases like $\exp(-n/n_*)$. The total intensity of the radiation at angle $\theta\ll1$ is

$$dP/d\Omega\approx\sum^{n_*}dP_n/d\Omega\sim G\mu^2n_*{}^{1/3}\propto\theta^{-1}\ . \tag{2.19}$$

The case when $\mathbf{k}$ is in the xz plane can be analyzed similarly. The results are still given by Eqs. (2.16) and (2.19), but now $a=[1-\cos^2(\phi/2)\sin^2\theta]^{1/2}$ and $n_*=3\,|\sin\theta\cos(\phi/2)|^{-3}$.

From Eq. (2.19) we see that the angular distribution of the radiation diverges like θ^{-1} as $\theta\to0$. However, this singularity is integrable and the total power, $P=\int(dP/d\Omega)d\Omega$, is finite. We can also use Eqs. (2.16) and (2.18) to find the asymptotic behavior of

$$P_n=\int(dP_n/d\Omega)d\Omega \tag{2.20}$$

at large n. We can write $P_n\sim0.7G\mu^2n^{-2/3}\Delta\theta_{xy}\Delta\theta_{xz}$, where $\Delta\theta_{xy}\sim3/n^{1/3}\sin(\phi/2)$ and $\Delta\theta_{xz}\sim3/n^{1/3}\cos(\phi/2)$ are the angular widths of the peak of $dP_n/d\Omega$ in xy and xz directions, respectively. This gives

$$P_n\sim10G\mu^2(\sin\phi)^{-1}n^{-4/3}\ . \tag{2.21}$$

Equation (2.21) applies only for $n\gg(\sin\phi)^{-3}$ and is not valid for $\phi=0$ and π. These are the singular cases in which the total power diverges.

We have done a computer calculation of the total power dE/dt for several values of ϕ. We first calculated P_n for $n=1,2,\ldots$ until the asymptotic regime (2.21) was reached and then estimated the remainder of the series using Eq. (2.21). Figure 1 shows P_n as a function of n for some values of ϕ. The closer ϕ gets to 0 or π, the longer it takes to reach the asymptotic region. The total radiation power

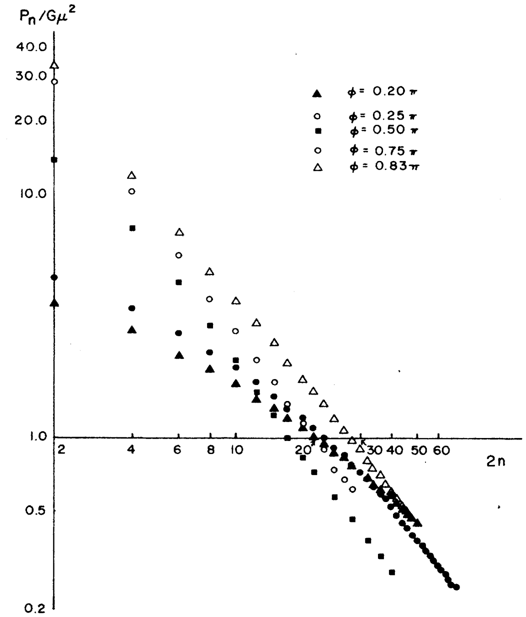

FIG. 1. The spectral power P_n for several loops of Eq. (2.9) with $\alpha=0$.

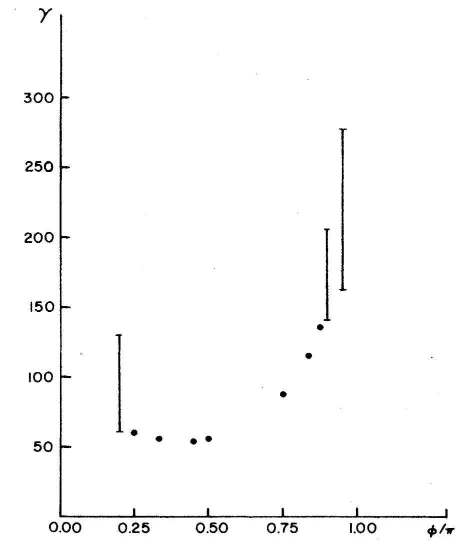

FIG. 2. Total radiated power as a function of ϕ for loops of Eq. (2.9) with $\alpha=0$. Error bars show the uncertainty in the value of γ in cases where the asymptotic regime (2.21) has not been reached.

can be written as

$$dE/dt=\gamma G\mu^2 \ , \tag{2.22}$$

where γ is a numerical coefficient. Figure 2 shows γ as a function of ϕ. For values of ϕ close to 0 or π we had to stop the calculation before the asymptotic region was reached and could only find upper and lower bounds for the remainder of the series. These are shown by error bars in the figure.

