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Deformation effects in the Skyrmion-Skyrmion interaction

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In the framework of the Skyrme model we calculate the Skyrmion-Skyrmion interaction allowing for deformations in the chiral field. Significant changes in the potentials compared to the ones with the pure hedgehog *Ansatz* appear. This strongly suggests that no convincing calculation of the static N-N potential as constructed from the Skyrmion-Skyrmion interaction has been done up to now. Our results therefore indicate that a reliable N-N potential can only be gained by more elaborate methods such as, e.g., lattice calculations.

Following an old idea of Skyrme,[1] baryons emerge as chiral solitons in a nonlinear meson field theory. Including the Wess-Zumino term[2] in the nonlinear σ model, Witten[3] has shown the connection between Skyrme's model and QCD in the large-N_C expansion (N_C is the number of colors). Witten proved that Skyrme's suggestion of identifying the integrated topological current W_μ with the baryon number is indeed correct. So topology is the main ingredient to stabilize the soliton. Static properties of the low-lying baryons[4] as well as the Skyrmion-Skyrmion interaction[5] have been worked out in detail. One of the main assumptions in these calculations is the hedgehog shape of the underlying chiral field due to the principal of maximal symmetry.

Our main concern in this Rapid Communication is to investigate whether the Skyrmion-Skyrmion interaction is crucially dependent on deformations of the chiral field θ. In the work of Jackson, Jackson, and Pasquier,[5] no deformations except scale changes were allowed; here we want to go one step further. This can also be motivated from many-Skyrmion systems where cylindrically symmetric solutions considerably lower the energy per particle as compared to the spherical case.[6]

The underlying Lagrangian of the Skyrme model[1] is

$$\mathscr{L}=-\frac{f_\pi^2}{4}\mathrm{Tr}(\partial_\mu U\,\partial^\mu U^\dagger)-\frac{\epsilon^2}{4}\mathrm{Tr}[\partial_\mu U^\dagger,\partial_\nu U]^2\ , \tag{1}$$

where the quaternion field $U(\mathbf{r})$ incorporates the underlying mesonic degrees of freedom, the scalar field σ and the isotriplet $\boldsymbol{\pi}$. The quadratic term in Eq. (1) gives the common current-algebra results, $f_\pi=93$ MeV being the pion decay constant. The quartic term in the Lagrangian (1) stabilizes the soliton; the value of $\epsilon^2=0.005\,52$ follows from the Goldberger-Treiman relation to give $g_A=1.25$. The most general *Ansatz* for $U(\mathbf{r})$ is given by

$$U(\mathbf{r})=e^{i\boldsymbol{\tau}\cdot\hat{\mathbf{n}}(\mathbf{r})\theta(\mathbf{r})}\ , \tag{2}$$

where $\hat{\mathbf{n}}(\mathbf{r})$ is some unit vector and $\theta(\mathbf{r})$ is the chiral field related to the σ and $\boldsymbol{\pi}$ degrees of freedom. In the case of the simple hedgehog, $\hat{\mathbf{n}}=\mathbf{r}/|\mathbf{r}|$ and $\theta(\mathbf{r})=\theta(|\mathbf{r}|)$, i.e., the isospin points radially in space and the chiral field depends on one variable, the distance r. Decomposing the quaternion field $U(\mathbf{r})$ in the form

$$U(\mathbf{r})=a(\mathbf{r})+i\boldsymbol{\tau}\cdot\mathbf{b}(\mathbf{r})\ , \tag{3}$$

the energy functional $E(a,b)$ is[7]

$$E=\sum_{i=1}^{3}\frac{f_\pi^2}{2}[(\partial_i a)^2+(\partial_i\mathbf{b})^2]+4\epsilon^2\sum_{i,j=1}^{3}[(\partial_i\mathbf{b})^2(\partial_j\mathbf{b})^2-(\partial_i\mathbf{b}\cdot\partial_j\mathbf{b})^2+2(\partial_i a)^2(\partial_i\mathbf{b})^2-2\partial_i a\,\partial_j a\,\partial_i\mathbf{b}\cdot\partial_j\mathbf{b}]\ , \tag{4}$$

where the functions a and b contain the complete geometrical information about the Skyrmion-Skyrmion potential. One can also express the baryon number B in terms of the a's and b's; we have[7]

$$B=\int d^3x\,a\,(\tfrac{3}{2}+a^2)\epsilon_{ijk}\partial_i\mathbf{b}\times\partial_j\mathbf{b}\cdot\partial_k\mathbf{b}\ . \tag{5}$$

Equation (5) will serve as a check for the numerical accuracy in the determination of the static Skyrmion-Skyrmion potential from the field energy given by Eq. (4).

