Harmonic expansions for $(4+K)$-dimensional Rarita-Schwinger fields on coset spaces and effective Lagrangian in four dimensions

Guy Fogleman*
TRIUMF, 4004 Wesbrook Mall, Vancouver, British Columbia, Canada V6T 2A3

K. S. Viswanathan
Department of Physics and Theoretical Science Institute, Simon Fraser University, Burnaby, British Columbia, Canada V5A 1S6

(Received 1 June 1984)

Harmonic expansions on the internal compact coset manifold G/H for the $(4+K)$-dimensional Rarita-Schwinger fields are developed. The dimensional reduction of the Rarita-Schwinger Lagrangian coupled to gravity in $4+K$ dimensions is carried out using these expansions. The resulting four-dimensional effective Lagrangian describes an infinite tower of massive spin-$\frac{3}{2}$ and spin-$\frac{1}{2}$ fields, coupled minimally and nonminimally to gauge fields. The masses of the Dirac fields are not given by the eigenvalues of the internal Rarita-Schwinger operator as is usually supposed.

I. INTRODUCTION

The theory of harmonic expansion on compact coset spaces of the form G/H for fields that occur in the Kaluza-Klein (KK) theories has been discussed by Salam and Strathdee[1] and by other authors.[2–5] The harmonic expansion for the Dirac fields and the dimensional reduction of the Dirac Lagrangian coupled to gravity were discussed in Ref. 1. Awada and Toms[6] have recently discussed the quantum effects of $(4+K)$-dimensional Dirac fields. In supergravity theories, one encounters the Rarita-Schwinger (RS) fields $\Psi_M(z)$ rather than the Dirac fields. Thus it would be of great interest in pondering the problem of harmonic expansions for the RS fields and with the help of these expansions obtain the effective Lagrangian in four dimensions. To the best of our knowledge this has not been done so far. Most of the work in the literature has been on the 11-dimensional supergravity ($d=11$ supergravity) theory involving spontaneous compactification on suitable seven-dimensional spaces of constant curvature.[7–12] In this article we address ourselves to the RS fields coupled only to gravity in $4+K$ dimensions. We develop harmonic expansions on G/H where G is a compact Lie group and $H\subset G$ for the $(4+K)$-dimensional fields $(\Psi_m(x,y),\Psi_\mu(x,y))$ ($m=0,1,2,3$ and $\mu=1,2,\ldots,K$) and $x\in R^4$ and $y\in G/H$. The four-dimensional spin-$\frac{3}{2}$ and the Dirac fields arise as coefficients in the expansion. It is observed that the four-dimensional RS fields are defined not just by the fields $E_a{}^M(z)\Psi_M(x,y)$ ($a=0,1,2,3$) but rather identified as the field

$$\Psi_a^{\rm RS}(x,y)=E_a{}^M\Psi_M(x,y)+\tfrac{1}{2}\Gamma_a(\Gamma^\alpha\Psi_\alpha)(x,y)\ .$$

This result is known[7] in the literature in connection with $d=11$ supergravity. It is shown here that this is true in any dimension. Furthermore, it is shown that in order to decouple the four-dimensional spin-$\frac{1}{2}$ and spin-$\frac{3}{2}$ quadratic Lagrangian one must impose a gauge condition

$$\Gamma^m\Psi_m^{\rm RS}(x,y)=0 \quad (1)$$

on the spin-$\frac{3}{2}$ fields. This gauge condition ensures that the fields Ψ_a do not contain the $(\frac{1}{2},0)\oplus(0,\frac{1}{2})$ representation of SO(1,3) and contain only the irreducible component $(1,\frac{1}{2})\oplus(\frac{1}{2},1)$. This is to be contrasted with the frequently discussed gauge condition[8,13]

$$\Gamma^M\Psi_M(z)=0 \quad (2)$$

$(M=0,1,2,\ldots,3+K)$ imposed on the $(4+K)$-dimensional RS fields. We find that it is sufficient to impose only the condition (1). It is shown below that the RS fields in four dimensions arise as coefficients in the harmonic expansion of

$$\Psi_a^{\rm RS}=\Psi_a+\tfrac{1}{2}\Gamma_a(\Gamma^\alpha\Psi_\alpha)\ . \quad (3)$$

If we require

$$\Gamma^a\Psi_a^{\rm RS}=0 \quad (a=0,1,2,3) \quad (4)$$

then the four-dimensional RS fields will belong to the $(1,\frac{1}{2})\oplus(\frac{1}{2},1)$ representation of SO(1,3). The harmonic expansion for the spin-$\frac{1}{2}$ Dirac particles are in terms of reducible representations of the group G.

The masses for the Rarita-Schwinger fields are, as is to be expected, given by the eigenvalues of the internal Dirac operator. It will be shown that masses of the Dirac fields are not given by the eigenvalues of the internal Rarita-Schwinger operator $\Gamma^{\mu\nu\rho}\nabla_\nu(y)$ but rather by the eigenvalues of the operator

$$M^{\mu\rho}=(\Gamma^\mu g^{\nu\rho}+\Gamma^\nu g^{\rho\mu}+\Gamma^\rho g^{\mu\nu})\nabla_\nu(y)\ . \quad (5)$$

In the next section we discuss the harmonic expansions for the Rarita-Schwinger fields in $4+K$ dimensions. In Sec. III the dimensional reduction of the RS Lagrangian coupled to gravity is carried out using these harmonic expansions and the Kaluza-Klein ansatz for the vielbein fields. We find that, in addition to the four-dimensional quadratic Rarita-Schwinger and Dirac Lagrangians minimally coupled to the gauge field, there are nonminimal coupling terms in the dimensionally reduced Lagrangian. These nonminimal "Pauli-moment" terms include a term coupling spin-$\frac{1}{2}$ and spin-$\frac{3}{2}$ fields through the field strength F_{mn}.

II. HARMONIC EXPANSIONS

Our treatment of harmonic expansions follows closely that of Salam and Strathdee.[1] We employ the notations contained therein. Coordinates in $4+K$ dimensions are denoted by $z^M \equiv (x^m, y^\mu)$. x^m refer to space-time coordinates and y^μ the K remaining coordinates. World indices are taken from the middle of the latin alphabet and the tangent-frame indices from the early part. Thus the index A in the vielbein $E_M{}^A(z)$ represents the SO(1,3+K) index. We split $A=(a,\alpha)$; a refers to SO(1,3) and α to SO(K) indices. We shall try to work with frame indices where possible. The RS field $\Psi_M(z)$ transforms under $(4+K)$-dimensional general coordinate and frame rotations as follows:

$$\Psi_M(z) \to \frac{\partial z^N}{\partial z'^M} S(a^{-1}) \Psi_N(z) \ . \tag{6}$$

Equivalently,

$$\Psi_A(z) \equiv E_A{}^M(z) \Psi_M(z) \to a_A{}^B S(a^{-1}) \Psi_B(z) \ . \tag{7}$$

$\Psi_M(z)$ thus is a frame spinor and a coordinate vector. Here $S(a)$ is the matrix of the spinorial representation and $a_A{}^B$ is the matrix of the vector representation of SO(1,3+K). The dimensionally reduced theory exhibits the following symmetries: (i) four-dimensional general covariance, (ii) local SO(1,3) invariance, and, (iii) local gauge invariance under G. Gauge transformations arise as coordinate transformations $y^\mu \to y'^\mu(y,x)$ in the extra K dimensions. These transformations ($y^\mu \to y'^\mu$ are induced by (left) translations by $g \in G$ and the associated tangent-frame rotations[1] described by $D_\alpha{}^\beta(h)$ ($h \in H \subset G$).

