Constraints on stress-energy perturbations in general relativity

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Conditions are found for the existence of integral constraints on stress-energy perturbations in general relativity. The integral constraints can be thought of as a general-relativistic generalization of the conservation of energy and momentum of matter perturbations in special relativity. The constraints are stated in terms of a vector field $\vec{V}$, and the Robertson-Walker spacetimes are shown to have such constraint vectors. Although in general $\vec{V}$ is not a Killing vector, in a vacuum spacetime the constraint vectors are precisely the Killing vectors.

I. INTRODUCTION

We are used to associating conserved quantities with symmetries. In special relativity there are ten conserved currents, corresponding to the invariance of flat spacetime under space and time translations, rotations, and boosts. Conservation laws restrict the possible motions of matter. For example, suppose that at some initial time the mass density ρ is uniform, and then there is a small explosion. A possible density perturbation, in the theory of special relativity, is

$$\delta\rho(\vec{x},t) = -2\epsilon\delta(\vec{x}) + \epsilon\delta(\vec{x}-\vec{v}t) + \epsilon\delta(\vec{x}+\vec{v}t) \ .$$

However, a single jet

$$\delta\rho(\vec{x},t) = -\epsilon\delta(\vec{x}) + \epsilon\delta(\vec{x}-\vec{v}t)$$

is not allowed; it conserves mass but not momentum. More generally, whenever a stress-energy perturbation vanishes on the boundary of a volume G, then conservation of energy and momentum in special relativity implies

$$\int_G dv\, \delta\rho = 0 \ , \qquad \int_G dv\, \delta\rho \vec{x} = 0 \ . \tag{1}$$

(This also assumes that $\delta\rho$ is zero at some initial time.)

In general relativity energy and momentum are not conserved, in general. Heuristically, this is because energy is exchanged between the matter and gravitational fields. Precisely, the stress energy $T_{\mu\nu}$ is covariantly conserved, $T^{\mu\nu}{}_{;\mu}=0$ rather than $T^{\mu\nu}{}_{,\mu}=0$. Symmetries of the spacetime, or Killing vectors, still correspond to conserved currents: if $\vec{\xi}$ is a Killing vector, then $J^{\mu}{}_{;\mu} = (T^{\mu}{}_{\nu}\xi^{\nu})_{;\mu}=0$.

However, the same is not true for stress-energy perturbations $\delta T^{\mu}{}_{\nu}$. This is because $\delta T^{\mu}{}_{\nu}$ is not even covariantly conserved, so a Killing vector does not lead to a conserved quantity. In this paper, we look for Gauss's laws for stress-energy perturbations in general relativity. The problem is to find integral conditions which $\delta T^{\mu}{}_{\nu}$ must satisfy, as a generalization of the special-relativistic statements (1). The integral constraints are formalized in terms of a vector field $\vec{V}$, which can be thought of as a generalization of a Killing vector. $\vec{V}$ is defined in analogy with Gauss's law in special relativity. Let G be a spacelike volume with normal $\vec{n}$. Then we will require that the integral of $\delta T^{\alpha}{}_{\mu}V^{\mu}n_{\alpha}$ over the volume G is equal to a surface integral over the boundary of G.

We find the conditions for the existence of a constraint vector $\vec{V}$ in a general spacetime. The conditions depend on the background geometry and on the spacetime splitting, but are independent of the choice of gauge for the metric perturbations and of the equation of state for $\delta T^{\mu}{}_{\nu}$.

This is of interest because the Robertson-Walker spacetimes each have ten constraint vectors $\vec{V}$. Six of these are the Killing vectors which lie in the spatial hypersurfaces (in standard comoving coordinates) and the other four are not Killing vectors. In different flat-space limits the integral constraints reduce to the special-relativity statements (1). For example, the open Robertson-Walker universe is asymptotically flat, as $t\rightarrow\infty$. In this limit, the four "other" constraint vectors become precisely the four space and time translation Killing vectors of Minkowski space.

It will be shown that for vacuum spacetimes the constraint vectors are precisely the Killing vectors. When matter is present, there are conditions on the stress energy such that a Killing vector is also a constraint vector. The de Sitter stress energy satisfies these conditions, and the ten de Sitter Killing vectors are also constraint vectors.

One astrophysical application of the integral constraints is the Sachs-Wolfe effect. Density fluctuations cause perturbations in the null geodesics along which photons propagate. The Sachs-Wolfe[1] effect is the contribution to the anisotropy in the microwave background due to the resulting change in the photon's four-momentum. Sachs and Wolfe calculate the anisotropic contribution to the temperature change, in a flat pressureless Robertson-Walker universe. They find $\delta T/T \sim \frac{1}{10}A(\vec{x}_E)$, where A is the gravitational potential for $\delta\rho/\rho$ and $\vec{x}_E$ is the emission point. When $\vec{x}_E$ is outside of the source $\delta\rho$, the integral constraints imply that the monopole and dipole moments of the source vanish [as in (1)]. Hence the magnitude of $\delta T/T$ with the constraints is decreased by a factor of order $(1+z_E)^{-1}$ (Ref. 2).

In Sec. II, we derive the conditions for the existence of a constraint vector in a general spacetime. Section III gives some general results about constraint vectors, and Sec. IV discusses the connection with Killing vectors. The cases of the Robertson-Walker and de Sitter spacetimes are solved in Sec. V. Section VI treats various flat-space limits, and Sec. VII the behavior of sound waves.

