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Polarization phenomena in collinear reactions

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It is shown for a collinear reaction containing four particles with arbitrary spins which amplitudes remain nonzero and how they are related to the observables. In terms of primary observables all submatrices relating products of amplitudes to observables either vanish or turn into one-by-one submatrices, except the 8_i types which may turn into three-by-three submatrices, but these latter submatrices are mostly avoidable when determining amplitudes. In terms of the secondary observables the 1_M and 2_i submatrices are slightly larger. Specifically, it is shown that in collinear reactions all observables in which only one particle is polarized (no matter how) vanish. Since reactions at very high energies are expected to be predominantly very close to being collinear, the smallness of such observables in such reactions can be expected on general grounds but polarization effects involving observables with more than one polarized particle can very well be very large. An iterative approximation method for the polarization analysis of reactions at very high energies is suggested. The results of this paper are also applicable to all models in which helicity conservation holds, since they are, for all t values, formally identical with collinear reactions.

I. INTRODUCTION

This paper offers a general discussion of the polarization structure of collinear reactions, that is, reactions for which the final-state particles emerge in the same direction as the initial particles came in at (in the c.m. system).

The motivation for such a discussion is, first, that for any given reaction the spin structure is considerably simplified in the collinear case as compared to the general kinematic case. Collinearity reduces the number of amplitudes, thus easing all the tasks polarization analysis aims at: the testing of conservation laws, the phenomenological determination of reaction amplitudes, the checking of theoretical models, the searching for dynamical clues, etc.

Collinearity, however, is a very special and restricted kinematic configuration, and so the feasibility and practical utilization may still be questioned even in the light of the above discussed advantages. The case for the utility of collinearity is, however, greatly strengthened by the fact that very-high-energy reactions strongly tend to be bunched at very small reaction angles. Thus collinearity for such reactions is a realistic and good approximation, in which deviations from this approximation can then be expected to produce corrections in terms of some small additional amplitudes which are zero in the exactly collinear case. Thus we can approach all high-energy polarization phenomena in terms of this simpler, collinear approximation, with small correction terms to be included at a later, more refined stage. With this argument the polarization structure of collinear reactions assumes not only a substantial but in fact a possibly central role in the study of particle reactions at very high energies.

In addition, any theoretical model in which helicity conservation holds, that is, in which the sum of the helicities of the initial particles is equal to the sum of the helicities of the final-state particles, is formally identical, at all t values, to a collinear reaction. Thus the results of this paper are applicable to all such models also. This lends this paper an even broader range of applicability.

The spin structure of collinear reactions has been investigated in a paper[1] some twelve years ago, and many of those results remain valid and useful. In particular, the *number* of amplitudes in a collinear reaction involving particles of arbitrary spins was then determined and hence need not be presented here. The shortcoming of that previous paper was, however, in its using a polarization formalism which, by today's standards, was not maximally advantageous, and therefore when it came to specifying *which* amplitudes survive in the collinear case and how the observable-amplitude relationships simplify, only partial specifications could be given for the case of the arbitrary spins. Thus the main improvement in the present paper is the use of the optimal formalism[2] which will allow us to specify the observable-amplitude relationship completely and hence offer guidelines for the design of experimental programs.

II. REDUCTION OF AMPLITUDES BY COLLINEARITY

In using the optical formalism to describe collinear reactions, we first need to choose a quantization direction best suited for the purpose. In the present case the choice is obvious: the quantization axis should be the direction of collinearity, so that we use the planar optimal frame[3] with $\beta=0$ which coincides with the helicity frame.

In this frame the condition of collinearity can be satisfied exceedingly simply. This condition is [see Eq. (2.2) of Ref. 1]

$$s_{az}+s_{bz}=s_{cz}+s_{dz} \, , \tag{2.1}$$

where the s_i's ($i=a,b,c,d$) are the spins of particles a, b, c, and d, respectively, in the reaction

$$a+b\rightarrow c+d \tag{2.2}$$

and the z subscript denotes the component in the quantization direction. The amplitudes in the optimal formalism are denoted by $D(\lambda,l,\Lambda,L)$, where $\lambda=s_{cz}$, $l=s_{az}$, $\Lambda=s_{dz}$, and $L=s_{bz}$. Thus the collinearity condition requires that only those optimal amplitudes survive for which

$$l+L=\lambda+\Lambda\ . \tag{2.3}$$

This condition holds regardless of whether the amplitudes are or are not also constrained by symmetry conditions due to parity conservation, time-reversal invariance, identical-particle constraints, etc. This condition is therefore responsible for the reduction in the *number* of nonzero amplitudes as given by Eqs. (2.3)—(2.5) of Ref. 1.

III. SIMPLIFICATION IN THE OBSERVABLE-AMPLITUDE RELATIONS. PRIMARY OBSERVABLES

The relationship between the observables and the bilinear combinations (products) of reaction amplitudes (called "bicoms") for a four-particle reaction containing particles with arbitrary spins was given in Table I of Ref. 4, and Figs. 1 and 2 of that reference give the amplitude structure. That these results hold for any reaction regardless of the value of the spins was shown in Ref. 5.

The above-cited table indicates that for any reaction, there will be one type of eight-by-eight matrices, four types of four-by-four matrices, six types of two-by-two matrices, four types of one-by-one matrices, and finally one type of a special kind of one-by-one matrix containing only magnitude squares of amplitudes. These groups of matrices are denoted by 8_i, 4_i, 2_i, 1_i, and 1_M, respectively.

From the point of view of the impact of collinearity on these submatrices, the important point to recall is that in 1_M the bicoms contain the same amplitude twice; in 1_i the bicoms contain two amplitudes which differ from each other only in one of the four arguments; in 2_i the two amplitudes in the bicoms differ in two arguments; in 4_i in three; and finally in 8_i in all four.