As shown by Jackson *et al.*,[5] there are three independent sets of potentials in the spherically symmetric case, according to the relative spin/isospin rotations of the two solitons. Here we are interested in the change of these potentials if one allows for nonspherical deformations of the chiral field. Any of these rotations being specified by three Euler angles (α,β,γ), the potentials v_A, v_B, v_C are defined by

$$\begin{aligned}v_A(R;[\alpha])&=V(R;\alpha=0,\beta=0,\gamma=0)\ ,\\ v_B(R;[\alpha])&=V(R;\alpha=90^\circ,\beta=0,\gamma=0)\ ,\\ v_C(R;[\alpha])&=V(R;\alpha=0,\beta=180^\circ,\gamma=0)\ .\end{aligned} \tag{6}$$

Let us mention that the connection between these so-called potentials and the N-N potential is by no means trivial. The definitions (6) correspond to somehow classical quantities, namely, energies [see Eq. (9)] of classical field configurations with baryon number 2, when the energies of two noninteracting solitons have been subtracted. It should be kept in mind that the quartic term of Eq. (1) mocks up quantum effects, so the meaning of "classical" is already uncertain.[8]

Since there is no proof that the overlap of two spherically symmetric solitons minimizes the energy functional (4) (note that even for $B=1$ it has not yet been proven that the hedgehog *Ansatz* minimizes the energy) we investigate the Skyrmion-Skyrmion potential with a slightly more complicated *Ansatz*.

A generalization of the spherically symmetrical *Ansatz* is by no means evident. Certainly, Eq. (2) should be the starting point, but it gives no hint concerning the location of each soliton. Therefore, we cannot define their distance reliably. Usually, the position of the single solitons is defined by the value -1 of the quarternion field $U(\mathbf{r})$. Although this is not very satisfactory for multisoliton configurations and might even be doubtful, there is almost no alternative, e.g., analyzing the baryon density in Eq. (5). So we restricted ourselves onto a subspace of *Ansätze* for the field $U(\mathbf{r})$, which allows us to define properly the distance R and relative spin/isospin rotations of two Skyrmions. We make an *Ansatz* of the form

$$U(\mathbf{r}) = e^{i\tau\cdot\hat{\mathbf{r}}_+\theta(\mathbf{r}_+)}e^{i\tau\cdot\hat{\mathbf{r}}_-\theta(\mathbf{r}_-)} \ , \tag{7}$$

with $\hat{\mathbf{r}}_\pm = \mathbf{r}_\pm/|\mathbf{r}_\pm|$ and $\mathbf{r}_\pm = (x \pm R/2, y, z)$. The chiral field is given by

$$\theta(\mathbf{r}_\pm) = C(R)\tfrac{1}{2}\theta^{B=2}(\lambda|\mathbf{r}_\pm|) + [1-C(R)]\theta^{B=1}(\lambda|\mathbf{r}_\pm|[1+h(R)g(x,\rho)]), \quad [\alpha]=0, \tag{7a}$$

$$\theta(\mathbf{r}_\pm) = \theta^{B=1}(\lambda|\mathbf{r}_\pm|\{1+h(R)g(x,\rho)[1+s(R)\cos^2(\vartheta+\vartheta_0)]\}), \quad [\alpha]\neq 0 \ . \tag{7b}$$

Although this seems to correspond only to a minor change of the spherically symmetrical *Ansatz,* it will already give significant effects.