II. CONSTRAINTS ON STRESS-ENERGY PERTURBATIONS

Let M be a four-dimensional manifold with metric $g^{(0)}_{\mu\nu}$ [signature $(-+++)$]. Assume that $g^{(0)}_{\mu\nu}$ with stress energy $T^{\mu\nu}_{(0)}$ is a solution to the Einstein equations $G^{\mu}_{(0)\nu}=8\pi T^{\mu}_{(0)\nu}$. Let $\{S(t)\}$ be a slicing of M with induced metric g_{ij}. Let $\vec{n}$ be the unit normal ($\vec{n}\cdot\vec{n}=-1$) on the hypersurface S. $w^{\hat{0}}=-\vec{w}\cdot\vec{n}$ is the normal component of $\vec{w}$.

$$K_{ij}=-\tfrac{1}{2}\mathscr{L}_{\vec{n}}g_{ij}=-\frac{1}{2}\frac{\partial}{\partial x^{\hat{0}}}g_{ij}$$

is the extrinsic curvature of S, and $\pi^{ij}=\sqrt{g}\,(g^{ij}K-K^{ij})$ is the canonical momentum. Here $g=\det(g_{ij})$, and $K=\mathrm{Tr}K=K^{i}{}_{i}$ and $\mathscr{L}_{\vec{n}}$ denotes the Lie derivative. Also, we will write $V_{\alpha;\beta}$ for covariant differentiation on M and $D_i\beta_k\equiv\beta_{k\,|\,i}$ for differentiation on S. Greek indices run from 0 to 3 and latin indices from 1 to 3.

Consider perturbations from the background spacetime

$$\begin{aligned} g_{ij}&=g^{(0)}_{ij}+h_{ij}\ ,\\ \pi^{ij}&=\pi^{ij}_{(0)}+p^{ij}\ ,\\ T^{\mu}{}_{\nu}&=T^{\mu}_{(0)\nu}+\delta T^{\mu}{}_{\nu}\ . \end{aligned} \tag{2}$$

Definition (see Fig. 1): Let $G\subset S$ with boundary ∂G. Then $\vec{V}$ is an integral constraint vector (ICV) for the hypersurface $(S,g^{(0)}_{ij})$ of the spacetime $(M,g^{(0)}_{\mu\nu})$ if the integral of $V^{\mu}\delta T^{\alpha}{}_{\mu}n_{\alpha}$ over the volume G is equal to a surface integral over ∂G, for all perturbations h_{ij} and p_{ij} which satisfy the linearized Einstein equations:

$$\int_G dv\, V^{\mu}\delta T^{\hat{0}}{}_{\mu}=\int_{\partial G} da_l B^l\ .$$

One could think of other generalization of (1). This definition was chosen because Robertson-Walker (RW) spacetimes have ICV's as defined here, and the integral constraints reduce to flat-space energy-momentum conservation in limiting cases. First, we prove the following

Theorem. Let $\vec{V}=F\vec{n}+\vec{\beta}$, $\vec{\beta}\cdot\vec{n}=0$. Then $\vec{V}$ is an ICV if and only if

$$2L^{ij}F=g^{ij}(\beta^l K_{lk})^{|k}-(\beta^l K^{ij})_{|l} \tag{3a}$$

and

$$\beta_{i\,|\,j}+\beta_{j\,|\,i}=2FK_{ij}\ . \tag{3b}$$

Here $L_{ij}\equiv\frac{1}{2}(-D_iD_j+g_{ij}\Delta+R_{ij})$, $\Delta\equiv D_iD^i$, and R_{ij} is the Ricci tensor of S.

Corollary. If $K_{ij}=-\frac{1}{2}\varphi(\vec{x},t)g_{ij}$ [so at least locally,

$$ds^2=-dt^2+\Psi^2(\vec{x},t)^{(3)}g_{ij}(\vec{x}),\quad \varphi=2\dot{\Psi}/\Psi]\ ,$$

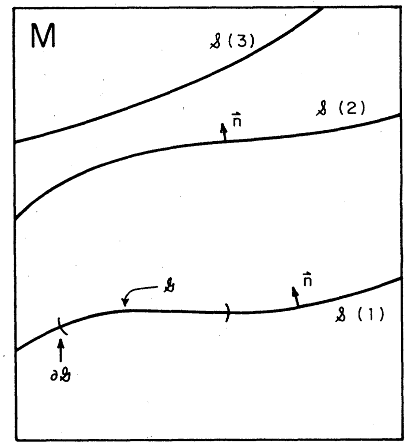

FIG. 1. $(M,g^{0}_{\mu\nu})$ is a spacetime satisfying the Einstein equations with stress energy $T^{0}_{\mu\nu}$. $(S(t),g_{ij})$ is a slicing of M with unit normal $\vec{n}$. G is a subset of S with boundary ∂G. The idea of an integral constraint vector V^{μ} is that the integral of $V^{\mu}\delta T^{a}_{\mu}n_a$ over the volume G is equal to a surface integral over ∂G.

then $\vec{V}$ is an ICV is and only if

$$L_{ij}F=0 \tag{4a}$$

and

$$\beta_{i\,|\,j}+\beta_{j\,|\,i}=-F\varphi g_{ij}\ . \tag{4b}$$

So when $K_{ij}=-\frac{1}{2}\varphi g_{ij}$, a Killing vector for the three-surface is always an ICV with $F=0$.