The family of the loop trajectories (2.13) is degenerate in that all loops collapse to a double line at $t=L/4$. Cosmologically, the most interesting loops are those which never intersect themselves. We have calculated numerically the gravitational radiation power for several nonintersecting loops, with similar results. For example, the lower and upper bounds for γ for the loop (2.9) with $\alpha=0.5$ and $\phi=0$ are 32.4 and 64.4, respectively. The lower bound is $\sum_1^{n_c} P_n$, where n_c is the value of n at which the calculation was stopped. (In this particular case $n_c=30$. Pursuing the calculation to substantially greater values of n would take an enormous amount of computer time.) We cannot improve this estimate, since we do not know the asymptotic behavior for $\alpha\neq 0$. The upper bound is based on estimating the remainder of the series using the average slope ($\simeq -1.17$) of P_n at $n\sim n_c$. The graph of P_n for this loop is shown in Fig. 3. (The oscillations of P_n at $n \gtrsim 15$ suggest that the asymptotic form of P_n may not be a simple power law.) For all loops we considered the values of γ are rather large, typically between 50 and 100.

The rate at which the loops radiate momentum can be found from

$$\frac{d\mathbf{P}}{dt}=\sum_n \int \frac{dP_n}{d\Omega}\mathbf{k}\, d\Omega \ . \tag{2.23}$$

The loops described by Eq. (2.9) have mirror symmetry and do not radiate momentum. We have calculated dE/dt and $d\mathbf{P}/dt$ for a few asymmetric loop trajectories belonging to the family

$$\mathbf{x}(\sigma,t)=\frac{L}{4\pi}(\hat{\mathbf{e}}_1[(1-\alpha)\sin\sigma_- - \tfrac{1}{3}\alpha\sin 3\sigma_- + \sin\sigma_+] - \hat{\mathbf{e}}_2[(1-\alpha)\cos\sigma_- + \tfrac{1}{3}\alpha\cos 3\sigma_- + \cos\phi\cos\sigma_+]$$
$$+\hat{\mathbf{e}}_3\{[\alpha(1-\alpha)]^{1/2}\sin 2\sigma_- - \sin\phi\cos\sigma_+\}) \ . \tag{2.24}$$

Here, α and ϕ are constant parameters, $0\le\alpha\le 1$, $-\pi<\phi\le\pi$. The shape of the loop with $\alpha=0.5$ and $\phi=\pi/2$ is shown in Fig. 4 at $t=0$ and $t=L/4$. The results for the gravitational radiation from loops with $\alpha=0.5$ and a few values of ϕ are shown in Table I, where γ_P is defined as

$$|\,d\mathbf{P}/dt\,| =\gamma_P G\mu^2 \ . \tag{2.25}$$

For these solutions the series (2.12) and (2.23) are rapidly converging ($P_n \propto n^{-3}$) and the errors are small.

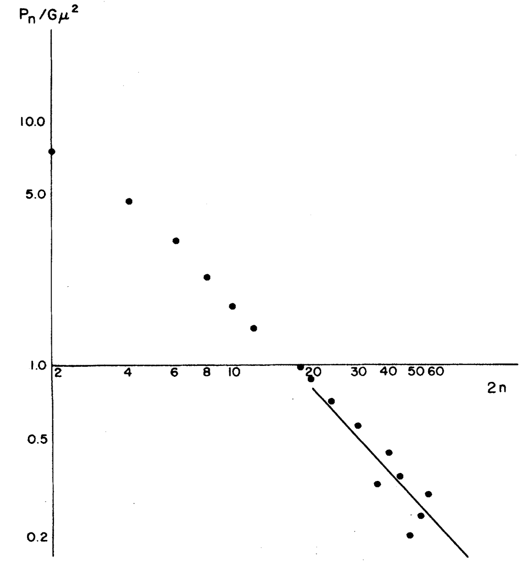

FIG. 3. The spectral power P_n for nonintersecting loop of Eq. (2.9) with $\alpha=0.5$ and $\phi=0$. The average slope used to estimate the remainder of the series is shown by a solid line.