Let us now look at these submatrices individually. The 1_M type one-by-one matrices will either survive unchanged or vanish when we impose the collinearity constraints, depending on whether the amplitude that a particular one of them contains satisfies Eq. (2.3) or not.

Turning now to the 1_i's, these contain bicoms of the form

$$D(\lambda,l;\Lambda,L)D^*(\lambda',l;\Lambda,L) \tag{3.1}$$

or similar ones where not the λ index but one of the other indices changes. If in these bicoms $l+L\neq\lambda+\Lambda$ and/or $l+L\neq\lambda'+\Lambda$, then the whole 1_i vanishes. The 1_j would not vanish if we had simultaneously

$$l+L=\lambda+\Lambda \ \text{ and } \ l+L=\lambda'+\Lambda \tag{3.2}$$

with $\lambda\neq\lambda'$, but that is impossible. Thus all 1_i's vanish in the collinear case. Since the 1_i's contain all those observables in which three of the four particles are unpolarized and only one particle is polarized somehow (it does not matter how and in which direction), we obtain the following completely general and important theorem.

In any four-particle reaction with particles with arbitrary spins, in a collinear configuration all those experimental observables vanish identically in which three particles are unpolarized and the spin state of only one particle is specified.

The importance of this theorem in particle physics at very high energies needs to be stressed. It has often been said that polarization phenomena are expected to play a small and unimportant role in particle physics at very high energies.[6] Implicitly this statement is thought to be equivalent to the expectation that the simple vector polarization of one particle in a four-particle reaction with the other three particles being unpolarized is very small. This statement can be faulted on several counts. For one thing, some experiments now indicate[7] that at least in some kinematic domains for some reactions, even this simple polarization quantity is substantial even at tens of GeV energies. Second, our theorem indicates that for small reaction angles (where most of the events are for very-high-energy reactions) the simple polarization quantities should indeed be fairly small on general grounds, but this by no means indicates that *all* polarization effects are negligibly small. On the contrary, it is quite legitimate to expect that observables involving the simultaneous polarizations of two or more particles in the reaction could be quite large. Until, therefore, such observables are measured, *a priori* statements on the magnitude of polarization effects are purely speculative, even if some data show that the simple individual vector polarizations are small.

We now turn to the 2_i submatrices. The two bicoms in the first of the 2_i's, as evident from Table I of Ref. 4, need to satisfy, for the collinear case, the following four conditions:

$$l+L=\lambda+\Lambda,\ \ l+L=\lambda'+\Lambda'\ , \tag{3.3}$$

$$l+L=\lambda'+\Lambda,\ \ l+L=\lambda+\Lambda'\ , \tag{3.4}$$

with $\lambda\neq\lambda'$ and $\Lambda\neq\Lambda'$. It is possible to satisfy the two conditions in Eq. (3.3) simultaneously, though it is also possible that one or both of them are violated. Even if both of them are satisfied, and hence that bicom is nonzero, the two conditions in Eq. (3.4) are at the same time violated. Similar results hold for the other five types of 2_i's also. Thus we arrive at the conclusion that the 2_i's in the collinear case either vanish entirely or only one of their two bicoms survive, that is, the 2_i turns into a one-by-one matrix.

We now turn to the 4_i's. Looking at the first of the four 4_i's in Table I of Ref. 4, we see, as we did for the 2_i's, that eight conditions must be satisfied:

$$l+L=\lambda+\Lambda,\ \ l+L'=\lambda'+\Lambda'\ , \tag{3.5}$$

$$l+L=\lambda'+\Lambda,\ \ l+L'=\lambda+\Lambda'\ , \tag{3.6}$$

$$l+L'=\lambda+\Lambda,\ \ l+L=\lambda'+\Lambda'\ , \tag{3.7}$$

$$l+L'=\lambda'+\Lambda,\ \ l+L=\lambda+\Lambda'\ , \tag{3.8}$$

with $\lambda\neq\lambda'$, $L\neq L'$, and $\Lambda\neq\Lambda'$. An inspection of these conditions yields the same result that we had for the 2_i's, namely, that each 4_i either vanishes altogether or collapses into a one-by-one matrix.

Finally, we turn to the 8_i's. Here we obtain 16 relations that need to be satisfied in the collinear case, namely,

$$l+L=\lambda+\Lambda,\quad l'+L'=\lambda'+\Lambda'\,, \qquad (3.9)$$

$$l'+L=\lambda'+\Lambda,\quad l+L'=\lambda+\Lambda'\,, \qquad (3.10)$$

$$l'+L=\lambda+\Lambda,\quad l+L'=\lambda'+\Lambda'\,, \qquad (3.11)$$

$$l+L=\lambda'+\Lambda,\quad l'+L'=\lambda+\Lambda'\,, \qquad (3.12)$$

$$l+L'=\lambda+\Lambda,\quad l'+L=\lambda'+\Lambda'\,, \qquad (3.13)$$

$$l'+L'=\lambda'+\Lambda,\quad l+L=\lambda+\Lambda'\,, \qquad (3.14)$$

$$l'+L'=\lambda+\Lambda,\quad l+L=\lambda'+\Lambda'\,, \qquad (3.15)$$

$$l+L'=\lambda'+\Lambda,\quad l'+L=\lambda+\Lambda'\,, \qquad (3.16)$$

with $l\neq l'$, $L\neq L'$, $\lambda\neq\lambda'$, and $\Lambda\neq\Lambda'$.

An inspection of these eight pairs of conditions shows that, for example, if Eq. (3.9) is satisfied, Eqs. (3.10) and (3.16) are also, but the others are not. So for the 8_i's, either all bicoms vanish, or five of eight do, thus reducing the eight-by-eight to a three-by-three matrix.

In summary, therefore, we see that if we use the primary optimal observables, that is, those in which the spin states of all four particles are specified, then each 1_M either vanishes or remains an unchanged one-by-one matrix; all 1_i's vanish; each 2_i and 4_i either vanishes completely or turns into a one-by-one matrix; and each 8_i either vanishes completely or turns into a three-by-three matrix.