Under the reduced symmetry mentioned above, $\Psi_A(x,y)$ can be separated into $(\Psi_a(x,y), \Psi_\alpha(x,y))$ ($a=0,1,2,3, \alpha=1,2,\ldots,K$). Now $\Psi_a(x,y)$ transforms under (i) and (ii) as

$$\Psi_a(x,y) \to a_a{}^b(x) S(a^{-1}) \Psi_b(x,y) \ , \quad a_a{}^b \in \mathrm{SO}(1,3) \ , \tag{8}$$

while

$$\Psi_\alpha(x,y) \to a_\alpha{}^\beta(y) S(a^{-1}) \Psi_\beta(x,y) \ , \quad a_\alpha{}^\beta \in \mathrm{SO}(K) \ . \tag{9}$$

Since frame rotations in extra dimensions are associated with the transformations $D_\alpha{}^\beta(h)$ on G/H, we identify $a_\alpha{}^\beta$ with $D_\alpha{}^\beta(h)$ and thus $S(a^{-1}) \equiv \mathscr{D}(h)$, where $\mathscr{D}(h)$ is the matrix of the spinorial representation of H. Thus we have

$$\Psi_{\alpha,i}(x,y) \to D_\alpha{}^\beta(h) \mathscr{D}_{ij}(h) \Psi_{\beta j}(x,y) \tag{10}$$

for the transformation of $\Psi_{\alpha,i}$ under $y \to y'(y,x)$. The index i on Ψ_α is to remind us that Ψ_α, which is a $2^{2+[K/2]}$-component spinor, is decomposed into representations of H as outlined in Ref. 1. $\Psi_{a,i}$ is a four-component Dirac spinor. Similarly $\Psi_{a,i}(x,y)$ transforms under $y^\mu \to y'^\mu(y,x)$ as follows:

$$\Psi_{a,i}(x,y) \to \mathscr{D}_{ij}(h) \Psi_{a,j}(x,y) \ . \tag{11}$$

Note that, whereas Ψ_a transforms homogeneously under left translations, Ψ_m transforms as [see Eq. (6)]

$$\Psi'_m(x,y') = D(h) \left[\Psi_m(x,y) + \frac{\partial y^\nu}{\partial x'^m} \Psi_\nu \right] , \tag{12}$$

and since[1]

$$\frac{\partial y^\nu}{\partial x'^m} = -(g^{-1}\partial_m g)^{\hat\beta} K_{\hat\beta}{}^\nu(y) \ , \tag{13}$$

where $g \in h$ and $K_{\hat\beta}{}^\nu$ is a Killing vector, it can be seen that Ψ_m does not transform homogeneously. $\Psi_a = E_a{}^m \Psi_m + E_a{}^\mu \Psi_\mu$, however, transforms homogeneously as in (11) because of the gauge field contained in $E_a{}^\mu$ through the Kaluza-Klein ansatz. Ψ_μ does transform homogeneously since $E_\mu{}^a = 0$.

We now turn to the harmonic expansions for $\Psi_{a,i}(x,y)$ and $\Psi_{\alpha,i}(x,y)$. Consider the ansatz

$$\Psi_{a,i}(x,y) = \sum_n \left[\frac{d_n}{d_{\mathscr{D}}} \right]^{1/2} D^n_{ip}(L^{-1}(y)) \Psi^{(n)}_{a,p}(x) \tag{14}$$

and

$$\Psi_{\alpha,i}(x,y) = \sum_n D^n_{ip}(L^{-1}(y)) D_\alpha{}^{\hat\beta}(L^{-1}(y)) \phi^{(n)}_{\hat\beta,p}(x) \ , \tag{15}$$

where the sum on n in (14) and (15) is over all those irreducible representations of G whose restriction to H contains $\mathscr{D}(h)$, viz., the spinorial representation of H. For instance, take $G/H = \mathrm{SO}(8)/\mathrm{SO}(7)$. Then the sum over n in (14) and (15) includes all those irreducible representations of SO(8) that contain the $\underline{8}$ of SO(7). d_n and $d_{\mathscr{D}}$ are the dimensionalities of the representations (n) and $\mathscr{D}$, respectively. $D^n_{ip}(L^{-1}(y))$ is the matrix element of the nth representation of G. $L(y)$ is a representative point of a (left) coset of G. The expansion coefficients $\Psi^{(n)}_{a,p}(x)$ and $\phi^{(n)}_{\hat\alpha,p}(x)$ are the four-dimensional fields which carry a group index (n). $\Psi^{(n)}_{a,p}(x)$ transforms as a RS field in four dimensions. The fields $\phi^{(n)}_{\hat\alpha,p}(x)$ carry both an adjoint representation and a representation n of G. Thus $\phi^{(n)}_{\hat\alpha,p}(x)$ defines, in general, a reducible representation [ad$G \times n$]. $\phi^{(n)}_{\hat\alpha,p}$ transforms as a spin-$\frac{1}{2}$ field under local SO(1,3). Let us now consider their transformation properties under G. We first verify that (14) and (15) are consistent with (10) and (11).

Under $y^\mu \to y'^\mu$ induced by (left) translations by $g \in G$ (G acts transitively on G/H),

$$\left.\begin{aligned} L(y) &\to L(y') = gL(y)h^{-1} \\ L^{-1}(y) &\to L^{-1}(y') = hL^{-1}(y)g^{-1} \end{aligned}\right\} \left[\begin{matrix} g \in G \\ h \in H \end{matrix}\right] . \tag{16}$$

Then it follows from (14) that

$$\Psi_{a,i}(x,y) \underset{y\to y'}{\longrightarrow} \sum_n \left[\frac{d_n}{d_{\mathscr{D}}}\right]^{1/2} D^n_{ip}(L^{-1}(y'))\Psi'^{(n)}_{a,p}(x) = \mathscr{D}_{ij}(h)\sum \left[\frac{d_n}{d_{\mathscr{D}}}\right]^{1/2} D^n_{jq}(L^{-1}(y))D^n_{qp}(g^{-1})\Psi'^{(n)}_{a,p}(x) . \tag{17}$$

Thus (17) is consistent with (11) provided that $\Psi^{(n)}_{a,p}(x)$ transforms under G as

$$\Psi^{(n)}_{a,p}(x) \underset{G}{\rightarrow} \Psi'^{(n)}_{a,p}(x) = D^n_{pq}(g)\Psi^{(n)}_{a,q}(x) . \tag{18}$$

The field $\tilde{\Psi}_m \equiv E_m{}^a(x)\Psi_a(x)$ can obviously be expanded in a similar manner. $\Psi_m = E_m{}^A\Psi_A$ cannot, however, be expanded as Ψ_a is in Eq. (14) since Ψ_m does not transform homogeneously under left translations.