ρ and ϑ are the polar coordinates in the y-z plane. Note that we still keep up the hedgehog shape $\tau\cdot\hat{\mathbf{r}}$ but we allow for deformations in the chiral field $\theta(\mathbf{r})$. Let us now discuss in some detail the *Ansätze* (7a),(7b). For the case of no rotations, i.e., $[\alpha]=0$, we know that the product *Ansatz* $U^{B=2} = U_1^{B=1}U_2^{B=1}$ works well for large separations R; for small separations we know the exact solution to baryon number $B=2$. Calculating $\epsilon_A(R)$ for both *Ansätze,* we gain an accurate two-body potential by matching both solutions at $R=0.3$ fm (see also Ref. 7). This matching is described by the function $c(R)$ in (7a); $h(R)$ gives the strength of the deformation; $g(x,\rho)$ (the two solitons are aligned along the x axis) is a superposition of two Gaussian functions in the x and ρ directions, giving the geometrical deformation of the chiral field. $c(R)$, $h(R)$, and the two parameters in $g(x,\rho)$ have to be fitted for each separation R so as to minimize the energy. In the case of a spin/isospin rotation, the $B=2$ solution always gives a higher energy than the product of two $B=1$ solutions. So the matching procedure described above can be avoided. For the case $[\alpha]\neq 0$ there is no cylindrical symmetry in the problem as there is for the unrotated potential $v_A(R)$, so that an additional angular dependence enters. $s(R)$ and ϑ_0 have to be fitted for each separation R. The parametrization of the ϑ dependence was done in such a way as to minimize the number of adjustable parameters. Anyway, all our results are almost independent of the choice of ϑ_0. The parameter λ describes global scale changes as discussed in Ref. 5. It turns out to be unimportant in the unrotated case because all the gain in field energy is lost due to the fact that the relative distance of the Skyrmions is thereby increased.

Let us now discuss our results. Figure 1 shows the results for the unrotated case $[\alpha]=0$. The dashed-dotted curve corresponds to the *Ansatz* (7a) with $c=h=0$, $\lambda=1$ (product *Ansatz* of two $B=1$ hedgehogs without deformations), the dashed curve gives the result for $c=\lambda=1$, $h=0$ (undeformed $B=2$ hedgehog). Minimizing the field energy with respect to the set of parameters discussed above amounts to the potential shown by the solid curve of Fig. 1. Significant changes in the geometrical shape of the configuration $U(\mathbf{r})$ can be inferred from the fact that the strength parameter $h(R)$ is typically of the order of 0.5. The shape parameters in $g(x,\rho)$ [see Eq. (7a)] depend very weakly on the separation R. Summarizing, the deformation effects turn out to be of great importance for relative distances $0.4 \leq R \leq 2.0$ fm. They have a maximum at $r=0.6$ fm where the field energy is lowered by 250 MeV. Figure 2 shows equipotential lines of the chiral field for a separation

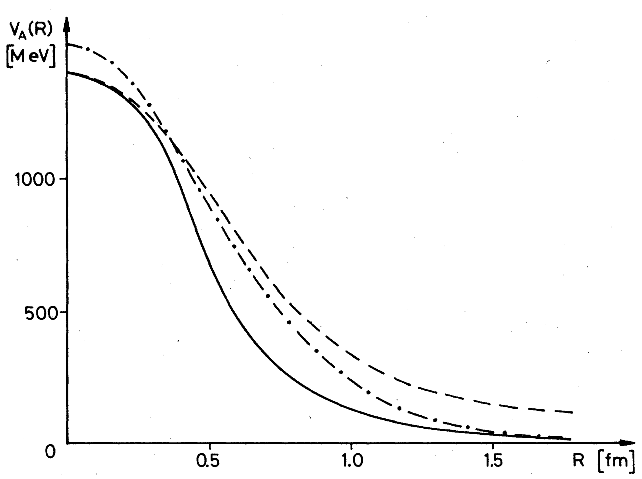

FIG. 1. Different approximations of $v_A(R)$ are compared. The dashed curve corresponds to $\lambda=1$, $c=1$, $h=0$ in Eq. (7a), the dashed-dotted curve to $\lambda=1$, $c=h=0$. Optimal parameters using the full *Ansatz* (7a) produce the solid curve.