We see then that except for the 8_i's, the relationship between the observables and the bicoms is entirely diagonal (i.e., is in terms of one-by-one submatrices). Furthermore, the "blemish" caused by the three-by-three matrices from the 8_i's is in a practical sense unimportant, since the observables in the 8_i's never contain D (in the notation of Table I of Ref. 4) and thus no index can be averaged over. Thus the 8_i's contain only observables which genuinely and unalterably require the specification of the polarization direction of all four particles. Such observables are not only very inaccessible experimentally but are also known to be altogether avoidable even in the most ambitious of all possible experimental programs, namely, in the one determining all reaction amplitudes unambiguously. Thus we can say that in a practical sense, in any collinear reaction, the observable-bicom matrix is completely diagonal if we use the primary observables.

IV. THE OBSERVABLE-AMPLITUDE RELATIONS IN AN ALTERNATE VIEW

In this brief section the results of Sec. III are justified in another, more visualizable way. Although much simpler than the method used in the previous section, this method is probably more suitable as an *a posteriori* plausibility argument than as a rigorous proof.

In the primary observables of the optimal formalism the arguments in the observables are D, R, and I. These three arguments correspond to[3] various polarization quantities in the three directions of the quantization direction (D) and the two directions perpendicular to it (R and I). Now in a collinear reaction these latter two directions are indistinguishable, and hence two observables which go into each other under the transformation

$$R\leftrightarrow I \qquad (4.1)$$

simultaneously for every argument in the observable, must be either equal to each other or perhaps may differ in sign only.

Using this rule, one can obtain all of the results of Sec. III. This is left as an exercise for the reader.

V. THE OBSERVABLE-AMPLITUDE RELATIONSHIP. SECONDARY OBSERVABLES

For technical reasons, experimental programs of polarization measurements prefer to use not the primary observables of Sec. III but secondary observables[8] in which the polarization state of each particle is either averaged

TABLE I. The amplitudes for $\frac{1}{2}+\frac{1}{2}\rightarrow\frac{1}{2}+\frac{1}{2}$ under various symmetries and for the collinear case. For notation and explanation, see text.

No.	L	$L+E$	$L+T$	$L+E+T$	$L+T+E+P$
1	$(++++)$	$(++++)$	$(++++)$	$(++++)$	$(++++)$
2	$+++-$	$+++-$	$+++-$	$+++-$	$+++-$
3	$++-+$	$++-+$	$=-[2]$	$=-[2]$	$=-[2]$
4	$+-++$	$=-[2]$	$+-++$	$=-[2]$	$=-[2]$
5	$-+++$	$=-[3]$	$=-[4]$	$=[2]$	$=[2]$
6	$(++--)$	$(++--)$	$(++--)$	$(++--)$	$(++--)$
7	$+-+-$	$+-+-$	$+-+-$	$+-+-$	$+-+-$
8	$(+--+)$	$(+--+)$	$(+--+)$	$(+--+)$	$(+--+)$
9	$(-++-)$	$=[8]$	$=[8]$	$=[8]$	$=[8]$
10	$-+-+$	$-+-+$	$=[7]$	$=[7]$	$=[7]$
11	$(--++)$	$=[6]$	$(--++)$	$=[6]$	$=[6]$
12	$+---$	$+---$	$+---$	$+---$	$=-[2]$
13	$-+--$	$-+--$	$=-[12]$	$=-[12]$	$=[2]$
14	$--+-$	$=-[12]$	$--+-$	$=-[12]$	$=[2]$
15	$---+$	$=-[13]$	$=-[14]$	$=[12]$	$=-[2]$
16	$(----)$	$(----)$	$(----)$	$(----)$	$=[1]$

TABLE II. The relationship between observables and amplitudes for collinear elastic pp scattering with $L+P+T+I$. For notation, see text.

				Re			Im		
	$\lvert a\rvert^2$	$\lvert c\rvert^2$	$\lvert e\rvert^2$	ac^*	ae^*	ce^*	ac^*	ae^*	ce^*
$AAAA$	2	2	2						
$\Delta\Delta AA$	2	−2	2						
$\Delta A\Delta A$	2	2	−2						
$ARAR$				2					
$AR\Delta I$							−2		
$ARRA$					2				
$ARI\Delta$								−2	
$AARR$						2			
$A\Delta RI$									−2

over (i.e., the particle is unpolarized) or it is made into a composite satisfying the null criterion. (The null criterion requires that the sum of the coefficients of the primary arguments forming the secondary designation vanish.) It is therefore useful to describe the polarization structure of collinear reactions also in terms of secondary observables.

To do this, one needs the analog of Table I of Ref. 4, in terms of secondary observables. This has not appeared in the literature before and so its structure is described in the Appendix.

Using that structure, one can readily see that the situation for collinear reactions is as follows.

The 1_M matrix in terms of secondary observables is an N-by-N matrix where N is the number of amplitudes in the collinear reaction.

The 1_i's all vanish even in terms of the secondary observables.

The 2_i's in general (see Appendix) have $\eta_\alpha\eta_\beta$ terms (if we consider the 2_i of type $2_{\alpha\beta}$). In the collinear case this either reduces to η_γ terms (where η_γ is the smaller of η_α and η_β), or the 2_i vanishes altogether. Here $\eta=2s+1$, where s is the spin of the particle.

The 4_i's, even in terms of secondary observables, will turn into one-by-one submatrices or vanish altogether.

Finally, the 8_i's are the same whether we use primary or secondary observables and hence they will, here also, either vanish altogether or turn into three-by-three matrices.

Thus we see that the difference between using primary or secondary observables is in the 1_M and 2_i's only. Both change from the simple and convenient one-by-one matrices into something larger (though in many practical cases not into something huge). This is the inevitable price one must pay the experimental convenience of using secondary observables.