A similar argument shows that

$$\Psi_{a,i}(x,y) \underset{y\to y'}{\longrightarrow} \mathscr{D}_{ij}(h)D_a{}^\beta(h)\Psi_{\beta,j}(x,y) , \tag{19}$$

provided that

$$\phi^{(n)}_{\hat{\alpha},p}(x) \underset{G}{\rightarrow} \phi'^{(n)}_{\hat{\alpha},p}(x) = D^{\hat{\beta}}_{\hat{\alpha}}(g)D^n_{p,q}(g)\phi^{(n)}_{\hat{\beta},q}(x) . \tag{20}$$

In (20) $D_{\hat{\alpha}}{}^{\hat{\beta}}(g)$ is the matrix of the adjoint representation of G defined by

$$g^{-1}Q_{\hat{\alpha}}g = D_{\hat{\alpha}}{}^{\hat{\beta}}(g)Q_{\hat{\beta}} , \tag{21}$$

where $Q_{\hat{\alpha}}$ are the generators of G. The indices $\hat{\alpha}$ which run over the dimensionality of G are split as $Q_{\hat{\alpha}} = (Q_\alpha, Q_{\bar{\alpha}})$ where $Q_{\bar{\alpha}}$ are the generators of H and Q_α the remaining ones. In arriving at (19) from (16), we have made use of the fact that H is fully reducible; namely, $D_\alpha{}^{\bar{\beta}}(h) = 0$.

It is seen from (20) that $\phi^{(n)}_{\hat{\alpha},p}(x)$ transforms reducibly under G. Thus the massive spin-$\frac{1}{2}$ fields can be classified according to the Clebsch-Gordan (CG) decomposition $D^{(n)}\otimes D^{\rm ad}$ of G.

It will be useful later to have an expansion for $\Gamma^\alpha\Psi_\alpha(x,y)$ ($\alpha = 1,2,\ldots,K$). It can be readily seen from (19) that

$$(\Gamma^\alpha\Psi_\alpha(x,y)) \underset{y\to y'}{\longrightarrow} \mathscr{D}(h)(\Gamma^\alpha\Psi_\alpha(x,y)) , \tag{22}$$

where in arriving at (22) we have made use of the relation

$$\mathscr{D}(h)\Gamma^\beta\mathscr{D}(h^{-1})D_\beta{}^\alpha(h^{-1}) = \Gamma^\alpha \tag{23}$$

and of the fully reducible nature of H. Thus the harmonic expansion for $\Gamma^\alpha\Psi_\alpha(x,y)$, consistent with (22), may be written as

$$(\Gamma^\alpha\Psi_\alpha(x,y))_i = \sum_n D^n_{ip}(L^{-1}(y))\Omega^{(n)}_p(x) ; \tag{24}$$

$\Omega^{(n)}_p(x)$ are four-dimensional spinors. On the other hand, it follows from (15) that

$$(\Gamma^\alpha\Psi_\alpha(x,y))_i = \sum_{n'} \Gamma^\alpha_{ij}D_\alpha{}^{\hat{\beta}}(L^{-1}(y))D^{n'}_{jp}(L^{-1}(y))\phi^{(n')}_{\hat{\beta},p}(x) . \tag{25}$$

Thus we may derive the relationship between $\phi^{(n)}_{\hat{\alpha}\beta}(x)$ and $\Omega^{(n)}_p(x)$ in the form

$$\Omega^{(n)}_p(x) = \sum A^{n,n'}_{p,q,\hat{\beta}}\phi^{(n')}_{\hat{\beta},q}(x) , \tag{26}$$

where

$$A^{nn'}_{p,q,\hat{\beta}} = \frac{(d_n/d_{\mathscr{D}})}{V_K}\int_{G/H} d\mu(y)D^n_{pi}(L(y))\mathscr{D}_{ij}(\Gamma^\alpha)D^{n'}_{jq}(L^{-1})D_\alpha{}^{\hat{\beta}}(L^{-1}(y)) . \tag{27}$$

We have made use of the following result in arriving at (27):

$$\sum_i \int_{G/H} d\mu(y)D^n_{qi}(L(y))D^{n'}_{ip}(L^{-1}(y)) = V_K\frac{d_{\mathscr{D}}}{d_n}\delta_{nn'}\delta_{pq} . \tag{28}$$

V_K is the volume of G/H.

It is possible to give a group-theoretic interpretation to (26). It can be shown that

$$A^{nn'}_{p,q,\hat{\beta}} = a(n,n')\begin{bmatrix} {\rm ad}G & n' & n \\ \hat{\beta} & q & p\end{bmatrix} , \tag{29}$$

where

$$\begin{bmatrix} {\rm ad}G & n' & n \\ \hat{\beta} & q & p\end{bmatrix}$$

is a *CG* coefficient for the product representation $({\rm ad}G\otimes n')$ that contains n and

$$a(n,n') = \frac{\sqrt{d_n}}{d_{\mathscr{D}}}\Gamma^\alpha_{ij}\begin{bmatrix} {\rm ad}G & n' & n \\ \alpha & i & j\end{bmatrix}^* . \tag{30}$$

Thus

$$\Omega^n_p(x) = \sum_{n'} a(n,n')\chi^{(n,n')}_p(x) , \tag{31}$$

where

$$\chi^{(n,n')}_p \equiv \frac{1}{\sqrt{d_n}}\sum_{\hat{\beta},q}\begin{bmatrix} {\rm ad}G & n' & n \\ \hat{\beta} & q & p\end{bmatrix}\phi^{(n')}_{\hat{\beta},q}(x) . \tag{32}$$

Thus $\chi^{(n,n')}_p(x)$ is an irreducible basis for the representation n contained in $({\rm ad}G)\otimes(n')$. It should be remarked that the sum over both (n) and (n') (in the notation above) must be over those representations of G that contain $\mathscr{D}(h)$ of H. For instance, in $d = 11$, supergravity on $S^7 \equiv {\rm SO}(8)/{\rm SO}(7)$ both (n') and (n) in (31) refer to those representations of SO(8) that contain the $\underline{8}$ of SO(7) and $D^{\rm ad}\otimes D^{n'} = \sum D^{(n)}$.

Note that the internal Rarita-Schwinger gauge condition $\Gamma^{\alpha}\Psi_{\alpha}(x,y)=0$ implies that [see Eq. (15)]

$$[\Gamma^{\alpha}_{ij}D^{(n)}_{jp}(L^{-1})D_{\alpha}{}^{\hat\beta}(L^{-1})]\phi^{(n)}_{\hat\beta,p}(x)=0$$

for all representations (n) of the group G. This cannot hold in general for nontrivial $\phi^{(n)}_{\hat\beta,p}(x)$.

