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Evaluation of the derivative quartic terms of the meson chiral Lagrangian from forward dispersion relations

T. N. Pham and Tran N. Truong
Centre de Physique Théorique de l'Ecole Polytechnique, Plateau de Palaiseau, 91128 Palaiseau, Cedex, France
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Using the forward dispersion relations for $\pi\pi$ scattering, we show that the coefficients of the derivative quartic terms in the chiral Lagrangian are positive. The Skyrme term is mainly given by the ρ-meson contribution to the $I=1$ total cross section and gives a proton mass of 1.27 GeV. The non-Skyrme term is found of comparable strength from the $I=0$ S-wave $\pi\pi$ data.

The standard minimal $SU(3)\times SU(3)$ nonlinear effective Lagrangian for the pseudoscalar-meson octet is known to reproduce all the current-algebra results for processes involving soft mesons in an elegant manner.[1,2] For hard-meson processes, one expects on general grounds deviations from the current-algebra prediction as found in K_{l4} form factors,[3] $\eta\to 3\pi$ decay rates,[4] and the S-wave $I=0$ $\pi\pi$ scattering length.[5,6] Thus correction terms must be present to account for the experimental data.

There are two interrelated approaches to account for the correction terms. The first one consists in using current-algebra results as low-energy theorems which are supplemented by the use of the unitarity and analyticity to calculate the correction terms.[3,5] This is a nonperturbative approach, and good agreement with experimental data is obtained.[3,5] The second method involves the introduction of terms involving higher powers of field derivatives to the minimal Lagrangian; the strength of these terms are new parameters which must be determined either from independent experiments or from the matrix elements given by the first method.[6] Because the effective-Lagrangian method is a power-series expansion of the matrix element in terms of the invariant variables, it can only account for the singularities induced by the unitarity condition in a perturbative way.[6]

In the region where the power-series expansion is valid[3] (e.g., below the unitarity cut), the effective-Lagrangian method should give good results, as it was recently shown that the inclusion of the symmetry-breaking terms with derivative coupling is needed[7] to account for f_K/f_π and the K_{l3} Callan-Treiman relation. (It has also recently been shown that the derivative quartic terms are needed to account for the possible discrepancy with data for $K\to 3\pi$ rate.[8] Unfortunately, this is not the whole story because the calculated $K\to 2\pi$ and $K\to 3\pi$ are purely real, while the physical $K\to 2\pi$ amplitude has a phase of 40–45° which cannot be neglected. This illustrates some difficulty of using the effective Lagrangian above the unitarity cut.[5]) From this standpoint we see that similar terms in higher powers of derivatives (quartic terms, etc.) must also be present for pion-pion scattering. A possible source for these terms may be found in the coupling of the pseudoscalar meson with heavy particles (vector, scalar mesons $\rho,\sigma\ldots$), which in the limit of low-external momenta can be eliminated from the theory, leaving a meson Lagrangian with higher powers of field derivatives. For example, in the σ model,[9] by expanding the σ propagator in powers of k^2/m_σ^2 (k^2 being a typical external momentum), one finds that the k^2 term corresponds to the minimal Lagrangian and the k^4 terms gives rise to derivative quartic terms of the form $(\partial_\mu\Phi\,\partial_\mu\Phi)^2$, etc.

Our interest in these derivative higher-order Lagrangians is kindled by a recent revival of the Skyrme model,[10] started by Witten[11] and others,[12] who extended Skyrme's idea and showed that the nucleon may be described as a topologically stable soliton of the chiral Lagrangian. The presence of higher powers of field derivatives in the chiral Lagrangian is required for a stable soliton configuration with its mass related to the strength of these terms.[10] If we consider only terms quartic in the field derivative, then in the chiral-symmetry limit there are only two quartic terms, one of them being the term originally used by Skyrme in his soliton solution. At low energy these terms give rise to an energy-dependent $\pi\pi$ scattering amplitude, which in the forward direction can be determined directly from the dispersion relation. The purpose of this Rapid Communication is to evaluate the strength of the quartic terms using the forward dispersion relation for $\pi\pi$ scattering. In this way, we can see immediately that the coefficients of the two quartic terms are positive as a consequence of the positivity of the cross section. Our results show that the Skyrme term depends essentially on the P-wave $\pi\pi$ data while the non-Skyrme term depends mainly on the $I=0$ S-wave data in the low-energy region. We find