It should be remembered that since Table I of Ref. 4 and the Appendix of the present paper both deal with the case when only Lorentz invariance is imposed on the reaction, the results of Secs. III and V represent upper limits of the complexity of collinear reactions. For collinear which are also constrained by additional symmetries like parity conservation, time reversal invariance, identical particles, etc., the above outlined structure may further simplify. A specific example of this is given in the next section.

VI. EXAMPLES

A. pp elastic scattering

The amplitudes for the reaction $\frac{1}{2}+\frac{1}{2}\to\frac{1}{2}+\frac{1}{2}$ are listed in Table I, under L, $L+E$, $L+T$, $L+E+T$, and $L+E+T+P$, where L denotes Lorentz invariance, E identical-particle constraints, T time-reversal invariance, and P parity conservation. In each column the amplitudes which survive in the special case of collinear kinematics are enclosed in parentheses. We see that the number of collinear amplitudes is 6 (L), 4 ($L+E$), 5 ($L+T$), 4 ($L+E+T$), and 3 ($L+E+T+P$). The last of these represents the actual reaction of pp elastic scattering.

The observable-bicom relationship for the noncollinear $L+E+T+P$ case is given in Table V of Ref. 3. We can obtain the collinear case from that table by imposing

$$\genfrac{}{}{0pt}{}{+\;+}{+\;-}=\genfrac{}{}{0pt}{}{+\;-}{+\;-}=0\,. \tag{6.1}$$

Using the notation of Ref. 3, namely,

$$\genfrac{}{}{0pt}{}{+\;+}{+\;+}\equiv a,\quad \genfrac{}{}{0pt}{}{+\;+}{-\;-}\equiv c,\quad \genfrac{}{}{0pt}{}{+\;-}{-\;+}\equiv e\,, \tag{6.2}$$

we then get the observable-bicom relationship for collinear pp elastic scattering given in Table II.

In this case the 8_i ends up containing magnitude squares of amplitudes just as 1_M does which, however, contains simpler observables so the 8_i can be ignored altogether. We note, however, that, in accordance with our prediction, it does collapse into a three-by-three matrix. The 1_i's vanish altogether as our general investigation requires. The 2_i's in this case turn into one-by-one matrices also even in terms of the secondary observables. In general we would expect them to turn into two-by-two ma-

TABLE III. The amplitudes for $pp\to d\pi$ under various symmetries. Amplitudes in parentheses are those which survive in the collinear case. For notation, see text.

No.	1	2	3	4	5	6	7	8	9	10	11	12
L	$\left(\genfrac{}{}{0pt}{}{+\;+}{+}\right)$	$\genfrac{}{}{0pt}{}{+\;-}{-}$	$\genfrac{}{}{0pt}{}{+\;+}{-}$	$\genfrac{}{}{0pt}{}{+\;-}{+}$	$\genfrac{}{}{0pt}{}{0\;+}{+}$	$\genfrac{}{}{0pt}{}{0\;-}{-}$	$\left(\genfrac{}{}{0pt}{}{0\;+}{-}\right)$	$\left(\genfrac{}{}{0pt}{}{0\;-}{+}\right)$	$\genfrac{}{}{0pt}{}{-\;+}{+}$	$\left(\genfrac{}{}{0pt}{}{-\;-}{-}\right)$	$\genfrac{}{}{0pt}{}{-\;+}{-}$	$\genfrac{}{}{0pt}{}{-\;-}{+}$
$L+P$	$\left(\genfrac{}{}{0pt}{}{+\;+}{+}\right)$	$\genfrac{}{}{0pt}{}{+\;-}{-}$	$\genfrac{}{}{0pt}{}{+\;+}{-}$	$\genfrac{}{}{0pt}{}{+\;-}{+}$	$\genfrac{}{}{0pt}{}{0\;+}{+}$	=[5]	$\left(\genfrac{}{}{0pt}{}{0\;+}{-}\right)$	=−[7]	=−[2]	=−[1]	=[4]	=[3]

TABLE IV. The relationship between observables and amplitudes for collinear reaction of $pp \to \pi d$, with only Lorentz invariance. For notation, see text.

	$\lvert A \rvert^2$	$\lvert B \rvert^2$	$\lvert C \rvert^2$	$\lvert D \rvert^2$	ReCB^*	ImCB^*	ReBD^*	ImBD^*	ReCD^*	ImCD^*
AAA	$+$	$+$	$+$	$+$						
$A\Delta A$	$+$	$-$	$+$	$-$						
ΔAA	$+$	$+$	$-$	$-$						
$\Delta\Delta A$	$+$	$-$	$-$	$+$						
RRA					$+$					
RIA						$+$				
RAR^+							$+$			
$RAI^{+}_{(-)}$								$+$		
ARR^+									$+$	
$ARI^{+}_{(-)}$										$+$

trices, but parity conservation in this case simplifies that. The 4_i's were predicted to either vanish or turn into one-by-one matrices. In our example they all vanish. Finally, the 1_M is indeed an N-by-N matrix, where in this case $N=3$.

The observables in Table II are given in the standard notation of the optimal formalism. The correspondence between this and the old notation traditional in pp scattering (P, A, C_{NN}, D_{LS}, etc.) is given in Table VI of Ref. 3.

B. $pp \to d\pi$

The amplitudes for this and spinwise similar reactions are given in Table III under L and $L+P$. The amplitudes which survive in the collinear case are in parentheses. The numbers, of collinear amplitudes are 9 (L) and 2 ($L+P$).

The observables-bicom relationship for the noncollinear case is given in Table I of Ref. 9. Using the notations of Ref. 9 and

$$\underset{+}{+\,+} \equiv A, \quad \underset{-}{0\,+} \equiv B ,$$

$$\underset{-}{0\,-} \equiv C, \quad \underset{-}{-\,-} \equiv D ,$$

then we get the observables-bicom relationships for this reaction as presented in Table IV with only the Lorentz invariance. Table V represents the observable-bicom relationships with $L+P$ invariance.