It is important to note that the existence of an ICV is independent of the choice of gauge of the perturbations. The constraints are also independent of the equation of state that is assumed for $\delta T^{\mu}{}_{\nu}$. Also, we will see that if $h_{ij}\,|\,_{\partial G}=0$ and $p_{ij}\,|\,_{\partial G}=0$, then the boundary term vanishes. The existence of constraints does depend on the background geometry and the slicing. Also, note that condition (3b) can be restated as $\mathscr{L}_{\vec{V}}g_{ij}=0$.

Proof of theorem. We use the Arnowitt-Deser-Misner (ADM) formalism.[3,4] Let

$$H=2\sqrt{g}\,G^{\hat{0}}{}_{\hat{0}}=\frac{1}{\sqrt{g}}(\pi^{ik}\pi_{ik}-\tfrac{1}{2}\pi^2)-\sqrt{g}R\ ,$$

$$H_k=2\sqrt{g}\,G^{\hat{0}}{}_{k}=-2\pi_k{}^{i}{}_{|i}\ ,$$

and $\Phi=(H,H_k)$. Perform the variation of $G^{\hat{0}}{}_{\mu}$ and integrate by parts:

$$\begin{aligned} 2\sqrt{g}\,V^{\mu}\delta G^{\hat{0}}{}_{\mu}&=FDH+\beta^k DH_k-h\sqrt{g}\,V^{\mu}G^{\hat{0}}{}_{\mu}\\ &=h_{ij}\left[\frac{\delta H^*}{\delta g_{ij}}\cdot F+\frac{\delta H^*_k}{\delta g_{ij}}\cdot\beta^k\right]\\ &\quad+p^{ij}\left[\frac{\delta H^*}{\delta\pi^{ij}}\cdot F+\frac{\delta H^*_k}{\delta\pi^{ij}}\cdot\beta^k\right]\\ &\quad-h_{ij}\sqrt{g}\,V^{\mu}G^{\hat{0}}{}_{\mu}+D_i(\sqrt{g}B^i)8\pi\ . \end{aligned}$$

Therefore, using the linearized Einstein equations,

$$\int_G dv\, V^\mu \delta T^{\hat 0}{}_\mu = \int_{\partial G} d\alpha_l B^l$$

for all h_{ij}, p^{ij} if and only if

$$D\Phi^* \cdot (F,\beta^k) - \sqrt{g}\, 8\pi (g^{ij} T^{\hat 0}{}_\mu V^\mu, 0) = (0,0) \ . \qquad (5)$$

The adjoints are computed explicitly, for example, in Ref. 4. So explicitly (5) is

$$\sqrt{g}\, 8\pi g^{ij} T^{\hat 0}{}_\mu V^\mu = \sqrt{g}\,(R^{ij} - \tfrac{1}{2} g^{ij} R)F - S_g(\pi,\pi)F - \sqrt{g}\,(F^{|i|j} - g^{ij}\Delta F) - L_{\vec\beta}\pi^{ij} \qquad (6a)$$

and

$$0 = \beta_{i\,|\,j} + \beta_{j\,|\,i} - 2FK_{ij} \ . \qquad (6b)$$

Here

$$S_g(\pi,\pi) = + \frac{1}{2\sqrt{g}} g^{ij}(\pi^{kl}\pi_{kl} - \tfrac{1}{2}\pi^2) - \frac{2}{\sqrt{g}}(\pi^{ik}\pi_k{}^j - \tfrac{1}{2}\pi^{ij}\pi) \ ,$$

$$L_{\vec\beta}\pi = -\beta^i{}_{|k}\pi^{jk} - \beta^j{}_{|k}\pi^{ik} + (\pi^{ij}\beta^k)_{|k} \ .$$

We find for the boundary term

$$16\pi B^i = \frac{1}{\sqrt{g}}(\beta^i \pi^{jk} h_{jk} - 2\beta^i h_{ij}\pi^{ji}) + Fh^{\,|i} - F^{\,|i} h + F_{|l} h^{li} - F h^{il}{}_{|l} - \frac{2}{\sqrt{g}}\beta_k p^{ki} \qquad (7)$$

Finally, Eqs. (6a) and (6b) can be rewritten as in (3a) and (3b). Q.E.D.

III. GENERAL RESULTS

The ICV's are of interest because RW and de Sitter spacetimes actually have such vectors. We have not yet found any other nonvacuum spacetimes which have a constraint vector (in vacuum ICV's turn out to be precisely the Killing vectors). Still, there are some general results which give a better understanding of the 12 coupled partial differential equations (3).

(i) The existence of ICV's can be stated in terms of the existence of solutions to the linearized Einstein constraint equations, for arbitrary matter sources. The constraint equations are

$$(H,H_k) - \sqrt{g}\, 16\pi (T^{\hat 0}{}_0, T^{\hat 0}{}_k) = 0 \ , \qquad (8)$$

and the linearized equations are

$$M\cdot(h,p) \equiv D\Phi\cdot(h,p) - \sqrt{g}\, h\, 8\pi (T^{\hat 0}{}_{\hat 0}, T^{\hat 0}{}_k) = \sqrt{g}\, 16\pi (\delta T^{\hat 0}{}_{\hat 0}, \delta T^{\hat 0}{}_k) \ . \qquad (9)$$

Proposition I. Given any (suitably bounded)[5] sources $(\delta T^{\hat 0}{}_{\hat 0}, \delta T^{\hat 0}{}_k)$, then there exist (h_{ij}, p^{ij}) such that $(h_{ij}, p^{ij}, \delta T^{\hat 0}{}_{\hat 0}, \delta T^{\hat 0}{}_k)$ is a solution to (9) if and only if M is onto, if and only if M^* is one to one, if and only if there exist no (suitably bounded) ICV's.