III. DIMENSIONAL REDUCTION

The RS Lagrangian in $4+K$ dimensions coupled to gravity is given by

$$\mathscr{L}_{\rm RS}=-\frac{i}{2}\overline{\Psi}_M\Gamma^{MNP}\nabla_N\Psi_P+\text{H.c.}\ , \tag{33}$$

where

$$\Gamma^{MNP}=\Gamma^M\Gamma^N\Gamma^P-\Gamma^M g^{NP}+\Gamma^N g^{MP}-\Gamma^P g^{MN} \tag{34}$$

and

$$\{\Gamma^M,\Gamma^N\}=2g^{MN}\ . \tag{35}$$

Due to antisymmetry of Γ^{MNP} only the spin-$\frac{1}{2}$ connection is needed in (33):

$$\nabla_N\Psi_P=(\partial_N+\tfrac{1}{2}B_{N[AB]}\Sigma^{AB}_{1/2})\Psi_P \tag{36}$$

and

$$\Sigma^{AB}_{1/2}=-\tfrac{1}{4}[\Gamma^A,\Gamma^B]\ . \tag{37}$$

Equation (33) may be rewritten in frame indices. Defining

$$\Psi_A=E^M_A\Psi_M\ , \tag{38}$$

$$\mathscr{L}_{\rm RS}=-\frac{i}{2}\overline{\Psi}_A\Gamma^{ABC}\nabla_B\Psi_C+\text{H.c.}\ , \tag{39}$$

where

$$\{\Gamma^A,\Gamma^B\}=2\eta^{AB},\ \ \Gamma^A=E_M{}^A\Gamma^M\ , \tag{40}$$

$$\eta^{AB}=\text{diag}(+1,-1,-1,\ldots,-1)\ , \tag{41}$$

$$\nabla_B\Psi_C=(\partial_B+\tfrac{1}{2}B_{B[DE]}\Sigma^{DE}_{1/2})\Psi_C-B_{B[CD]}\Psi^D\ , \tag{42}$$

$$\partial_B=E_B{}^M\partial_M\ , \tag{43}$$

and

$$E_A{}^M E_M{}^B=\delta_A{}^B\ . \tag{44}$$

$E_A{}^M$ are the vielbein fields.

In order to implement the dimensional reduction scheme, we use the Kaluza-Klein ansatz[14]

$$E_A{}^M(z)=\begin{bmatrix} E_a{}^m(x) & A_a{}^{\hat\alpha}(x)K_{\hat\alpha}{}^{\mu}(y)\\ 0 & e_\alpha{}^{\mu}(y)\end{bmatrix}\ , \tag{45}$$

where

$$A_a{}^{\hat\alpha}(x)\equiv A_m{}^{\hat\alpha}(x)E_a{}^m(x) \tag{46}$$

are the gauge fields for G. $K_{\hat\alpha}{}^{\mu}(y)$ are the Killing vectors on G/H. The coordinate dependence is as specified in the ansatz.

The inverse vielbein can be seen to be given by

$$E_M{}^A(x,y)=\begin{bmatrix} E_m{}^a(x) & -A_m{}^{\hat\alpha}(x)D_{\hat\alpha}{}^{\beta}(L(y))\\ 0 & e_\mu{}^{\beta}(y)\end{bmatrix}\ . \tag{47}$$

The matrix $D_{\hat\alpha}{}^{\beta}(L(y))$ (defined in 21) is related to $K_{\hat\alpha}{}^{\mu}(y)$ by

$$K_{\hat\alpha}{}^{\mu}(y)e_\mu{}^{\beta}(y)=D_{\hat\alpha}{}^{\beta}(L(y))\ . \tag{48}$$

The ansatz (46) for the vielbein coefficients is consistent with the symmetry of $M^{(4)}\otimes B_K$ [M^4 is four-dimensional Einstein space with vielbein $E_m{}^a(x)$ and B_K a coset space G/H with K-bein $e_\alpha{}^{\mu}(y)$].

It is also necessary to know the spin-connection coefficients. We take these as solutions to zero-torsion constraints. These are given in Ref. 1. We quote these below (note some sign differences with Ref. 1):

$$B_{a[b,c]}(x)=-(\tfrac{1}{2}E_{[a}{}^mE_{b]}{}^n\partial_mE_{nc}-\tfrac{1}{2}E_{[b}{}^mE_{c]}{}^n\partial_mE_{na}$$
$$+\tfrac{1}{2}E_{[c}{}^mE_{a]}{}^n\partial_mE_{nb})\ , \tag{49a}$$

$$B_{a[b\gamma]}=-B_{\gamma[ab]}=\tfrac{1}{2}F_{ab}{}^{\hat\alpha}(x)D_{\hat\alpha\gamma}(L(y))\ , \tag{49b}$$

$$B_{a[\beta\gamma]}=-A_a{}^{\hat\delta}[D_{\hat\delta}{}^{\bar\alpha}(L(y))-D_{\hat\delta}{}^{\alpha}(L(y))\pi_\alpha{}^{\bar\alpha}(y)]C_{\bar\alpha\beta\gamma}\ , \tag{49c}$$

$$B_{\alpha[\beta,c]}=0\ , \tag{49d}$$

$$B_{\alpha[\beta,\gamma]}=\tfrac{1}{2}C_{\alpha\beta\gamma}+\pi_\alpha{}^{\bar\alpha}(y)C_{\bar\alpha\beta\gamma}\ , \tag{49e}$$

where the following notations have been used:

$$E_{[a}{}^mE_{b]}{}^n=E_a{}^mE_b{}^n-E_b{}^mE_a{}^n\ , \tag{50a}$$

$$F_{ab}{}^{\hat\alpha}(x)=E_a{}^mE_b{}^n(\partial_mA_n{}^{\hat\alpha}-\partial_nA_m{}^{\hat\alpha}-C_{\hat\beta\hat\gamma}{}^{\hat\alpha}A_m{}^{\hat\beta}A_n{}^{\hat\gamma})\ , \tag{50b}$$

$$\pi_\alpha{}^{\bar\alpha}(y)=e_\alpha{}^{\mu}(y)e_\mu{}^{\bar\alpha}(y)\ . \tag{50c}$$

The embedding of H into SO(K) is given by

$$D_{ij}(Q_{\bar\alpha})=\tfrac{1}{2}C_{\bar\alpha\beta\gamma}\Sigma^{\beta\gamma}_{ij}\ , \tag{50d}$$

where the structure constants $C_{\hat\alpha\hat\beta\hat\gamma}$ are defined by

$$[Q_{\hat\alpha},Q_{\hat\beta}]=C_{\hat\alpha\hat\beta}{}^{\hat\gamma}Q_{\hat\gamma}\ . \tag{50e}$$

It remains for us to substitute (45) and (49) and the expansions (14) and (15) into the Lagrangian (33) and integrate over the extra K coordinates. The resultant four-dimensional Lagrangian describes an infinite tower of spin-$\frac{3}{2}$ and spin-$\frac{1}{2}$ fields which are coupled to the gauge fields $A_m{}^{\hat\alpha}(x)$ both minimally and nonminimally. The two main problems that arise are (a) to identify the four-dimensional RS and Dirac fields and (b) to make sure that the four-dimensional RS field transforms as $(1,\frac{1}{2})+(\frac{1}{2},1)$ representation of SO(1,3).