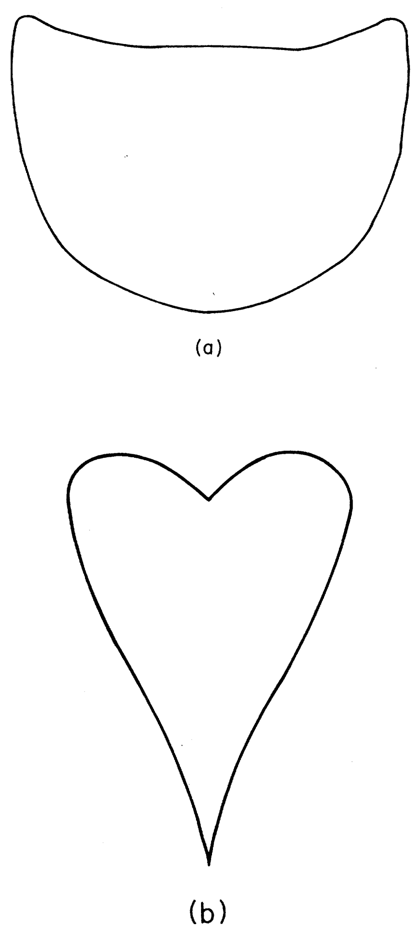

FIG. 4. The xy projections of the asymmetric loop (2.24) (a) at $t=0$ and (b) at $t=L/4$.

TABLE I. Energy and momentum radiation by asymmetric loops (2.24) for $\alpha=0.5$ and several values of ϕ.

ϕ/π	γ	γ_P
0.25	47.5	6.4
0.50	54.0	12.0
0.75	47.0	5.0

III. STOCHASTIC GRAVITATIONAL BACKGROUND

The gravitational radiation background is produced by loops of different shapes and sizes; some of them may still be around, but most of them have already decayed. Loops are formed by intercommuting of intersecting strings. It is usually assumed[1-4] that loops formed at cosmic time t have initial length

$$L_0 \sim \alpha t \tag{3.1}$$

and the rate of loop formation is

$$dn/dt \sim \beta t^{-4} . \tag{3.2}$$

Here, dn/dt is the number of loops formed per unit volume per unit time, and α and β are numerical coefficients which can be determined by a computer simulation of the evolution of strings. If the intercommuting probability for intersecting strings is ~ 1 (and the numerical analysis by Shellard[12] suggests that it is), then one expects that $\alpha \sim \beta \sim 1$. We shall assume that a substantial fraction of the loops are of non-self-intersecting variety. The lifetime of such loops is $\tau = L_0/\gamma G\mu$. Each loop emits gravitational waves at a discrete set of frequencies $f_n = 2n/L(t)$, where $L(t) = L_0 - \gamma G\mu t$. (Note that for strings of cosmological interest $\gamma G\mu \sim 10^{-4} \ll 1$.) The frequencies grow as the length of the loop decreases. Radiation emitted by the loops is red-shifted by the cosmological expansion, so that the present frequency of the radiation can be much smaller than the original one.

Let us calculate the intensity of the radiation in the frequency range from f to $f+df$ at present. If the radiation is emitted at time t, it should have the initial frequency

$$f(t) = fa(t_0)/a(t) , \tag{3.3}$$

where $a(t)$ is the scale factor and t_0 is the present cosmic time. Loops radiating at this frequency have the initial lengths

$$L_n = 2n/f(t) + \gamma G\mu t \tag{3.4}$$

with $dL_n = 2n[a(t)/a(t_0)]df/f^2$. Using Eqs. (3.1) and (3.2) we can find the number density of such loops at time t:

$$dn_n(t) = \alpha^3 \beta \left[\frac{a(L_n/\alpha)}{a(t)} \right]^3 \frac{dL_n}{L_n{}^4} . \tag{3.5}$$

Loops of initial length L_n were formed at $t \sim L_n/\alpha$, and so Eq. (3.5) applies only for

$$L_n < \alpha t . \tag{3.6}$$

The present energy density of the radiation in the frequency interval df is

$$d\rho_g = \sum_n P_n \int^{t_0} dt\, dn_n(t)[a(t)/a(t_0)]^4 \ . \tag{3.7}$$

The lower limit of the time integration is determined from the condition (3.6). The spectral power P_n is, of course, different for loops of different configurations, and here we have assumed some kind of averaging over configurations.