In this case the largest submatrices are four-by-four which collapse into one-by-one submatrices containing only magnitudes squared of amplitudes just as 1_M does,

TABLE V. Observable-bicom relationship for $pp \to d\pi$ and spinwise-similar reactions with the imposition of discrete symmetries.

	$\lvert A \rvert^2$	$\lvert B \rvert^2$	ReAB^*	ImAB^*
AAA	2	2		
$\Delta\Delta A$	2	-2		
RAR^-			$+$	
RAI^-				$+$

but containing simpler observables. The 2_i's turn into one-by-one matrices and the 1_i's vanish altogether. Finally, the 1_M which is a three-by-three matrix reduces to a two-by-two matrix under parity conservation.

C. πd elastic scattering

The amplitudes for this reaction are given in Table VI under L, $L+P$, and $L+P+T$. Again, the amplitudes indicated in parentheses are those that survive in the collinear reaction. The numbers of collinear amplitudes are 3 (L), 2 ($L+P$), 2 ($L+P+T$).

Using the following notations for the amplitudes $D(c,a)$,

$$(+\,+)=A \text{ and } (0\,0)=D ,$$

we get the observable-bicom relationship for $\pi d \to \pi d$ as given in Table VII. Table VIII is the same as Table VII but with the imposition of discrete symmetries.

For this reaction the largest submatrices are two-by-two submatrices which collapse into one-by-one matrices in the collinear case. The 1_M becomes a three-by-three submatrix. With the imposition of $L+P+T$ 1_M further shrinks to a two-by-two matrix as shown in Table VIII.

TABLE VI. The amplitudes for πd elastic scattering under various symmetries. Amplitudes in parentheses are those which survive in the collinear case. For notation see text.

No.	L	$L+P$	$L+P+T$
1	$(+\,+)$	$(+\,+)$	$(+\,+)$
2	$+\,-$	$+\,-$	$+\,-$
3	$+\,0$	$+\,0$	$+\,0$
4	$0\,+$	$0\,+$	$=-[3]$
5	$0\,-$	$=-[4]$	$=-[4]$
6	$(0\,0)$	$(0\,0)$	$(0\,0)$
7	$-\,+$	$=[2]$	$=[2]$
8	$(-\,-)$	$=[1]$	$=[1]$
9	-0	$=-[3]$	$=-[3]$

TABLE VII. The relationship between observables and amplitudes for collinear πd elastic scattering with only Lorentz-invariance consideration. For notation see text.

	$\lvert A \rvert^2$	$\lvert C \rvert^2$	$\lvert D \rvert^2$	$\mathrm{Re}AC^*$	$\mathrm{Im}AC^*$	$\mathrm{Re}AD^*$	$\mathrm{Im}AD^*$	$\mathrm{Re}CD^*$	$\mathrm{Im}CD^*$
A,A	$+$	$+$	$+$						
Δ,Δ	$+$	$+$	0						
A,Δ	$+$	$-$	0						
R^+,I^+									-2
R^0,I^0					-2				
R^-,I^-							-2		
R^+,R^+								$+2$	
R^0,R^0				$+2$					
R^-,R^-						$+2$			

TABLE VIII. Observable-bicom relationship with the imposition of Lorentz invariance plus parity conservation and time-reversal invariance for $\pi d \rightarrow \pi d$ elastic scattering.

	$\lvert A \rvert^2$	$\lvert D \rvert^2$	$\mathrm{Im}AD^*$	$\mathrm{Re}AD^*$
A,A	2	$+$		
Δ,Δ	2			
A,Δ	$=0$			
R^+,I^+			-2	
R^+,R^+				$+2$

TABLE IX. The amplitudes for $\pi N \rightarrow \pi N$ and spinwise-similar reactions under various symmetries. Amplitudes in parentheses are those which survive in the collinear case.

No.	L	$L+T$	$L+T+P$
1	$(+\,+)$	$(+\,+)$	$(+\,+)$
2	$+\,-$	$+\,-$	$+\,-$
3	$-\,+$	$=-[2]$	$=-[2]$
4	$(-\,-)$	$(-\,-)$	$=[1]$

TABLE X. The relationship between observables and amplitudes for collinear reaction of $\pi N \rightarrow \pi N$ and spinwise-similar reactions with only Lorentz invariance. For notation see text.

	$\lvert a \rvert^2$	$\lvert b \rvert^2$	$\mathrm{Re}ab^*$
A,A	4	4	
A,Δ	4	-4	
A,R^0			4

D. πN elastic scattering

The amplitudes for the reactions of $0+\frac{1}{2}\rightarrow 0+\frac{1}{2}$ are given in Table IX under L, $L+T$, and $L+T+P$. The surviving amplitudes in the collinear case are again in parentheses. The numbers of such amplitudes are 2 (L), 2 ($L+T$), 1 ($L+T+P$). Denoting the amplitude $D(c,a)$ as $+\,+=a$ we get the observable-bicom relationships for πN elastic scattering as given in Table X. With the imposition of discrete symmetries Table X shrinks to a single relation as

$$AA \equiv \sigma = 8\,\lvert a \rvert^2 \,.$$

VII. SUMMARY AND CONCLUSIONS

We see that it is easy to determine which amplitudes will be nonzero for collinear reactions containing particles with arbitrary spins, and how the bicoms of those amplitudes will relate to experimental observables. In particular, we saw that in terms of primary observables the 1_M submatrix vanishes or remains one-by-one, the 1_i submatrices all vanish, the 2_i and 4_i submatrices either vanish or turn into one-by-one matrices, and only the 8_i submatrices (which can actually be avoided when determining the reaction amplitudes from the polarization data) are somewhat more involved in that they either vanish or turn into three-by-three matrices.