We first separate the indices A,B,C into those that belong to SO(1,3) and to SO(K), respectively. The RS field Ψ_A can be split as

$$\Psi_A=(\Psi_a,\Psi_\alpha),\ \ \Psi_a=E_a{}^M\Psi_M\ , \tag{51}$$

and

$$\Psi_a = E_a{}^M \Psi_M = e_a{}^\mu(y)\Psi_\mu \ . \tag{52}$$

Retaining only the kinetic and mass terms in the Lagrangian, we obtain, after some algebra, the following:

$$\begin{aligned}\mathcal{L}_{\text{quad}} = -\frac{i}{2}[&\overline{\Psi}_a\Gamma^{abc}\nabla_b(x)\Psi_c + \overline{\Psi}_a\Gamma^{ab\gamma}\nabla_b(x)\Psi_\gamma \\ &+\overline{\Psi}_\alpha\Gamma^{\alpha bc}\nabla_b(x)\Psi_c + \overline{\Psi}_\alpha\Gamma^{\alpha b\gamma}\nabla_b\Psi_\gamma \\ &+\overline{\Psi}_a\Gamma^{a\beta c}\nabla_\beta(y)\Psi_c + \overline{\Psi}_a\Gamma^{a\beta\gamma}\nabla_\beta(y)\Psi_\gamma \\ &+\overline{\Psi}_\alpha\Gamma^{\alpha\beta c}\nabla_\beta(y)\Psi_c + \overline{\Psi}_\alpha\Gamma^{\alpha\beta\gamma}\nabla_\beta(y)\Psi_\gamma] \\ &+\text{H.c.} \ ,\end{aligned} \tag{53}$$

where

$$\Psi_a = E_a{}^M(x)\Psi_M \ ,$$

$$\nabla_b(x)\Psi_c = (\partial_b + \tfrac{1}{2}B_{b[de]}\Sigma^{de}_{1/2})\Psi_c - B_{b[cd]}\Psi^d \ , \tag{54a}$$

$$\widetilde{\partial}_b = E_b{}^m(x)\partial_m \ , \tag{54b}$$

$$\nabla_b(x)\Psi_\gamma = (\widetilde{\partial}_b + \tfrac{1}{2}B_{b[de]}\Sigma^{de}_{1/2})\Psi_\gamma \ , \tag{54c}$$

$$\nabla_\beta(y)\Psi_a = (\partial_\beta + \tfrac{1}{2}B_{\beta[\gamma\delta]}\Sigma^{\gamma\delta}_{1/2})\Psi_a \ , \tag{54d}$$

$$\nabla_\beta(y)\Psi_\gamma = [(\partial_\beta + \tfrac{1}{2}B_{\beta[\delta\epsilon]}\Sigma^{\delta\epsilon}_{1/2})\Psi_\gamma - B_{\beta[\gamma\delta]}\Psi^\delta] \ . \tag{54e}$$

The first four terms in Eq. (53) represent the kinetic energy while the rest represent the mass of the RS and Dirac fields in four dimensions. It is clear from above that Ψ_a and Ψ_α are coupled in both the kinetic and mass terms. Thus Ψ_a (upon harmonic expansion) cannot be identified as the four-dimensional RS field.

However it is easy to check that it is possible to rewrite the kinetic terms in the following form:

$$\begin{aligned}\mathcal{L}_{\text{KE}} = -\frac{i}{2}&(\overline{\Psi}_a + \tfrac{1}{2}\overline{\Psi}_\gamma\Gamma^\gamma\Gamma_a)\Gamma^{abc}\nabla_b(x)(\Psi_c + \tfrac{1}{2}\Gamma_c\Gamma^\delta\Psi_\delta) \\ &+\overline{\Psi}_\alpha[\Gamma^b\nabla_b(x)](\eta^{\alpha\beta} + \tfrac{1}{2}\Gamma^\alpha\Gamma^\beta)\Psi_\beta + \text{H.c.}\end{aligned} \tag{55}$$

Thus we define

$$\Psi^{\text{RS}}_a(x,y) = \Psi_a(x,y) + \tfrac{1}{2}\Gamma_a(\Gamma^\alpha\Psi_\alpha) \tag{56}$$

and in terms of Ψ^{RS}_a and Ψ_α, $\mathcal{L}_{\text{KE}}$ is completely decoupled:

$$\begin{aligned}\mathcal{L}_{\text{KE}} = -\frac{i}{2}[&\overline{\Psi}^{\text{RS}}_a\Gamma^{abc}\nabla_b(x)\Psi^{\text{RS}}_c + \overline{\Psi}_\alpha\Gamma^b\nabla_b(x) \\ &\times(\eta^{\alpha\beta} + \tfrac{1}{2}\Gamma^\alpha\Gamma^\beta)\Psi_\beta] \ .\end{aligned} \tag{57}$$

Cremmer and Julia[7] have already, in their $d=11$ supergravity study, introduced this definition for Ψ_a and we are here simply establishing its validity for the general case. Note that the Dirac part of the Lagrangian is not diagonal. Cremmer and Julia diagonalize it by introducing, in the case of $d=11$ supergravity,

$$\lambda_{ijk} = \left[\frac{3}{\sqrt{2}}\right]\Gamma^\alpha{}_{[ij}\Psi_{k]\alpha} \ , \tag{58}$$

where the right-hand side of (58) is antisymmetrized with respect to i,j, and k; i,j,k refer to SO(7) spinor indices. It can be easily checked that (58) does not work for general G/H and works only for the 11-dimensional case.