Proof. The adjoint is

$$M^*\cdot(F,\beta^k) = D\Phi^*\cdot(F,\beta^k) - 8\pi\sqrt{g}\,(FV^\mu T^{\hat 0}{}_\mu, 0) \qquad (10)$$

Comparing this with (5), $(F,\beta^k) \in \mathrm{Ker} M^*$ if and only if $\vec V = F\vec n + \vec\beta$ is an ICV, with F,β^k suitably bounded. (M and M^* map tensors which are candidates for perturbations. So it is possible to have an unbounded ICV, and yet $\mathrm{Ker}M^*$ empty. In fact, this is the situation in the $k=0,-1$ RW universes.) Fischer and Marsden[4] show that M^* is elliptic in the vacuum case, and from (10) it is clear that M^* is elliptic with the matter term. Finally, M^* elliptic and one to one is equivalent to M onto (Ref. 4). Q.E.D.

(ii) In general, the existence of an ICV on one spatial slice $S(t_0)$ does not guarantee the existence of an ICV on another slice $S(t)$. However, if the metric can be written as

$$ds^2 = -dt^2 + a^2(t)^{(3)}g_{ij}(\vec x)dx^i dx^j$$

so that $K_{ij} = -(\dot a/a)g_{ij}$ globally, then if the surface $t=t_0$ has an ICV it is of the form

$$\vec V(\vec x, t_0) = -F(\vec x)\frac{\partial}{\partial t} + \dot a a f^k(\vec x)\frac{\partial}{\partial x^k} \ ,$$

where $L_{ij}F = 0$ and

$$f_{k\,|\,j} + f_{j\,|\,k} = 2^{(3)}g_{jk}(\vec x)F \ .$$

Therefore, $\vec V(\vec x, t)$ is an ICV for any slice $t=$constant.

(iii) Equation (3a) is equivalent to $V_{i;j} + V_{j;i} = 0$. Recall here that $\vec\beta$ belongs to the tangent space of S and $\vec V$ is in the tangent space of M.

When $K_{ij} = -\frac{1}{2}\varphi(\vec x, t)g_{ij}$ we have the following:

(iv) $L_{ij}F = 0$ implies that $D_k R = 0$. Therefore, a necessary condition for the existence of an ICV is that S has constant curvature.

(v) For compact, S, $L_{ij}F = 0$ is equivalent to the statement that there exists a function $F(\vec x)$ such that δR is orthogonal to F for all perturbations of g_{ij}. This follows because if $g_{ij} \to g_{ij} + h_{ij}$, then $R \to R - 2L_{ij}h^{ij}$, and

$$\int_S dv\, F\delta R = -2\int dv\, h^{ij} L_{ij} F = 0 \ .$$

(vi) The existence of a (bounded) solution F to $LF = 0$ is equivalent to the existence of a metric which is (infinitesimally) conformal to g_{ij}, and has the same curvature scalar. Simply note that if $\hat g_{ij} = (1+\epsilon F)g_{ij}$, then $\hat R = R - 2\epsilon LF$.

IV. CONNECTION WITH KILLING VECTORS

The Killing vectors $\vec\xi$ of a spacetime generate conserved currents: if $J^\mu = \xi^\nu T_{(0)}{}^\mu{}_\nu$, then $J^\mu{}_{;\mu} = 0$. However, the same is not true for the perturbed current $\delta J^\mu = \xi^\nu \delta T^\mu{}_\nu$. This is because $\delta T^\mu{}_\nu$ is not covariantly conserved, $\delta T^\mu{}_{\nu;\mu} = T^\mu_{(0)\lambda}\Gamma^\lambda{}_{\nu\mu} - T^\lambda_{(0)\nu}\Gamma^\mu{}_{\lambda\mu}$. Therefore

$$\delta J^\mu{}_{;\mu} = \xi_{\nu;\beta} h^\nu{}_\alpha T^{\alpha\beta}_{(0)} + \xi^\nu (T^\mu_{(0)\lambda}\delta\Gamma^\lambda{}_{\nu\mu} - T^\lambda_{(0)\nu}\delta\Gamma^\mu{}_{\lambda\mu}) \ .$$

In a vacuum these expressions do reduce to $\delta T^\mu{}_{\nu;\mu} = 0$ and $\delta J^\mu{}_{;\mu} = 0$. So in a vacuum the Killing vectors do generate conservation laws for perturbations.

Even when $T_{(0)}{}^\mu{}_\nu \neq 0$, some ICV's are Killing vectors. The following discussion will help to show the connection between Killing vectors and constraint vectors.

Proposition 2. Let $(M^{(4)},g_{\mu\nu},T_{\mu\nu})$ be a solution to the Einstein equations with a Killing vector $\vec{\xi}$. Then there exists a hypersurface S such that $\vec{\xi}$ is an ICV for S if and only if

$$\xi^\mu T^{\hat{0}}{}_\mu g^{ij}=\xi^{\hat{0}}T^{ij} . \tag{11}$$

The condition (11) on $T^{\mu\nu}$ can be restated as

$$T^{ij}=p(x)g^{ij} \text{ and } \xi^{\hat{0}}p=\xi^\mu T^{\hat{0}}{}_\mu,\ \ \xi^{\hat{0}}\neq 0$$

$$\xi^\mu T^{\hat{0}}{}_\mu=0,\ \ \xi^{\hat{0}}=0 .$$

Note that this implies that in vacuum a Killing vector is always an ICV for any slicing.