In terms of secondary observables (i.e., the ones experimentalists prefer to work with), the situation for the 1_i, 4_i, and 8_i's is the same as it was for the primary observables. The 1_M's, however, combine into an N-by-N submatrix (where N is the number of amplitudes in the collinear reaction), and the 2_i's of the type $2_{\alpha\beta}$ either vanish or turn into an η_γ-by-η_γ submatrix, where η_γ is the smaller of η_α and η_β, and $\eta=2s+1$.

Since reactions at very high energies are predominantly very close to collinear, they can be assumed to be dominated by those amplitudes which are nonzero collinearly and the other amplitudes can be expected to be small.

TABLE XI. The relationship between secondary observables and amplitude products. The number to the left of the parentheses is the number of submatrices of a given type for primary observables, the parenthetical number is the size of the submatrix with secondary observables, and the number to the right of the parentheses is the number of a given type for secondary observables. For further explanation, see text.

Type	$0+\frac{1}{2}\to 0+\frac{1}{2}$	$\frac{1}{2}+\frac{1}{2}\to\frac{1}{2}+\frac{1}{2}$	$0+\frac{1}{2}\to 0+\frac{3}{2}$	$\frac{1}{2}+\frac{1}{2}\to\frac{1}{2}+\frac{3}{2}$	$0+\frac{3}{2}\to 0+\frac{3}{2}$
8	0(8)0	2(8)2	0(8)0	12(8)12	0(8)0
4_1	0(4)0	4(8)2	0(4)0	24(8)12	0(4)0
4_2	0(8)0	4(8)2	0(8)0	24(8)12	0(16)0
4_3	0(4)0	4(8)2	0(4)0	24(8)12	0(4)0
4_4	0(8)0	4(8)2	0(16)0	8(16)2	0(16)0
2_{12}	0(4)0	8(8)2	0(4)0	48(8)12)	0(8)0
2_{13}	2(2)2	8(8)2	12(2)12	48(8)12	72(2)72
2_{14}	0(4)0	8(8)2	0(8)0	16(16)2	0(8)0
2_{23}	0(4)0	8(8)2	0(4)0	48(8)12	0(8)0
2_{24}	0(8)0	8(8)2	0(16)0	16(16)2	0(32)0
2_{34}	0(4)0	8(8)2	0(8)0	16(16)2	0(8)0
1_{123}	4(2)2	16(8)2	24(2)12	96(8)12	48(4)12
1_{124}	0(4)0	16(8)2	0(8)0	32(16)2	0(16)0
1_{134}	4(2)2	16(8)2	8(4)2	32(16)2	48(4)12
1_{234}	0(4)0	16(8)2	0(8)0	32(16)2	0(16)0
1_M	4(4)1	16(16)1	8(8)1	32(32)1	16(16)1

Type	$0+0\to 0+1$	$0+1\to 0+1$	$0+1\to 1+1$	$1+1\to 1+1$	$0+0\to 0+2$
8	0(8)0	0(8)0	0(8)0	162(8)162	0(8)0
4_1	0(4)0	0(4)0	54(4)54	162(12)54	0(4)0
4_2	0(4)0	0(12)0	0(12)0	162(12)54	0(4)0
4_3	0(4)0	0(4)0	0(12)0	162(12)54	0(4)0
4_4	0(12)0	0(12)0	0(12)0	162(12)54	0(20)0
2_{12}	0(2)0	0(6)0	54(6)18	162(18)18	0(2)0
2_{13}	0(2)0	18(2)18	54(6)18	162(18)18	0(2)0
2_{14}	0(6)0	0(6)0	54(6)18	162(18)18	0(10)0
2_{23}	0(2)0	0(6)0	0(18)0	162(18)18	0(2)0
2_{24}	0(6)0	0(18)0	0(18)0	162(18)18	0(10)0
2_{34}	0(6)0	0(6)0	0(18)0	162(18)18	0(10)0
1_{123}	6(1)6	18(3)6	54(9)6	162(27)6	20(1)20
1_{124}	0(3)0	0(9)0	54(9)6	162(27)6	0(5)0
1_{134}	0(3)0	18(3)6	54(9)6	162(27)6	0(5)0
1_{234}	0(3)0	0(9)0	0(27)0	162(27)6	0(5)0
1_M	3(3)1	9(9)1	27(27)1	81(81)1	5(5)1

Type	$0+1\to 0+2$	$1+1\to 0+2$	$\frac{1}{2}+\frac{1}{2}\to 0+1$	$\frac{1}{2}+\frac{1}{2}\to 1+1$	$0+\frac{1}{2}\to 1+\frac{3}{2}$
8	0(8)0	0(8)0	0(8)0	18(8)18	0(8)0
4_1	0(4)0	0(12)0	0(8)0	36(8)18	36(4)36
4_2	0(12)0	0(12)0	0(8)0	36(8)18	0(8)0
4_3	0(4)0	180(4)180	6(4)6	18(12)6	0(12)0
4_4	0(20)0	0(20)0	0(12)0	18(12)6	0(16)0
2_{12}	0(6)0	0(18)0	0(8)0	72(8)18	72(4)36
2_{13}	60(2)60	180(6)60	12(4)6	36(12)6	36(6)12
2_{14}	0(10)0	0(30)0	0(12)0	36(12)6	24(8)6
2_{23}	0(6)0	180(6)60	12(4)6	36(12)6	0(12)0
2_{24}	0(30)0	0(30)0	0(12)0	36(12)6	0(16)0
2_{34}	0(10)0	90(10)18	6(6)2	18(18)2	0(24)0
1_{123}	60(3)20	180(9)20	24(4)6	72(12)6	72(6)12
1_{124}	0(15)0	0(45)0	0(12)0	72(12)6	48(8)6
1_{134}	30(5)6	90(15)6	12(6)2	36(18)2	24(12)2
1_{234}	0(15)0	90(15)6	12(6)2	36(18)2	0(24)0
1_M	15(15)1	45(45)1	12(12)1	36(36)1	24(24)1

This suggests an iterative approximation for the polarization analysis of reactions at very high energies in which the initial step is an analysis into the collinearly nonzero amplitudes only, followed by an analysis in terms of all amplitudes using the values obtained in the previous step as initial values.