$$\frac{1}{e^2}=\frac{f_\pi^2}{m_\rho^2},\quad \frac{\gamma}{e^2}\simeq\frac{2}{3}\left(\frac{f_\pi^2}{m_\sigma^2}\right), \tag{1}$$

where $1/e^2$ and γ/e^2 are the strength of the Skyrme term and the other term, respectively, and the S-wave $I=0$ $\pi\pi$ data has been parametrized in terms of a possible broad "σ resonance." Before proceeding to the derivation of Eq. (1), we should mention that in previous works the quartic coefficients were expressed in terms of the D-wave $I=0$ and $I=2$ scattering lengths[13] or the deviation from the current-algebra prediction of the P-wave scattering length.[12] These quantities are unfortunately extremely difficult to measure due to the smallness of the P- and D-wave phase shifts at the threshold (the only reliable measurements of the scattering lengths available at present is that of the $I=0$ S-wave phase shift from the k_{l4} data). We avoid this problem by using the effective Lagrangian to obtain the low-energy forward $\pi\pi$ scattering amplitude and then compare it with that obtained from the dispersion relation at a point below the unitarity cut (in terms of the measured S- and P-wave

$\pi\pi$ data up to 1 GeV). The effective Lagrangian for meson-meson interactions is given by

$$\mathcal{L}=\mathcal{L}_0+\mathcal{L}_Q \ ,$$

where $\mathcal{L}_0$ is the standard minimal term. In the chiral-symmetry limit (massless pion), there are only two chiral-invariant quartic terms and the quartic term $\mathcal{L}_Q$ is given in the most general form as

$$\mathcal{L}_Q=\frac{1}{32e^2}\mathrm{Tr}([\partial_\mu M M^\dagger,\partial_\nu M M^\dagger]^2)+\frac{\gamma}{8e^2}[\mathrm{Tr}(\partial_\mu M\,\partial_\mu M^\dagger)]^2 \tag{2}$$

in standard notation. M is the meson coupling matrix of the exponential form in the nonlinear realization of chiral symmetry and is given by

$$M=\exp\left(\frac{2i\Phi}{f_\pi}\right),\quad f_\pi\simeq m_\pi \ .$$

Φ is the octet pseudoscalar-meson field operator:

$$\Phi=\sum_i\frac{\lambda_i\Phi_i}{\sqrt{2}},\quad i=1,\ldots,8 \ .$$

We are interested in evaluating the coefficients e^2 and γ in $\mathcal{L}_Q$. The first term in Eq. (2) is denoted as the Skyrme term and the other term is called the non-Skyrme term. The contribution to the $\pi\pi$ scattering amplitude from these quartic terms is obtained directly from all tree graphs generated by $\mathcal{L}_Q$. Let $T_{ab,cd}(s,t,u)$ be the invariant scattering amplitude for the process $\pi_a+\pi_b\to\pi_c+\pi_d$; then the parts obtained from $\mathcal{L}_Q$ for $\pi^+\pi^0\to\pi^+\pi^0$ and $\pi^0\pi^0\to\pi^0\pi^0$ elastic scatterings are given by

$$T^Q_{+0}(s,t,u)=\left(\frac{1}{e^2f_\pi^4}\right)[(s-2m_\pi^2)^2+(u-2m_\pi^2)^2-2(t-2m_\pi^2)^2]+\frac{\gamma}{e^2}\frac{1}{f_\pi^4}(t-2m_\pi^2)^2 \ , \tag{3a}$$

$$T^Q_{00}(s,t,u)=\frac{2\gamma}{e^2}\frac{1}{f_\pi^4}[(s-2m_\pi^2)^2+(t-2m_\pi^2)^2+(u-2m_\pi^2)^2] \ , \tag{3b}$$