In the vacuum case, using a result of Moncrief,[6] we also have the converse.

Proposition 3. Let $(M^{(4)},g_{\mu\nu})$ be a vacuum solution, and suppose there is a slicing S_t such that on each hypersurface S_t there are N linearly independent ICV's $\vec{V}^{(a)}(\vec{x},t)$, $a=1,\ldots,N$. Then the constraint vectors can always be chosen so that each ICV is a Killing vector.

The ambiguity or choice comes in because any time-dependent linear combination of ICV's is also an ICV.

Note that propositions 2 and 3 imply that in vacuum the constraint vectors are the Killing vectors.

The de Sitter spacetime has ten Killing vectors all which satisfy (11), since $T^\mu{}_\nu\propto g^\mu{}_\nu$. So all the Killing vectors are also ICV's. This is interesting because it is an example of a nonvacuum case where proposition 2 applies. It is easy to check directly that the Killing vectors also satisfy (4).

The RW spacetimes each have six Killing vectors which lie in the spatial hypersurfaces, in standard RW coordinates (14). So $\xi^{\hat{0}}=0$, and since $T^{\hat{0}}{}_j=0$, these spatial Killing vectors also satisfy (11). One can also check directly from (4) that whenever $K_{ij}\propto g_{ij}$, a spatial Killing vector is also an ICV.

Proof of proposition 2. Generalizing Moncrief's proof[6] to include spacetimes with matter, the following can easily be shown:

Given a solution to the Einstein equations $(M^{(4)},g_{\mu\nu},T_{\mu\nu})$ and a vector field $\vec{\xi}$, pick some hypersurface S with normal $\vec{n}$. Let $\vec{\xi}=F\vec{n}+\beta^k\vec{e}_k$, $\vec{n}\cdot\vec{e}_k=0$. Then

$$(-L_{\vec{\xi}}\pi^{ij},L_{\vec{\xi}}g_{ij})=D\Phi^*\cdot(F,\beta_k)-\sqrt{g}\,8\pi(FT^{ij},0) . \tag{12}$$

[Recall $\Phi=(H,H_k)$.] Therefore, if $\vec{\xi}$ is a Killing vector,

$$D\Phi^*\cdot(F,\beta_k)-\sqrt{g}\,8\pi(FT^{ij},0)=(0,0) . \tag{13}$$

Now, $\vec{\xi}$ is an ICV if and only if (5) is true. Comparing (5) and (13) gives the result. Q.E.D.

Proof of proposition 3. Let $\vec{V}^{(a)}=F^{(a)}\vec{n}+\vec{\beta}^{(a)}$. Then (5) becomes, for a vacuum,

$$D\Phi^*\cdot(F^{(a)},\beta_j^{(a)})=(0,0)$$

on S_t. Moncrief[6] proved that there exists a Cauchy development of (S_0,g,π) with a Killing vector $\vec{\xi}^{(a)}$, such that $\vec{\xi}^{(a)}=F^{(a)}\vec{n}+\vec{\beta}^{(a)}$ on S_0. Further, Fischer and Marsden (Ref. 4, Theorem 4.27) prove uniqueness (up to coordinate transformations) for vacuum spacetimes with g_{ij} and π^{ij} specified on an initial surface S_0, in a neighborhood of S_0. Therefore, the Cauchy development of (S_0,g,π) is the same spacetime as the one we started with.

By proposition 2, the fact that $\vec{\xi}^{(a)}$ is a Killing vector implies that $\vec{\xi}^{(a)}$ is an ICV. Therefore,

$$\vec{\xi}^{(a)}(\vec{x},t)=\sum_{b=1}^{N}\lambda^{ab}(t)\vec{V}^{(b)}(\vec{x},t),\ \ a=1,\ldots,N$$

and $\lambda^{ab}(t_0)=\delta^{ab}$. On each slice relabel the constraint vectors:

$$\vec{V}^{(a)^{\rm new}}(t)=\lambda^{ab}(t)\vec{V}^{(b)}(t)=\vec{\xi}^{(a)}.\ \ \text{Q.E.D.}$$

It seems plausible that the converse of proposition 2 is also true with matter: Given an ICV $\vec{V}$, then $\vec{V}$ is a Killing vector if and only if (11). However, so far we have not been able to prove this.

V. CONSTRAINT VECTORS IN ROBERTSON-WALKER SPACETIMES

RW spacetimes are of the form

$$ds^2=-dt^2+a^2(t)[d\chi^2+\Sigma^2(\chi,k)d\Omega^2] ,$$
$$\Sigma^2=\begin{cases}\chi^2, & k=0,\\ \sin^2\chi, & k=+1,\\ \sinh^2\chi, & k=-1.\end{cases} \tag{14}$$

We shall see that for any $a(t)$, there exist ten ICV's. Six of these are the Killing vectors that lie in the $t=$constant hypersurfaces. The other four have nonzero time components and so impose constraints on $\delta\rho$. First, we will find the solutions, and then show that the constraints reduce to energy-momentum conservation in different flat-space limits.