Let us first consider the contribution to (3.7) of the gravitational waves emitted during the radiation era, $t < t_{eq}$, where $t_{eq} = 4\times 10^{10}(\Omega h^2)^{-2}$ sec is the time of equal matter and radiation densities and Ω is the present density of the universe in units of ρ_c. Then $a(t)\propto t^{1/2}$. Introducing a new variable

$$x = \frac{2n}{ft}\frac{a(t)}{a(t_0)} \ , \quad dx = -\frac{na(t)}{fa(t_0)}\frac{dt}{t^2} \tag{3.8}$$

and using the relation

$$\rho_\gamma = \frac{3}{32\pi G t^2}\left[\frac{a(t)}{a(t_0)}\right]^4 , \tag{3.9}$$

where ρ_γ is the present energy density of thermal radiation, we can rewrite (3.7) as

$$d\rho_g = \frac{64\pi G}{3}\alpha^{3/2}\beta\rho_\gamma \frac{df}{f}\sum_n P_n \int_{x_n}^{\alpha} dx (x+\gamma G\mu)^{-5/2} \ . \tag{3.10}$$

Here,

$$x_n = 2na(t_{eq})/[a(t_0)ft_{eq})$$
$$= 7.5\times 10^{-8}\Omega h^2 n/f(\mathrm{yr}^{-1}) \ .$$

The lower limit of integration corresponds to $t = t_{eq}$ and the upper limit comes from the condition (3.6).

For $n \ll N = 1.3\times 10^7 \gamma G\mu(\Omega h^2)^{-1} f(\mathrm{yr}^{-1})$ the integral in Eq. (3.10) equals $2/3(\gamma G\mu)^{-3/2}$ and for $n \gg N$ it is $2/3(\gamma G\mu)^{-3/2}(N/n)^{3/2}$. Hence, we can write

$$\sum_n P_n \int_{x_n}^{\alpha} dx(x+\gamma G\mu)^{-5/2}$$
$$\approx \tfrac{2}{3}(\gamma G\mu)^{-3/2}\left[\gamma G\mu^2 - \sum_{n=N}^{\infty} P_n[1-(N/n)^{3/2}]\right] \ . \tag{3.11}$$

Obviously, the last term in Eq. (3.11) is negligible for sufficiently large N. We can introduce n_*, the value of n at which the series (2.12) can be truncated without substantially affecting the result. For the loop trajectories (2.13) truncating the series at $n_* \sim 100$ gives an accuracy of $\sim 20\%$. For $f \gg f_* \sim 10^{-7}(\gamma G\mu)^{-1}\Omega h^2 n_* \mathrm{yr}^{-1}$ Eq. (3.10) gives

$$\Omega_g(f) = \frac{128\pi}{9}\alpha^{3/2}\beta(G\mu/\gamma)^{1/2}\Omega_\gamma \ . \tag{3.12}$$

Here, $\Omega_\gamma = \rho_\gamma/\rho_c$ and $\Omega_g(f)$ is defined in Eq. (1.3).

To complete the calculation, one has to consider the contribution of $t > t_{eq}$ to the integral (3.7). After a straightforward calculation one finds that for $f \gg f_*$ this contribution is much smaller than (3.12). With $\gamma G\mu \sim 10^{-4}$, $\Omega h^2 \sim 0.25$, and $n_* \sim 100$ we have $f_* \sim 2.5\times 10^{-2}$ yr^{-1}, and the values of $f \gg f_*$ include the frequency range of cosmological interest, $f \gtrsim 1$ yr^{-1}. For $f \sim f_*$ Eq. (3.12) is still valid in the order-of-magnitude sense. We note that Eq. (3.12) is in good agreement with order-of-magnitude estimates in Refs. 2, 3, 8, and 9. The qualitative shape of the spectrum at $f \ll f_*$ has been discussed in Refs. 8 and 9. The upper bound of the spectrum (3.12) is at $f_{max} \sim \Omega_\gamma(t_{eq}t_f)^{-1/2}$, where t_f is the time at which the friction of strings due to their interaction with particles becomes unimportant.[1,2] For superheavy strings with $G\mu \sim 10^{-6}$, $t_f \sim 10^{-30}$ sec, and $f_{max} \sim 10^5$ sec^{-1}.