We also demonstrated that although at very high energies observables involving all but one unpolarized particles are likely to be small at many kinematic situations of practical significance, this does not mean that all polarization effects at high energies are small and unimportant. Large effects are still possible for observables involving more than one polarized particle.

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APPENDIX: THE STRUCTURE OF THE RELATIONSHIP BETWEEN SECONDARY OBSERVABLES AND AMPLITUDE PRODUCTS

In this appendix we describe the structure of the submatrices linking the secondary observables to the amplitude products (bicoms) for a four-particle reaction containing particles with arbitrary spins. The structure for primary observables is well known and serves as the basis for the results here.

In primary observables all four particles have their spin states specified. In the secondary observables one uses either unpolarized particles or a specification[8] of particle spin states which satisfy the null criterion, i.e., the sum of the coefficients of the primary spin states in each of these specifications is zero.

The reaction we consider is

$$s_1+s_2\rightarrow s_3+s_4\ , \tag{A1}$$

where s_i ($i=1,\ldots,4$) is the spin of the ith particle. We will also use the notation

$$\eta_i=2s_i+1\ . \tag{A2}$$

We will consider, one by one, the submatrices 8_i, 4_i, 2_i, 1_i, and 1_M.

8_i

These remain unchanged as we go from primary to secondary observables. Thus we continue to have

$$\tfrac{1}{8}\prod_{i=1}^{4}\eta_i(\eta_i-1) \tag{A3}$$

such submatrices, each still eight-by-eight.

4_i

There are four different types of 4_i, depending on which of the four indices is contracted in the bicoms. The four types are accordingly labeled 4_1, 4_2, 4_3, and 4_4.

Consider, for example, 4_1. This type is represented the following number of times:

$$\tfrac{1}{4}\eta_1[\eta_2(\eta_2-1)\eta_3(\eta_3-1)\eta_4(\eta_4-1)]\ , \tag{A4}$$

a result consistent with Eq. (2.34) of Ref. 2.

The submatrices of a given type can be divided into sets (each set containing η_1 such submatrices). In each set we have fixed values for the set of arguments ξ, U, Ξ, ω, V, and Ω, but all values of $u=v$ appear. We call the matrices in a given set sister submatrices.

We can then deduce from this array of submatrices that in terms of secondary observables the four-by-four matrices will turn $4\eta_1$-by-$4\eta_1$ in size. Their structure can be described if we denote the secondary observables by $A^{(1)}(XYZ), S_1^{(1)}(XYZ),\ldots,S_{\eta_1-1}^{(1)}(XYZ)$, where A denotes averaging over the spin states of the particle appearing in the superscript, and the $S_i^{(1)}$'s are secondary observables satisfying the null criterion mentioned earlier. The (XYZ) after the $A^{(1)}$ and the $S_i^{(1)}$'s remind us of the fixed but arbitrary specifications of the polarization states of the other three particles. These secondary observables have the following form in terms of the primary observables $\mathcal{O}_j$:

$$\begin{aligned} A^{(1)}(XYZ)&=\sum_{j=1}^{\eta_1}\mathcal{O}_j\ ,\\ S_1^{(1)}(XYZ)&=\sum_{j=1}^{\eta_1}{}^{(1)}\alpha_j^{(1)}\mathcal{O}_j\ ,\\ &\vdots\\ S_{\eta-1}^{(1)}(XYZ)&=\sum_{j=1}^{\eta}{}^{(1)}\alpha_j^{(\eta_1-1)}\mathcal{O}_j\ . \end{aligned} \tag{A5}$$

The original four-by-four matrix for the primary observables, as shown in Table I of Ref. 4, is

$$M_4=\begin{bmatrix}1&1&1&1\\1&-1&1&-1\\1&1&-1&-1\\1&-1&-1&1\end{bmatrix}\ . \tag{A6}$$

The $4\eta_1$-by-$4\eta_1$ matrix of the secondary observables will then be

$$
\begin{matrix}
\begin{cases} A^{(1)}(XYZ) \\ A^{(1)}(XYZ) \\ A^{(1)}(XYZ) \\ A^{(1)}(XYZ) \end{cases} \\
\begin{cases} S_1^{(1)}(XYZ) \\ S_1^{(1)}(XYZ) \\ S_1^{(1)}(XYZ) \\ S_1^{(1)}(XYZ) \end{cases} \\
\vdots \\
\begin{cases} S_{\eta_1-1}^{(1)}(XYZ) \\ S_{\eta_1-1}^{(1)}(XYZ) \\ S_{\eta_1-1}^{(1)}(XYZ) \\ S_{\eta_1-1}^{(1)}(XYZ) \end{cases}
\end{matrix}
\qquad
\begin{bmatrix}
M_4 & M_4 & \cdots & M_4 \\
{}^{(1)}\alpha_1^{(1)}M_4 & {}^{(1)}\alpha_2^{(1)}M_4 & \cdots & {}^{(1)}\alpha_{\eta_1}^{(1)}M_4 \\
\vdots & \vdots & & \vdots \\
{}^{(1)}\alpha_1^{(\eta_1-1)}M_4 & {}^{(1)}\alpha_2^{(\eta_1-1)}M_4 & \cdots & {}^{(1)}\alpha_{\eta_1}^{(\eta_1-1)}M_4
\end{bmatrix} . \tag{A7}
$$

The four observables bracketed together correspond to the four observables in the original four-by-four for the primary observables. Thus the first four would be $A^{(1)}(RRR)$, $-A^{(1)}(RII)$, $A^{(1)}(IRI)$, and $A^{(1)}(IIR)$, if we consider the real parts of the bicoms.