The $t=$constant surfaces of (14) all have Ricci tensor

$$R_{jk}=k\frac{2g_{jk}}{a^2} ,$$

and Eqs. (4) become

$$D_iD_jF=-k\frac{g_{ij}}{a^2}F , \tag{15a}$$

$$\beta_{i\,|\,j}+\beta_{j\,|\,i}=-2\frac{\dot{a}}{a}g_{ij}F . \tag{15b}$$

Consider (15) for $k=\pm1$. When $F\neq0$, let $\beta_i=k\dot{a}aD_iF$, and (15b) reduces to (15a). This is a great simplification, since the system of 12 partial differential equations has been reduced to six, and is possible precisely when $R_{ij}\propto g_{ij}$. We are left with the equations for a conformal killing vector $\vec{D}F$ on the three-sphere S^3 or pseudosphere H^3. When $k=+1$ the solutions for F are the four second-order spherical harmonics Q^{2lm} on S^3 (Ref. 7). We choose the following linear combinations of Q^{2lm} as solutions:

$$k=+1:\ Q^{(0)}=\cos\chi\ ,\quad Q^{(1)}=\sin\chi\cos\Theta\ ,\quad Q^{(2)}=\sin\chi\sin\Theta\cos\varphi\ ,\quad Q^{(3)}=\sin\chi\sin\Theta\sin\varphi\ , \tag{16a}$$

$$k=-1:\ \text{substitute } \cosh\chi \text{ for } \cos\chi \text{ and } \sinh\chi \text{ for } \sin\chi\ . \tag{16b}$$

Therefore, for $k=\pm 1$, we have the following four ICV's:

$$\vec{V}_{(a)}=Q^{(a)}\frac{\partial}{\partial t}+k\dot{a}aD^jQ^{(a)}\frac{\partial}{\partial x^j},\quad a=0,1,2,3\ . \tag{17}$$

In addition, the six spatial Killing vectors of $S^3(H^3)$ are ICV's. It is easy to see that these ten vectors are all of the ICV's. If $F\neq 0$, $\vec{D}F$ is a conformal Killing vector on $S^3(H^3)$, and if $F=0$, the ICV must be a spatial Killing vector. A three-dimensional surface has at most ten conformal Killing vectors; $S^3(H^3)$ has six KV's and the four conformal Killing vectors $\vec{D}Q^{(a)}$ just found.

When $k=0$ the solutions with $F\neq 0$ are easily found to be

$$\vec{V}_{(0)}=\frac{\partial}{\partial t}-\frac{\dot{a}}{a}x^i\frac{\partial}{\partial x^i}\ ,\qquad \vec{V}_{(k)}=x^k\frac{\partial}{\partial t}+\frac{\dot{a}}{a}\left(\tfrac{1}{2}\delta^{ki}r^2-x^ix^k\right)\frac{\partial}{\partial x^i},\quad k=1,2,3\ . \tag{18}$$

In addition, there are the six spatial Killing vectors.

The de Sitter universe is the solution to Einstein's equations for

$$T^{\mu\nu}=-\frac{\Lambda g^{\mu\nu}}{8\pi G},\quad \left[\frac{\dot{a}}{a}\right]^2=\frac{\Lambda}{3}\ ,$$

which implies $a=a_0e^{\sqrt{\Lambda/3}t}$. In this case the ten ICV's are the ten Killing vectors of de Sitter space. This tells us immediately that de Sitter has ten constraint vectors for any spacetime slicing.

For a closed universe the boundary term can always be taken to be zero. So when $k=+1$ and $\delta T^{\hat{0}}{}_k=0$, the constraint, vectors (17) imply the integral conditions on $\delta\rho$:

$$\int dv\,\delta\rho\cos\chi=0\ ,$$

$$\int dv\,\delta\rho\sin\chi\,Y_{1m}=0\ .$$

For small χ (e.g., if $\delta\rho$ is clustered around the origin) these reduce to the special-relativity statements (1) that the monopole and dipole moments of $\delta\rho$ vanish.

There are certainly other possible definitions one could make to generalize Gauss's law; for example, one could require

$$\int d^3x\,W^\mu\delta(\sqrt{g}\,T^{\hat{0}}{}_\mu)=0\ .$$

However, the RW spacetimes do not have any vectors which satisfy this. In the next section we will show in what sense the constraints are generalizations of energy-momentum conservation.

VI. FLAT-SPACE LIMITS AND THE DOMINANT ENERGY CONDITION

A. Flat-space limits

In the open $k=-1$ RW universe, ρ approaches zero as t goes to infinity, and the spacetime becomes flat,

$$ds^2\to -dt^2+t^2(d\chi^2+\sinh^2\chi\,d\Omega^2)\ . \tag{19}$$

This is Minkowski space written in comoving coordinates. Geodesic observers with constant spatial coordinates recede from each other at constant velocity. That is, the $t=$constant hypersurfaces of (19) are the hyperboloids in Minkowski space which are invariant under boosts. Let $\{x^\mu\}$ be standard Minkowski coordinates, and let

$$x^\mu=tQ^{(\mu)},\ \text{for } k=-1\ ,$$

where the $Q^{(\mu)}$ are defined in (16b). Then

$$ds^2_{\rm Mink}=-(dx^0)^2+\delta_{ij}dx^idx^j=-dt^2+t^2(d\chi^2+\sinh^2\chi\,d\Omega^2)\ .$$

The four time and space translation Killing vectors in flat space are

$$\vec{\xi}^{(a)}=-\vec{\nabla}x^a=-\vec{\nabla}tQ^{(a)},\quad a=0,1,2,3\ . \tag{20}$$

Now, if one takes the late-time limit of the four ICV's for $k=-1$ [Eq. (17)], they are just the Killing vectors (20), written in $\{t,\chi,\Theta,\varphi\}$ coordinates. Also,

$$\vec{\xi}^{(0)}=\frac{\partial}{\partial x^0}=\frac{\partial}{\partial t}+O(\chi)\ .$$

So a comoving observer also sees $\vec{\xi}^{(0)}$ as time translation in his local frame.