where s,t,u are the usual Mandelstam variables. Note that for $\pi^+\pi^0$ elastic scattering, in the forward direction $(t=0)$ the non-Skyrme term does not contribute to the energy-dependent quartic term while the Skyrme term does not contribute to $\pi^0\pi^0\to\pi^0\pi^0$ scattering. Hence by considering the amplitude in the forward direction $(t=0)$ and taking the derivative with respect to

$$\omega^2=(s-2m_\pi^2)^2\ (s+u=4m_\pi^2,\ t=0) \ ,$$

we can isolate the Skyrme and non-Skyrme terms. We get

$$\frac{\partial T^Q_{+0}}{\partial\omega^2}=\frac{2}{e^2f_\pi^4}\ ,\quad \frac{\partial T^Q_{00}}{\partial\omega^2}=\frac{4\gamma}{e^2f_\pi^4} \ . \tag{4}$$

Note that by taking the derivative of the amplitudes, we get rid also of the contribution from the derivative quadratic terms in the chiral-invariant Lagrangian or similar terms from the chiral-symmetry-breaking $(3,\bar{3})+(\bar{3},3)$ mass term, which are therefore not needed in our analysis. The two processes $\pi^0\pi^0\to\pi^0\pi^0$ and $\pi^+\pi^0\to\pi^+\pi^0$ are even under $s\leftrightarrow u$ crossing (even in ω). Using the Froissart bound, we can thus write a once-subtracted dispersion relation for $T(\omega)$. We have

$$T_{00}(\omega)=T_{00}(0)+\frac{\omega^2}{\pi}\int_{2m_\pi^2}^{\infty}\frac{\mathrm{Im}\,T_{00}(\omega')\,d\omega'^2}{\omega'^2(\omega'^2-\omega^2)} \ , \tag{5}$$

and a similar expression for $T_{+0}(\omega)$.

Using the optical theorem and taking the derivative of (5) and comparing it with (4), we obtain

$$\frac{1}{e^2}=\frac{1}{\pi}f_\pi^4\int_{4m_\pi^2}^{\infty}\frac{\sqrt{s(s-4m_\pi^2)}\sigma_{\rm tot}^{\pi^+\pi^0}(s)\,ds}{(s-2m_\pi^2)^3} \ , \tag{6a}$$

and similarly

$$\frac{\gamma}{e^2}=\frac{f_\pi^4}{2\pi}\int_{4m_\pi^2}^{\infty}\frac{\sqrt{s(s-4m_\pi^2)}\sigma_{\rm tot}^{\pi^0\pi^0}(s)\,ds}{(s-2m_\pi^2)^3} \ . \tag{6b}$$

From the positivity of the cross section we see that γ and e^2 must be positive. Also, obvious lower bounds for e^2 and γ can be obtained using only the measured cross sections in the available energy range. Furthermore, to get e^2 and γ from Eqs. (6), it is useful to note that because of the extra power in the energy denominator, the high-energy contribution is strongly suppressed so that most of the contribution to e^2 and γ comes from the low-energy dispersion integral. $\sigma_{\rm tot}^{\pi^+\pi^0}$ involves contributions from the $I=1$ and $I=2$ channels while $\sigma_{\rm tot}^{\pi^0\pi^0}$ receives contributions from the $I=0$ and $I=2$ channels. (We neglect the D-wave $I=0$ contribution.) At low energy, from the $\pi\pi$ data or from the current-algebra result, $\sigma_{\rm tot}^{I=2}$ is found negligible. Therefore, the Skyrme term receives mostly contribution from the ρ-meson resonance, and the non-Skyrme terms get the main contribution from the large $I=0$ S-wave scattering which could be interpreted as a possible broad scalar resonance (which however does not show up in the phase-shift analysis). We use the parametrization for the partial-wave amplitudes,

$$f_l^I(s)=\frac{e^{i\delta_l^I}\sin\delta_l^I}{\rho(s)} \ ,$$

which agrees with the experimental data and which gives[5] the correct Weinberg S- and P-wave amplitudes at $s=\frac{1}{2}m_\pi^2$ $[\rho(s)=(s-4m_\rho^2)^{1/2}/s^{1/2}]$,