2_i

The remaining kinds of submatrices can be handled in a way which is analogous to the treatment of the 4_i's just presented. In particular, we have six types of 2_i's depending on which two of the indices are contracted. They are 2_{12}, 2_{13}, 2_{14}, 2_{23}, 2_{24}, and 2_{34}. Let us take, for example, the 2_{12} type of which there are

$$\tfrac{1}{2}\eta_1\eta_2[\eta_3(\eta_3-1)\eta_4(\eta_4-1)] \ . \tag{A9}$$

The matrix for the secondary observables will be composed of the sister matrices with fixed values for the set ζ, Ξ, ω, and Ω, but with all possible values for $u=v$ and $U=V$. We then obtain a $2\eta_1\eta_2$-by-$2\eta_1\eta_2$ matrix for the secondary observables which has the form

$$
\begin{matrix}
\begin{cases} A^{(1)}A^{(2)}(XY) \\ A^{(1)}A^{(2)}(XY) \end{cases} \\
\begin{cases} A^{(1)}S_1^{(2)}(XY) \\ A^{(1)}S_1^{(2)}(XY) \end{cases} \\
\vdots \\
\begin{cases} S_{\eta_1-1}^{(1)}S_{\eta_2-1}^{(2)}(XY) \\ S_{\eta_1-1}^{(1)}S_{\eta_z-1}^{(2)}(XY) \end{cases}
\end{matrix}
\qquad
\begin{bmatrix}
M_2 & \cdots & M_2 \\
{}^{(2)}\alpha_1^{(1)}M_2 & \cdots & {}^{(2)}\alpha_{\eta_2}^{(1)}M_2 \\
\vdots & & \vdots \\
{}^{(1)}\alpha_1^{(\eta_1-1)}\,{}^{(2)}\alpha_1^{(\eta_2-1)}M_2 & \cdots & {}^{(1)}\alpha_{\eta_1}^{(\eta_2-1)}\,{}^{(2)}\alpha_{\eta_2}^{(\eta_2-1)}M_2
\end{bmatrix} . \tag{A10}
$$

There are

$$\tfrac{1}{2}\eta_3(\eta_3-1)\eta_4(\eta_4-1) \tag{A11}$$

such matrices for the secondary observables of the 2_{12} type. The corresponding results hold for the other five types of 2_i's.

1_i

There are four types of 1_i's, denoted by 1_{123}, 1_{124}, 1_{134}, and 1_{234}. Take, for example, 1_{123}, of which there are

$$\eta_1\eta_2\eta_3[\eta_4(\eta_4-1)] \ . \tag{A12}$$

For fixed values of the set Ξ and Ω, we consider the sister matrices with all possible values of $u=v$, $U=V$, and $\xi=\omega$. We then get matrices for the secondary observables which are $\eta_1\eta_2\eta_3$-by-$\eta_1\eta_2\eta_3$ and have the form

$$\begin{matrix} A^{(1)}A^{(2)}A^{(3)}(X) \\ A^{(1)}A^{(2)}S_1^{(3)}(X) \\ \vdots \\ S^{(1)}_{\eta_1-1}S^{(2)}_{\eta_2-1}S^{(3)}_{\eta_3-1}(X) \end{matrix} \begin{bmatrix} 1 & \cdots & 1 \\ {}^{(3)}\alpha_1^{(1)} & \cdots & {}^{(3)}\alpha_{\eta_3}^{(1)} \\ \vdots & & \vdots \\ {}^{(1)}\alpha_1^{(\eta_1-1)}\,{}^{(2)}\alpha_1^{(\eta_2-1)}\,{}^{(3)}\alpha_1^{(\eta_3-1)} & \cdots & {}^{(1)}\alpha_{\eta_1}^{(\eta_1-1)}\,{}^{(2)}\alpha_{\eta_2}^{(\eta_z-1)}\,{}^{(3)}\alpha_{\eta_3}^{(\eta_3-1)} \end{bmatrix} . \tag{A13}$$

There are

$$\eta_4(\eta_4-1) \tag{A14}$$

such matrices of the 1_{123} type. The corresponding results hold for the other three types of 1_i's.

1_M

Finally, we look at the 1_M type, of which there is only one. There are

$$\eta_1\eta_2\eta_3\eta_4 \tag{A15}$$

primary 1_M matrices. The secondary matrix, of which there is only one, will then be $\eta_1\eta_2\eta_3\eta_4$-by-$\eta_1\eta_2\eta_3\eta_4$, and its form is

$$\begin{matrix} A^{(1)}A^{(2)}A^{(3)}A^{(4)} \\ A^{(1)}A^{(2)}A^{(3)}A_1^{(4)} \\ \vdots \\ S^{(1)}_{\eta_1-1}S^{(2)}_{\eta_2-1}S^{(3)}_{\eta_3-1}S^{(4)}_{\eta_4-1} \end{matrix} \begin{bmatrix} 1 & \cdots & 1 \\ {}^{(4)}\alpha_1^{(1)} & \cdots & {}^{(4)}\alpha_{\eta_4}^{(1)} \\ \vdots & & \vdots \\ {}^{(1)}\alpha_1^{(\eta_2-1)}\,{}^{(2)}\alpha_1^{(\eta_2-1)} & \cdots & {}^{(1)}\alpha_{\eta_1}^{(\eta_1-1)}\,{}^{(2)}\alpha_{\eta_2}^{(\eta_2-1)} \\ \times{}^{(3)}\alpha_1^{(\eta_3-1)}\,{}^{(4)}\alpha_1^{(\eta_4-1)} & & \times{}^{(3)}\alpha_{\eta_3}^{(\eta_3-1)}\,{}^{(4)}\alpha_{\eta_4}^{(\eta_4-1)} \end{bmatrix} . \tag{A16}$$

This concludes the complete description of the structure of the submatrices for secondary observables. The results are tabulated for some simpler reactions in Table II.

The above results hold for reactions constrained only by Lorentz invariance. When other symmetries (e.g., parity conservation, time-reversal invariance, or identical-particle constraints) are imposed, additional simplifications materialize.[3]

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