In the $k=0$ universe, consider the case when $\delta T^{\hat{0}}{}_k=0$. (When $p=0$ and the flow is irrotational, one can choose coordinates which are simultaneously synchronous and comoving.[7] In this case it is not possible, in general, to build a perturbation which is localized in space.[2] However, if one starts with a local perturbation, the irrotational component dominates at late times.) If $\delta\rho$ is localized in space, then the boundary term vanishes and the integral constraints with the vectors (18) become the special-relativity statements (1).

B. Dominant energy condition

The positivity of $\delta\rho$ becomes an issue whenever $T_{(0)}{}^\mu{}_\nu=0$ and there exists a timelike ICV. This happens, for example, in the $k=-1$ RW universe as $t\to\infty$. Then, asymptotically, $\vec{V}_{(0)}$ is timelike over the entire hypersurface, and $T^\mu{}_\nu=\delta T^\mu{}_\nu$. The following comments apply, however, to any case where $T_{(0)}{}^\mu{}_\nu=0$ on G and $\vec{V}_{(0)}$ is timelike on G.

The dominant energy condition[8] states that $\delta T^{\hat{0}}{}_\mu=-T^\alpha{}_\mu n_\alpha$ is timelike or null, since n_α is timelike. Also, $\delta T^{\hat{0}}{}_\mu$ and $\vec{V}_{(0)}$ [Eq. (17)] are future pointing. Therefore, $\delta T^{0}{}_\mu V^\mu_{(0)}\le 0$, where equality is obtained only when either vector vanishes. So if $\delta T^{\hat{0}}{}_\mu$ is not identically zero, the dominant energy condition implies

$$\int_G dv\, V^{\mu}_{(0)}\delta T^{\hat{0}}{}_{\mu} < 0 \ . \tag{21}$$

Now suppose we have a perturbation which is created by causal processes over some finite region in space, and hence is localized. Then the boundary term in the IC vanishes, and the IC says

$$\int dv\, V^{\mu}_{(0)}\delta T^{\hat{0}}{}_{\mu} = 0 \ ,$$

in contradiction with (21).

To summarize, suppose a region of spacetime has a timelike ICV and $T_{(0)}{}^{\mu}{}_{\nu}=0$. Then stress-energy fluctuations cannot simultaneously satisfy the dominant energy condition, and be created by local causal processes. This is an example of Hawking and Ellis's conservation theorem.[8]

In standard Minkowski coordinates there is a simple example of a perturbation which satisfies the IC with zero boundary term, but consequently violates the dominant energy condition. Birkhoff's theorem implies that space is flat exterior to a spherical source $\delta\rho$ which satisfies

$$\int d^3x\, \delta\rho = 0 \ .$$

Since space is flat, $h_{\mu\nu}\equiv 0$, and the boundary term vanishes. Also, $\partial/\partial t$ is timelike, and the integral constraint which it generates is precisely the condition for flatness. And, of course, $\delta\rho$ is negative somewhere.

VII. THE BEHAVIOR OF SOUND WAVES

In the RW spacetimes (14), time translation $\partial/\partial t$ is not a Killing vector. However, the ICV's $\vec{V}_{(0)}$ [Eqs. (17) and (18)] look like time translations in a comoving observer's local frame. It is interesting to ask "how far out" on a spatial hypersurface $\vec{V}_{(0)}$ is "like" a time translation. In particular, where is $\vec{V}_{(0)}$ timelike? A first guess might be that $\vec{V}_{(0)}$ is timelike within the horizon

$$\chi_H(t) = \int_0^t \frac{dt}{a} \ .$$

However, this turns out not to be true. Consider instead the sound horizon

$$\chi_S(t) = \int_0^t \frac{c_S(t)}{a(t)} dt \ ,$$

where $\delta\rho = c_S{}^2(t)\delta p$. (It is assumed that c_S is independent of $\vec{x}$.) We will show that $\vec{V}_{(0)}$ is timelike within the sound horizon for several relevant examples of equations of state: (1) $p = c_S{}^2\rho$, $c_S{}^2=$constant, $0\le c_S{}^2\le 1$; (2) a relativistic mixture of photons and baryons with adiabatic perturbations; (3) an ideal gas.

Why should the sound horizon be the relevant boundary to consider? If there is a local perturbation in the matter and metric, gravitational waves will propagate to the horizon χ_H. But the perturbation in the matter $\delta\rho, v^i$ (and h) are nonzero only inside the forward sound cone of the initial disturbance.[2] Therefore, we consider whether $\vec{V}_{(0)}$ is timelike inside the region in which matter can be pushed around.

Let χ_T be the value of χ such that $\vec{V}_{(0)}$ is timelike for $\chi \le \chi_T$. From the expressions for $\vec{V}_{(0)}$,

$$\Sigma^2(k,\chi_T) = \frac{1}{\dot{a}^2 + k} \ , \tag{22}$$

where Σ^2 is defined in (14). For $k=0$ or $k=-1$, $\chi_T \to \infty$ as $t\to\infty$, and $\vec{V}_{(0)}$ becomes timelike everywhere. When $k=+1$, $\Sigma^2 = \sin^2\chi_T$. Then $\vec{V}_{(0)}$ is timelike on all of S^3 when $\dot{a}=0$, that is, at maximum expansion.