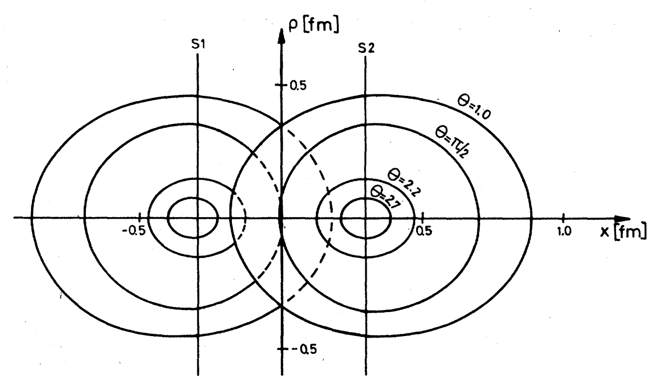

FIG. 2. Contour lines of the deformed chiral field $\theta(x,\rho)$. The solitons are located on the x axis with a separation of 0.6 fm.

TABLE I. Skyrmion-Skyrmion potentials v_B and v_C as defined in Eq. (6). $h=0$ and $\lambda=1$ give the undeformed case. Optimum stands for the variational procedure according to Eq. (7) giving the energy minimum. R is the relative separation of the two solitons.

R (fm)	v_B (MeV) $h=0$, $\lambda=1$	v_B (MeV) Optimum	v_C (MeV) $h=0$, $\lambda=1$	v_C (MeV) Optimum
0.0	1131	1057	769	586
0.1	1105	1023	746	577
0.2	1027	925	680	554
0.3	912	787	585	513
0.4	773	645	473	436
0.5	626	513	359	306
0.6	487	398	253	193
0.8	259	207	93	49
1.0	118	97	6	−15
1.2	44	24	−29	−34
1.4	12	4	−34	−37

of 0.6 fm. Each individual Skyrmion is squeezed in the overlap region and elongated on its other side.

Not such drastic changes can be seen for the rotated potentials v_B and v_C. In Table I we compare the undeformed potentials [$\lambda=1$, $h=0$ in (7b)] with the minimum of the variational procedure using the full *Ansatz* (7b). v_B is lowered by at most 130 MeV around $R\approx 0.4$ fm. For separations larger than 1 fm the effect can almost be ignored. Similar trends hold for v_C except for very small R where the global scale changes now significantly reduce the potential by about 180 MeV.

In summary, our calculations demonstrate the necessity of improved field configurations $U(r)$ to determine the N-N potential constructed from the Skyrme model more reliably. All potentials calculated with the deformed chiral field differ significantly from the ones using the spherical hedgehog *Ansatz.* This can explain the missing attraction in the central potential as calculated in Ref. 5. Similar observations have been made in Ref. 9. An answer to this question can only be given after a thorough investigation of all independent combinations of Euler angles. Furthermore, additional information about the ρ-vector and ω-tensor contributions can be gained from such an analysis.

Some doubts remain regarding how near our results are to the true minimum of the field energy $E(U(a,b))$. Therefore, we avoided a quantitative discussion at this stage of our investigation. In the case of v_B and v_C we strongly believe that the assumption of a radially pointing isospin vector $\mathbf{n}(\mathbf{r})=\hat{\mathbf{r}}$ should be given up. But there are no hints as to how a more general *Ansatz* for $\mathbf{n}(\mathbf{r})$ is to be chosen.

Recent lattice calculations[10] of the Skyrmion-Skyrmion interaction (unrotated case $[\alpha]=0$) seem to indicate that for strong overlap of the solitons the definition of the separation by the points $U(\mathbf{r})=-1$ breaks down. Therefore the trial function $U(\mathbf{r})$ minimizing the energy functional $E(U)$ might even look quite different from our result. But these lattice calculations suffer from two facts. Firstly, the rotated potentials require a huge amount of computer time due to the lack of cylindrical symmetry and, secondly, no convincing definition of the locations of the Skyrmions has been offered up to now.

Last but not least, we wish to mention that all existing calculations strictly rely on field energies gained from pure classical considerations. To extract a reliable N-N potential out of the Skyrme model, different approaches might be inevitable, such as a phase-shift analyses of the Skyrmion-Skyrmion scattering process, which is by no means trivial to calculate. This can clarify the very important question of whether the potential is really local and velocity independent as assumed in present calculations. It is uncertain that a static potential as extracted by different authors can survive such an investigation.

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