IV. GRAVITATIONAL ROCKET EFFECT

In this section we shall discuss how the motion of the loops is affected by the gravitational radiation recoil (or gravitational rocket effect). For simplicity we shall consider loops formed during the matter-dominated era, $t > t_{eq}$, when $a(t)\propto t^{2/3}$. The equation of motion for a loop of length L is

$$\dot{\mathbf{v}} + 2\mathbf{v}/3t = \gamma_P G\mu\mathbf{n}/L \ , \tag{4.1}$$

where $\mathbf{n}$ is the direction of the recoil. The velocity of the loop at formation $(t \sim L/\alpha)$ is mildly relativistic, $v_i \lesssim 1$. The solution of Eq. (4.1) with this initial condition is

$$\mathbf{v} = \mathbf{v}_i(L/\alpha t)^{2/3} + (3\gamma_P G\mu/5L)\mathbf{n}t \ . \tag{4.2}$$

(For time periods smaller than $\tau = L/\gamma G\mu$ we can treat L as a constant.) The rocket term becomes important at

$$t_r \sim (v_i/\gamma_P G\mu)^{3/5} L \ . \tag{4.3}$$

(Here and below we set $\alpha \sim 1$.) By the end of its life the loop reaches the velocity $v \sim \gamma_P/\gamma \sim 0.1$.

We note that angular momentum radiation can prevent the loop from accumulating a large velocity. On dimensional grounds, one expects that the angular momentum radiation rate is

$$dl/dt \sim G\mu^2 L \ . \tag{4.4}$$

If this torque causes the loop to rotate as a solid body, then the corresponding angular acceleration is $\ddot{\theta} \sim (2\pi)^2 G\mu L^{-2}$. The time it takes for the loop to rotate by an angle $\sim \pi/2$ is

$$\Delta t \sim \ddot{\theta}^{-1/2} \sim (G\mu)^{-1/2} L/2\pi \ ,$$

and the velocity accumulated during this time is $v \sim (G\mu)^{1/2}$. Note, however, that this conclusion is reached assuming that the loop rotates like a solid, which is far from obvious. Besides, even if it does, the rotation axis may not be at right angles to the direction of recoil, and then the loop will accelerate along the axis of rotation.

Another effect which can counteract the gravitational rocket force is the dynamical friction (or gravitational drag) due to small-angle scattering of particles.[9,13] With dynamical friction taken into account, the equation of motion for a loop is

$$\dot{\mathbf{v}} = -2\mathbf{v}/3t - \mathbf{v}/t_* + \gamma_P G\mu\mathbf{n}/L \ , \tag{4.5}$$

where

$$t_* \sim v^3/4\pi G^2 M\rho \sim 3v^3 t^2/2G\mu L \tag{4.6}$$

for $v > v_s$, $t_* \sim$const for $v < v_s$, and v_s is the velocity of sound. When v is sufficiently large, the dominant term on the right-hand side of (4.5) is the first term describing the deceleration of the loop due to expansion. The subsequent behavior of the loop depends on the initial velocity v_i which determines which of the two remaining terms in (4.5) first becomes dominant.[9] If the rocket term dominates first, dynamical friction never becomes important: the velocity grows and friction decreases like v^{-2}. If friction dominates first, the rocket effect is never important, since friction grows like v^{-2} as velocity decreases, while the rocket term remains constant. In this case, to estimate the timescale on which the loop is decelerated to a subsonic velocity we solve Eq. (4.5) without the last term and with t_* from Eq. (4.6):

$$v \sim v_i \left[\frac{L}{t}\right]^{2/3} \left[1-\frac{2G\mu}{v_i^3}\frac{t}{L}\right]^{1/3} . \tag{4.7}$$

Hence, the time scale of the frictional slow down is

$$t_f \sim v_i^3 L/2G\mu \, . \tag{4.8}$$

Friction dominates first if $t_f < t_r$, which gives

$$v_i < \gamma_P^{-1/4}(G\mu)^{1/6} \sim 0.1 \, , \tag{4.9}$$

where we have used the values $G\mu \sim 10^{-6}$, $\gamma_P \sim 10$.

The most interesting loops for cosmological applications are those which formed at $t < t_{\rm eq}$ but decayed at $t > t_{\rm eq}$. A similar analysis shows that such loops are decelerated to subsonic speeds if they have $v_i \lesssim (G\mu L/t_{\rm eq})^{1/6}$. For loops on the galactic scale, $L \sim 10^9$ sec and $G\mu \sim 10^{-6}$, this gives $v_i \lesssim 0.1$. The velocity distribution of loops at formation can be determined by a computer simulation of the evolution of strings. It is quite possible that a substantial fraction of all loops have $v_i \lesssim 0.1$ and will not rocket away. They will accrete matter giving rise to compact galactic nuclei and quasars, as discussed in Refs. 13 and 14.

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