The relation between χ_S and χ_T depends on the equation of state. The Einstein equations can be used to express $\dot{a}$ in terms of ρ and p. Let a_1 and ρ_1 be the values of a and ρ at t_1. Then

$$\begin{aligned} \Sigma_T{}^2(\rho) &= \frac{1}{\frac{8}{3}\pi G a_1}\frac{1}{\rho}\exp\left[\frac{2}{3}\int_{\rho_1}^{\rho}\frac{d\rho'}{\rho'+p'}\right] \ , \\ \chi_S(k,\rho) &= 2\int_0^{\Sigma_T(\rho)} \frac{\rho c_S}{\rho+3p}\frac{d\Sigma_T}{(1-k\Sigma_T{}^2)^{1/2}} \ . \end{aligned} \tag{23}$$

From (22) we see immediately that for $c_S=$constant,

$$\chi_S = \frac{2c_S}{1+3c_S{}^2}\chi_T < \chi_T, \text{ for } 0\le c_S \ .$$

A second example of interest is a relativistic gas of photons and baryons with adiabatic perturbations. Let n be the baryon number density, m the baryon mass, T the temperature, and σ the (constant) entropy. Then

$$\rho = a_{\rm SB}T^4 + nm, \quad p = \tfrac{1}{3}a_{\rm SB}T^4, \quad \sigma = \frac{4}{3}\frac{a_{\rm SB}T^3}{nk} \ ,$$

$a_{\rm SB}$ is the Stefan-Boltzmann constant. Let

$$\epsilon = \frac{nm}{a_{\rm SB}T^4} \ll 1 \ .$$

Then

$$\chi_S = \frac{1}{\sqrt{3}}\chi_T + \frac{\epsilon}{16\sqrt{3}} \times \begin{cases} 1, & k=0, \\ 2\dfrac{\cosh\chi_T - 1}{\sinh\chi_T}, & k=-1, \\ 2\dfrac{1-\cos\chi_T}{\sin\chi_T}, & k=+1. \end{cases}$$

It follows that χ_S is always less than χ_T, if $\epsilon < 1$.

A final example of interest is an ideal gas with equation of state $p = K^2\rho^{\gamma}$, $1\le\gamma\le\frac{5}{3}$. Let $u(\rho) = 2\rho c_S/(\rho+3p)$. Then

$$\chi_S = u(\rho)\chi_T - \int_0^{\rho} d\rho \frac{du}{d\rho}\chi_T(\rho;k) \ ,$$

where $\chi_T(\rho;k)$ is defined in (14) and (22). The integration cannot be done explicitly, but as long as $p < \frac{1}{3}\rho$, $du/d\rho > 0$ and $\chi_S < u(\rho)\chi_T < \chi_T$, for the allowed range of γ.

VIII. CONCLUSIONS

In this paper we have looked for Gauss's laws for stress-energy perturbations in general relativity. It was found that spacetimes with enough symmetry can be sliced in a way such that there is a generalized Gauss's law. On each spatial hypersurface S, there exists one or more constraint vectors $\vec{V}$ such that the integral of $V^{\mu}\delta T^{\alpha}{}_{\mu}n_{\alpha}$ over a spatial volume in S is equal to a boundary term, for arbitrary metric perturbations. ($\vec{n}$ is the normal to S.) The ICV's are independent of the gauge chosen for the perturbations, and of the equation of state for $\delta T^{\mu}{}_{\nu}$. $\vec{V}$ does depend on the slicing. Mathematically, the existence of constraint vectors tells if the linearized Einstein constraint equations are onto. That is, there exist solutions to the linearized constraint equations for arbitrary sources $\delta T^{\hat{0}}{}_{\mu}$, if and only if there exist no suitably bounded ICV's.

In vacuum, the ICV's are the Killing vectors. Further, if a nonvacuum spacetime has a stress-energy tensor of a special form and a Killing vector $\vec{\xi}$, then $\vec{\xi}$ is also an ICV. The de Sitter universe is an example of this, so the ten Killing vectors are also ICV's. The most interesting constraint vectors—since we already know to look for Killing vectors to find conserved quantities—are the ones which are not Killing vectors. The RW universes each have four ICV's which are not Killing vectors (plus six which are).

In what sense can we think of the resulting four constraint integrals as energy-momentum conservation? In different flat-space limits, the integrals reduce to the flat-space, special-relativity statements that energy and momentum of matter perturbations are conserved. Also, the ICV $\vec{V}_{(0)}$ looks like time translation near the origin of a $t=$constant surface, and is timelike within the sound horizon.

One application of the integral constraints is anisotropies in the microwave background. One contribution to anisotropies is the Sachs-Wolfe effect. In a previous paper[2] it was shown that the effect of the constraints in the $k=0$ universe was to decrease the magnitude of the Sachs-Wolfe effect, due to causal, uncorrelated perturbations. In a subsequent paper, we will apply the constraints to anisotropies in the microwave background in a closed universe.

Finally, it appears that there is a connection between the existence of constraint vectors and the existence of exact solutions to the Einstein constraint equations near some known solution. (D'Eath has discussed the existence of exact solutions near the RW solution.[9]) This will be explored in further work.

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