Consider next the mass terms. In terms of Ψ^{RS}_a and Ψ_α we find

$$\begin{aligned}\mathcal{L}_{\text{mass}} = -\frac{i}{2}[&2\overline{\Psi}^{\text{RS}}_a\Sigma^{ab}\nabla\!\!\!/(y)\Psi^{\text{RS}}_b - 3(\overline{\Psi}_\alpha\Gamma^\alpha)\nabla\!\!\!/(y)(\Gamma^\beta\Psi_\beta) \\ &-\tfrac{1}{2}\overline{\Psi}_a\Gamma^a\nabla\!\!\!/(y)(\Gamma^\alpha\psi_\alpha) - (\overline{\Psi}_a\Gamma^a)\nabla^\beta(y)\Psi_\beta \\ &-\tfrac{1}{2}(\overline{\Psi}_\alpha\Gamma^\alpha)\nabla\!\!\!/(y)\Gamma^c\Psi_c - \overline{\Psi}_a\nabla^\alpha(y)\Gamma^c\psi_c \\ &+\overline{\Psi}_\alpha\Gamma^{\alpha\beta\gamma}\nabla_\beta(y)\Psi_\gamma] + \text{H.c.}\end{aligned} \tag{59}$$

Here

$$\nabla\!\!\!/(y) = \Gamma^\alpha\nabla_\alpha(y) \ .$$

The mass terms contain coupling between the RS and Dirac fields. The first term in (59) has the structure of the mass term for the RS fields while the last term has the structure of the mass term for the Dirac fields. The coupling terms arise in (59) due to the fact that $\Psi^{\text{RS}}_a(x,y)$ or rather the fields $\Psi^{(n)}_{ap}(x)$ defined in (14) do not transform according to $(1,\frac{1}{2})\oplus(\frac{1}{2},1)$ representation of SO(1,3) but contain an admixture of $(\frac{1}{2},0)\oplus(0,\frac{1}{2})$ representation. Thus it will be necessary to impose a gauge condition on $\Psi^{(n)}_{a,p}(x)$ or equivalently on $\Psi^{\text{RS}}_a(x,y)$, so it contains only the spin-$\frac{3}{2}$ fields:

$$\Gamma^a\Psi^{\text{RS}}_a(x,y) = 0 \quad (a=0,1,2,3) \ . \tag{60}$$

From (56) and (60) we find

$$\Gamma^a\Psi_a = -2\Gamma^\alpha\Psi_\alpha \ . \tag{61}$$

We can now eliminate the terms containing Ψ_a in (59) and express $\mathcal{L}_{\text{mass}}$ in terms of Ψ^{RS}_a and Ψ_α only:

$$\begin{aligned}\mathcal{L}_{\text{mass}} = -\frac{i}{2}[&2\overline{\Psi}^{\text{RS}}_a\Sigma^{ab}\nabla\!\!\!/(y)\Psi^{\text{RS}}_b + \overline{\Psi}_\alpha(\Gamma^\alpha\eta^{\beta\gamma} + \Gamma^\beta\eta^{\gamma\alpha} \\ &+\Gamma^\gamma\eta^{\alpha\beta}) \\ &\times\nabla_\beta(y)\Psi_\alpha] + \text{H.c.}\end{aligned} \tag{62}$$

In (62) as well as in (57) we have not bothered to apply the gauge constraint $\Gamma^a\Psi^{\text{RS}}_a = 0$. Thus Eqs. (57) and (62) are the starting points of our harmonic expansion program. Let us discuss, first, the dimensional reduction of the RS Lagrangian given by

$$\mathcal{L}^{(4)}_{\text{RS}} = -\frac{i}{2}[\overline{\Psi}^{\text{RS}}_a\Gamma^{abc}\nabla_b(x)\Psi^{\text{RS}}_c + 2\overline{\Psi}^{\text{RS}}_a\Sigma^{ab}\nabla\!\!\!/(y)\Psi^{\text{RS}}_b] \ . \tag{63}$$

On substituting the harmonic expansion similar to (14) into (63), i.e.,

$$\Psi^{\text{RS}}_a(x,y) = \sum \left[\frac{d_n}{d_{\mathcal{D}}}\right]^{1/2} D^{(n)}_{ip}(L^{-1}(y))\Psi^{(n)}_{a,p}(x) \ , \tag{64}$$

and integrating over G/H we obtain

$$\mathscr{L}^{(4)}_{\mathrm{RS}}=-\tfrac{1}{2}V_K[\overline{\Psi}^{(n)}_{a,p}\Gamma^{abc}i\nabla_b(x)\Psi^{(n)}_{c,p}-\overline{\Psi}^{(n)}_{a,p}\Sigma^{ab}M^{nn'}_{pq}\Psi^{n'}_{b,q}]\,, \tag{65}$$

where

$$M^{nn'}_{p,q}=-\frac{2i}{V_K}\left[\frac{d_n d_{n'}}{d_{\mathscr{D}}{}^2}\right]^{1/2}\int_{G/H}d\mu(y)D^{(n)}_{pi}(L(y))[\Gamma^\alpha\nabla_\alpha(y)]_{ij}D^{(n)}_{jq}(L^{-1})\,. \tag{66}$$

The integral in (66) can be further simplified by using (49) and

$$\partial_\beta D(L^{-1})=-[D(Q_\beta)+\pi_\beta{}^{\overline{\beta}}D(Q_{\overline{\beta}})]D(L^{-1}(y))\,. \tag{67}$$

We skip the details of this calculation. The result is

$$M^{nn'}_{pq}=2i\frac{\delta_{nn'}\delta_{pq}}{d_{\mathscr{D}}}\,\mathrm{Tr}[\Gamma^\beta D^{(n)}(Q_\beta)+\tfrac{1}{8}C_{\alpha\beta\gamma}\Gamma^{\alpha\beta\gamma}]\,, \tag{68}$$

where $\Gamma^{\alpha\beta\gamma}$ is the antisymmetric combination of product of three Γ^α's. The trace in (68) is over the subspace of H which is embedded in SO(K) (Ref. 1). Thus the effective Lagrangian (65) describes an infinite tower of massive spin-$\frac{3}{2}$ multiplets which are characterized by representation n of G that contain the *spinor* representation of SO(K). The masses are in units of Planck mass scale M_p (typically inverse of a characteristic length scale of B_K). The masses are given explicitly by

$$M(n)=2\frac{i}{d_{\mathscr{D}}}\,\mathrm{Tr}_H[\Gamma^\alpha D^{(n)}(Q_\alpha)+\tfrac{1}{8}C_{\alpha\beta\gamma}\Gamma^{\alpha\beta\gamma}]\,. \tag{69}$$

The expression (69) for the masses of the four-dimensional spin-$\frac{3}{2}$ fields is the same as found by Salam and Strathdee for the Dirac spinor in Ref. 1. If the coset space G/H is of the form SO($K+1$)/SO(K), then $C_{\alpha\beta\gamma}=0$. [Note: α,β,γ refer to the generators not belonging to SO(K) subgroup of SO($K+1$).] Then the masses of the four-dimensional spin-$\frac{3}{2}$ fields are given by

$$M(n)=2\frac{i}{d_{\mathscr{D}}}\,\mathrm{Tr}_{\mathrm{SO}(K)}[\Gamma^\alpha D^{(n)}(Q_\alpha)]\,.$$