$$f_1^1(s)=\frac{(L/12)(s-4m_\pi^2)}{1-a(s-\frac{1}{2}m_\pi^2)+(L/12)(s-4m_\pi^2)[h(s)-h(m_\pi^2/2)-i\rho(s)]}=\frac{\gamma_1(s-4m_\pi^2)}{s_R-s+\gamma_1(s-4m_\pi^2)[h(s)-i\rho(s)]} \ , \tag{7}$$

$$f_0^0(s)=\frac{(L/2)(s-m_\pi^2/2)}{1+b_0(s-m_\pi^2/2)+(L/2)[s-(m_\pi^2/2)][h(s)-h(m_\pi^2/2)-i\rho(s)]} \ , \tag{8}$$

where

$$h(s)=\frac{2}{\pi}\left(\frac{s-4m_\pi^2}{s}\right)^{1/2}\ln\left(\frac{\sqrt{s}+\sqrt{s-4m_\pi^2}}{2}\right)$$

and $L=(4\pi f_\pi^2)^{-1}$. For $a=0.0357m_\pi^{-2}$, the P-wave phase shift passes through 90° at $m_\rho=775$ MeV and yields an experimental total width of 155 MeV. Similarly the S-wave $I=0$ phase shift can be fitted with $b_0=-0.04m_\pi^{-2} \simeq -m_\sigma^{-2}$, which agrees with the experimental data up to 700 MeV.

Putting the imaginary part of $f_l^I(s)$ as given by Eqs. (7) and (8) into Eqs. (6a) and (6b), we find

$$\frac{1}{e^2}\simeq\frac{f_\pi^2}{m_\rho^2} ,$$

and

$$\gamma\simeq\frac{2}{3}\frac{m_\rho^2}{m_\sigma^2} , \qquad (9)$$

where we have neglected the contributions from the unitarity correction, i.e., the function $h(s)$, which are small here. Note that the above results [Eq. (9)] can be obtained directly from the ρ and σ contribution at the tree level to the $\pi\pi$ scattering amplitudes which satisfy the low-energy theorem.

Our result clearly shows that the Skyrme and non-Skyrme terms are suppressed by a power of the mass of the heavy field (of the order k^2/m_ρ^2 relative to the minimal terms).

The value for e^2 we obtain is comparable to that of Donoghue, Golowich, and Holstein.[13] If we ignore the non-Skyrme term then we get a proton mass of 1.27 GeV (Ref. 14). For the non-Skyrme term we get however a value for γ much larger than that of Donoghue *et al.*[13]

There have been some recent attempts to compute the coefficients of these quartic terms as an effective potential generated by quarks at the one-loop level.[15] Our approach, on the other hand, shows the importance of nonperturbative effects associated with soft-gluon radiative corrections taken into account in the form of ρ and σ resonances.

In conclusion, in this simple analysis we have showed that the strengths of the quartic terms are quite large and must be taken into account in a computation of the proton mass. This is beyond the scope of this paper and will be discussed elsewhere.

Note added. Using Derrick's argument of scaling and some inequalities, it was recently shown by K. Fujii, S. Otsuki, and F. Toyoda [Kyushu University Report No. KYUSHU 84-HE-7 (unpublished)] that no stable soliton solution exists if

$$\gamma\geq\tfrac{1}{3} .$$

Comparing this with the result of Eq. (9) ($\gamma\simeq 0.7$), we see that there is no soliton solution with only the quartic derivatives in the effective Langrangian. We give here a simple proof of this result. Using Eq. (10) of Ref. 13, after a trivial rearranging of the contribution of the quartic terms to the energy integral, we have

$$\int d\tilde{r}\,\tilde{r}^2\left[(1-3\gamma)\left(\frac{\sin^4F}{\tilde{r}^4}+\frac{2F'^2\sin^2F}{\tilde{r}^2}\right)-\gamma\left(F'^2-\frac{\sin^2F}{\tilde{r}^2}\right)\right]>0 ,$$

which violates the positivity for $\gamma\geq\frac{1}{3}$.

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