IV. THE DIRAC LAGRANGIAN

Let us now turn to the dimensional reduction of the spin-$\frac{1}{2}$ part of the Lagrangian. From (57) and (62) we have

$$\begin{aligned}\mathscr{L}_{\mathrm{Dirac}}=\frac{-i}{2}[&\Psi_\alpha(\Gamma^a\nabla_a(x))(\eta^{\alpha\beta}+\tfrac{1}{2}\Gamma^\alpha\Gamma^\beta)\Psi_\beta\\ &+\overline{\Psi}_\alpha(\Gamma^\alpha\eta^{\beta\gamma}+\Gamma^\beta\eta^{\gamma\alpha}+\Gamma^\gamma\eta^{\alpha\beta})\nabla_\beta(y)\Psi_\gamma]\\ &+\mathrm{H.c.}\end{aligned} \tag{70}$$

We substitute the harmonic expansion (15) for Ψ_α and also the spin connection coefficients from (49e) and integrate over G/H. After simplification the resulting four-dimensional Lagrangian is given by

$$\mathscr{L}_{\mathrm{Dirac}}=-\tfrac{1}{2}V_K\{\overline{\phi}^{(n)}_{\hat{\alpha},p}[i\nabla\!\!\!/(x)I^{\hat{\alpha}\hat{\gamma}nn'}_{pq}-M^{\hat{\alpha}\hat{\gamma}nn'}_{pq}]\phi^{(n')}_{\hat{\gamma},q}\} +\mathrm{H.c.}\,, \tag{71}$$

where

$$I^{\hat{\alpha}\hat{\gamma}nn'}_{pq}=\frac{1}{V_K}\int_{G/H}d\mu(y)D^n_{pi}(L)D^{\hat{\alpha}}{}_\alpha(L)\times(\eta^{\alpha\beta}+\tfrac{1}{2}\Gamma^\alpha\Gamma^\beta)_{ij}D^{n'}_{jq}(L^{-1})D_\beta{}^{\hat{\gamma}}(L^{-1})\,, \tag{72}$$

and

$$M^{\hat{\alpha}\hat{\gamma}nn'}_{pq}=\frac{-i}{V_K}\int d\mu(y)D^{(n)}_{pi}(L)D^{\hat{\alpha}}{}_\alpha(L)\times(\Gamma^{\{\alpha}\eta^{\beta\gamma\}}\nabla_\beta(y))_{ij}D^{n'}_{jq}(L^{-1})D_\gamma{}^{\hat{\gamma}}(L^{-1})\,, \tag{73}$$

where

$$\Gamma^{\{\alpha}\eta^{\beta\gamma\}}\equiv\Gamma^\alpha\eta^{\beta\gamma}+\Gamma^\beta\eta^{\gamma\alpha}+\Gamma^\gamma\eta^{\alpha\beta}\,.$$

These integrals can be evaluated using the techniques of Ref. 1. The results are

$$I^{nn'}_{\hat{\alpha}\hat{\gamma},pq}=\frac{1}{d_N}\begin{bmatrix}\mathrm{ad}G & n & N\\ \hat{\alpha} & p & Q\end{bmatrix}^* K^N(n,n')\begin{bmatrix}\mathrm{ad}G & n' & N\\ \hat{\gamma} & q & Q\end{bmatrix}, \tag{74}$$

where

$$K^N(n,n')=\begin{bmatrix}\mathrm{ad}G & n & N\\ \alpha & i & p\end{bmatrix}(\eta^{\alpha\beta}+\tfrac{1}{2}\Gamma^\alpha\Gamma^\beta)_{ij}\begin{bmatrix}\mathrm{ad}G & n' & N\\ \beta & j & p\end{bmatrix}^*, \tag{75}$$

and

$$M^{\hat{\alpha}\hat{\gamma}nn'}_{pq}=\frac{1}{d_N}\begin{bmatrix}\mathrm{ad}G & n & N\\ \hat{\alpha} & p & Q\end{bmatrix}M^N(n,n')\begin{bmatrix}\mathrm{ad}G & n' & N\\ \hat{\gamma} & q & Q\end{bmatrix}, \tag{76}$$

where

$$M^N(n,n')=i\begin{bmatrix}\mathrm{ad}G & n & N\\ \alpha & i & Q\end{bmatrix}\Gamma^{\{\alpha}_{ik}\eta^{\beta\gamma\}}\times[D^{n'}(Q_\beta)-\tfrac{1}{4}C_{\beta\delta\epsilon}\Sigma^{\delta\epsilon}]_{kq}\begin{bmatrix}\mathrm{ad}G & n' & N\\ \gamma & q & Q\end{bmatrix}^*. \tag{77}$$

Note that n and n' are not summed over in (74) to (77) and N is not summed over in (75) and (77). The notation here is that $\Sigma^{\delta\epsilon}_{kq}=0$ unless q refers to the subgroup H index. The symbols

$$\begin{bmatrix} \mathrm{ad}G & n & N \\ \hat{\alpha} & p & Q \end{bmatrix}$$

denote the Clebsch-Gordan coefficients for G. N is a representation contained in the direct product $D^{\mathrm{ad}G}\otimes D^n$. The normalization here is such that

$$\sum_{N,Q} \begin{bmatrix} \mathrm{ad}G & n & N \\ \alpha & p & Q \end{bmatrix} \begin{bmatrix} \mathrm{ad}G & n' & N \\ \gamma & q & Q \end{bmatrix}^* = \eta_{\alpha\gamma}\delta_{pq}\delta^{nn'} . \quad (78)$$

Using the basis function $\chi_P^{(N,n)}(x)$ defined in Eq. (32), the Dirac Lagrangian may be written in the form

$$\mathscr{L}_{\mathrm{Dirac}} = -\tfrac{1}{2}V_K\{\bar{\chi}_P^N(n)[i\nabla\!\!\!/(x)K^N(n,n') - M^N(n,n')]\chi_P^N(n')\} + \mathrm{H.c.} \quad (79)$$

Note that the mass for the Dirac field is not given by the eigenvalues of the internal Rarita-Schwinger operator, $\Gamma^{\alpha\beta\gamma}\nabla_\beta(y)$, but rather by those of the operator

$$\Gamma^{\{\alpha}\eta^{\beta\gamma\}}\nabla_\beta(y) . \quad (80)$$

When integrated over the coordinates of the coset space, this operator gives the masses in Eq. (77). Using the orthonormality of the Clebsch-Gordan coefficients [Eq. (78)], the sum rule

$$\sum_N M_{\mathrm{Dirac}}^N(n,n') = \frac{d_{\mathscr{D}}}{2}(K+2)M_{\mathrm{RS}}(n)\delta^{nn'} , \quad (81)$$

which relates the masses of the Dirac fields to those of the Rarita-Schwinger fields, can be obtained. $M_{\mathrm{RS}}(n)$ is defined in Eq. (69) and K is the number of extra dimensions. The sum in Eq. (81) is over all representations N contained in the direct product $D^{\mathrm{ad}G}\otimes D^n$.

Note that the Dirac Lagrangian in the form of Eq. (79) is not diagonal in the group index.

V. COUPLING TERMS

The remaining terms in the dimensionally reduced Rarita-Schwinger Lagrangian couple the four-dimensional fields $\Psi_a^{\mathrm{RS}}(x)$, $\chi_p^{N,n}(x)$, and $A_a^{\hat{\alpha}}(x)$. In particular, the minimal coupling $\mathscr{L}'$ terms are given by (after a lengthy calculation)

$$\mathscr{L}' = -\frac{V_K}{2}[-i\bar{\Psi}_{a,p}^{(n)}(x)\Gamma^{abc}A_b{}^{\hat{\alpha}}(x)D_{pq}^{(n)}(Q_{\hat{\alpha}})\Psi_{c,q}^{(n)}(x) - i\bar{\chi}_p^N(n)\Gamma^b A_b{}^{\hat{\alpha}}(x)D_{pq}{}^N(Q_{\hat{\alpha}})K^N(n,n')\chi_q{}^N(n')] . \quad (82)$$

These two terms make the derivatives in (65) and (79) covariant with respect to the gauge group G.

The remaining coupling terms are nonminimal Pauli-moment type terms. They are given by (we skip the algebraic details)

$$\mathscr{L}_{\mathrm{Pauli}} = -\tfrac{1}{2}V_K\left[N_1 \begin{bmatrix} & n & n' \\ \hat{\alpha} & p & q \end{bmatrix} (\epsilon^{abcd}\bar{\Psi}_{a,p}^n\gamma_5 F_{bc}^{\hat{\alpha}}\Psi_{d,q}^{n'} + 2\bar{\Psi}_{a,p}^n F^{ab\hat{\alpha}}\Psi_{b,q}^{n'}) + N_2 \begin{bmatrix} \hat{\lambda} & \hat{\delta} & n & n' \\ \hat{\alpha} & & p & q \end{bmatrix} \bar{\phi}_{\hat{\lambda},p}^n (F_{ab}^{\hat{\alpha}}\Sigma^{ab})\phi_{\hat{\delta},q}^{n'} \right.$$
$$\left. + N_3 \begin{bmatrix} \hat{\delta} & n & n' \\ \hat{\lambda} & p & q \end{bmatrix} \bar{\Psi}_{a,p}^n \Gamma^{abc} F_{bc}^{\hat{\lambda}} \phi_{\hat{\delta},q}^{n'} + N_4 \begin{bmatrix} \hat{\delta} & n & n' \\ \hat{\lambda} & p & q \end{bmatrix} \bar{\phi}_{\hat{\delta},p}^n F_{ab}^{\hat{\lambda}} \Gamma^{abc}\Psi_{q,c}^{n'} \right] + \mathrm{H.c.} , \quad (83)$$

where

$$N_1 \begin{bmatrix} & n & n' \\ \hat{\alpha} & p & q \end{bmatrix} = \frac{1}{8V_k}\int D_{pi}^n(L)D_{\hat{\alpha}\beta}(L)\Gamma_{ij}^\beta D_{jq}^{n'}(L^{-1}) , \quad (84)$$

$$N_2 \begin{bmatrix} \hat{\lambda} & \hat{\delta} & n & n' \\ \hat{\alpha} & & p & q \end{bmatrix} = \frac{1}{4V_K}\int D_{pi}^n(L)D^{\hat{\lambda}}{}_\alpha(L)D_{\hat{\alpha}\beta}(L)\Gamma_{ij}^{\{\alpha}\eta^{\beta\gamma\}}D_\gamma{}^{\hat{\delta}}(L^{-1})D_{jq}^{n'}(L^{-1}) , \quad (85)$$

$$N_3 \begin{bmatrix} \hat{\delta} & n & n' \\ \hat{\lambda} & p & q \end{bmatrix} = -\frac{1}{4V_K}\int D_{pi}^n(L)D_{\hat{\lambda}\beta}(L)(\eta^{\beta\gamma} + \tfrac{1}{2}\Gamma^\beta\Gamma^\gamma)_{ij}D_{jq}^{n'}(L^{-1})D_\gamma{}^{\hat{\delta}}(L^{-1}) , \quad (86)$$

and

$$N_4 = N_3^+ . \quad (87)$$

Note that $\mathscr{L}_{\mathrm{Pauli}}$ contains Pauli-moment terms for Rarita-Schwinger fields, Dirac fields, and also a term coupling Rarita-Schwinger and Dirac fields. The integrals (84)—(87) can be evaluated using the techniques employed earlier. When this is done the Dirac fields are expressed in terms of the irreducible $\chi_p^N(n)$'s instead of the reducible $\phi_{\alpha,p}^n$'s.

VI. SUMMARY

We have implemented the harmonic expansions for the RS fields in Kaluza-Klein theories over coset spaces and the dimensional reduction of the RS Lagrangian using the

Kaluza-Klein ansatz for the vielbein fields. The complete four-dimensional effective Lagrangian for the tower of massive spin-$\frac{1}{2}$ and spin-$\frac{3}{2}$ fields coupled to gauge fields is presented. In addition to the gauge-invariant kinetic and mass terms there arise nonminimal gauge-invariant Pauli-moment terms. The masses for the four-dimensional fields have been computed in terms of the irreducible representation matrices of G.

ACKNOWLEDGMENTS

The authors gratefully acknowledge fruitful conversations with Bruce Campbell. One of us (G.F.) expresses his appreciation to the theory group of SLAC for their hospitality during his stay there. This work was partially supported by the Natural Sciences and Engineering Research Council of Canada.

---

*Present address: Department of Physics & Astronomy, San Francisco State University, San Francisco, CA 94132.

[1]A. Salam and J. Strathdee, Ann. Phys. (N.Y.) **141**, 316 (1982).

[2]W. Mecklenburg, Fortschr. Phys. **32**, 207 (1984).

[3]K. S. Viswanathan, in *Modern Kaluza-Klein Theories,* edited by H. C. Lee (World Scientific, Singapore, 1984).

[4]Sho. Tanaka, Prog. Theor. Phys. **70**, 563 (1983).

[5]D. Toms, report, 1984 (unpublished).

[6]M. A. Awada and D. Toms, Nucl. Phys. **B245**, 161 (1984).

[7]E. Cremmer and B. Julia, Nucl. Phys. **B159**, 141 (1979).

[8]F. Englert and H. Nicolai, CERN Report No. TH-3711, 1983 (unpublished).

[9]R. D'Auria and P. Fré, Report No. I.T.P.-SB-83-57 (unpublished).

[10]M. J. Duff and C. N. Pope, in *Supersymmetry and Supergravity '82,* edited by S. Ferrara, J. G. Taylor, and P. van Nieuwenhuizen (World Scientific, Singapore, 1983).

[11]A. Casher, F. Englert, and H. Nicolai, Nucl. Phys. **B243**, 173 (1984).

[12]B. de Wit, Report No. NIKHEF-H/83-18, 1983 (unpublished).

[13]E. Witten, Princeton report, 1983 (unpublished).

[14]J. F. Luciani, Nucl. Phys. **B135**, 111 (1978); E. Witten, *ibid.* **B186**, 412 (1981).