Special symmetries for the Utiyama Lagrangian with external Yang-Mills fields

David E. Betounes
Department of Mathematics, University of Southern Mississippi, Hattiesburg, Mississippi 39406-9207
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This paper demonstrates how conditions for the existence of symmetries for particles interacting with external background gauge fields arise easily from the standard symmetry equations. The general phenomenon is described by the Lagrangian L_{UA} obtained from Utiyama's original Lagrangian L_U by regarding the gauge fields A as a given set of fields. An infinitesimal transformation $X+Y$, consisting of certain combined infinitesimal coordinate and local gauge transformations, will be an infinitesimal symmetry of L_{UA} precisely when the Lie derivative $\mathscr{L}_{(X+Y)^1}(L_{UA})$ yields a Lagrangian with identically vanishing Euler-Lagrange equations. This Lie derivative is explicitly computed and is shown to contain the expressions derived recently by several papers in the literature, as well as a new expression with possible physical importance.

I. INTRODUCTION

This Comment provides mathematical details for a general treatment, using standard symmetry-theory techniques, of some special symmetries that recently have received attention by Beckers and Hussin[1] for particles interacting with external U(1) gauge fields (electromagnetic fields) and by Jackiw and Manton[2] and by Forgács and Manton[3] for particles interacting with general external gauge fields. The standard symmetry analysis not only yields, in a natural fashion, the Jackiw-Manton-Forgacs symmetry equation (JMF equation)[4] as a sufficient condition for the existence of such special symmetries, but also shows that this equation must be modified with an additional term to cover the more general Lagrangians considered here. This additional term may prove to be of physical importance in elementary particle and gravitational field theories. The analysis also indicates to what extent the modified JMF equation is necessary for the existence of such symmetries.

This paper uses the general Lagrangian L_U introduced by Utiyama in his fundamental paper[5] extending the ideas in the classic Yang-Mills paper.[6] To get L_U one begins with a global gauge-invariant Lagrangian L for the particles ψ and introduces the Yang-Mills fields into L to obtain a local gauge-invariant Lagrangian L^*. Adding on an appropriate Lagrangian M for the free Yang-Mills fields, one obtains the Utiyama Lagrangian $L_U=L^*+M$. The ideas developed in the cited references[1-3] (cf. Ref. 1 for further references on this topic) can be described in this general setting as follows.

By substituting a given gauge field $A(x)$ into L_U, one obtains a Lagrangian L_{UA} representing particles ψ interacting with $A(x)$, considered as an external field. In general some symmetry is lost in passing from L_U to L_{UA}; however, it has been observed that combining an infinitesimal coordinate-induced transformation $X=(\xi_C,\eta_N)$ with an infinitesimal local gauge transformation $Y=(0,\eta_W)$ will give an infinitesimal symmetry $X+Y$ of L_{UA} provided the JMF equation

$$DW-\mathscr{L}_\xi A=0 \tag{1.1}$$

holds. The demonstration of this in the particular situations previously considered has not been given in full detail, but has been reasoned (plausibly) by working with L_U instead of L_{UA} and obtaining the result for L_{UA} by the *ad hoc* imposition of the JMF equation. There are complicated details which arise when trying to implement this strategy more fully, principally due to the fact that L_U and L_{UA} are defined on different spaces (jet bundles). In addition, it is found that the JMF equation (1.1) is not sufficient in general, but rather one needs the modified equation

$$DW-\mathscr{L}_\xi A+[N,A]=0 \ . \tag{1.2}$$

For these reasons this paper works directly with L_{UA} and stresses the fact that, for the types of transformations considered, the standard symmetry equations for $X+Y$ arise directly from the condition that

$$\mathscr{L}_{(X+Y)^1}(L_{UA})$$

be a trivial Lagrangian. An important simplification of this follows from the observation that since M_A is a trivial Lagrangian, one can discard it in the symmetry analysis.

II. SYMMETRY EQUATIONS

In this section it will prove worthwhile to review briefly the standard symmetry theory which has developed in several forms and from various points of view in the physics and mathematics literature.[7] The main emphasis here will be placed on the idea that the algebra of all infinitesimal symmetries X of a Lagrangian L is completely determined, in principle, by a system of partial differential equations, the symmetry equations, for the component functions of X. This was discussed in several previous papers[8] and originated in Hermann's works[9] extending Cartan's classical treatment for dynamical systems.

Suppose $L(x,\psi,\psi')$ is a Lagrangian depending on the variables $x=\{x_\mu\}_{\mu=1}^{p}$, the fields $\psi=\{\psi^i\}_{i=1}^{m}$, and their partial derivatives $\psi'=\{\psi^i_\mu\}_{\mu=1,\dots,p}^{i=1,\dots,m}$. Mathematically an infinitesimal transformation X of the variables and fields is a vector field on an appropriate manifold E:

$$X(x,\psi)=\xi_\mu(x)\frac{\partial}{\partial x_\mu}+\eta^i(x,\psi)\frac{\partial}{\partial\psi^i} \ , \tag{2.1}$$

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which physicists often write in the heuristic form

$$x_\mu \to x_\mu + \xi_\mu(x) \ ,$$
$$\psi^i \to \psi^i + \eta^i(x,\psi) \ .$$

(Notation: Here and in the sequel there is implied summation over repeated indices.) The vector field X can be prolonged to a vector field X^1 (infinitesimal transformation of x, ψ, and ψ') according to the definition

$$X^1 = X + X^i_\mu \frac{\partial}{\partial \psi^i_\mu} \ , \tag{2.2}$$

where

$$X^i_\mu = \frac{\partial \eta^i}{\partial x_\mu} + \frac{\partial \eta^i}{\partial \psi^j}\psi^j_\mu - \frac{\partial \xi_\nu}{\partial x_\mu}\psi^i_\nu \ . \tag{2.3}$$

The expression for the Lie derivative of L along the prolonged vector field is defined by

$$\mathscr{L}_{X^1}(L) = \xi_\mu \frac{\partial L}{\partial x_\mu} + \eta^i \frac{\partial L}{\partial \psi^i} + X^i_\mu \frac{\partial L}{\partial \psi^i_\mu} \ . \tag{2.4}$$

In some previous work[8,10] it was shown that X is an infinitesimal symmetry of L if and only if $\mathscr{L}_{X^1}(L) + \operatorname{div}(\xi)L$ is a trivial Lagrangian, i.e., a Lagrangian whose Euler-Lagrange equations vanish identically. It was also shown how the condition that $\mathscr{L}_{X^1}(L) + \operatorname{div}(\xi)L$ be trivial furnishes the partial differential equation for the component functions ξ and η of X. These equations, *the symmetry equations,* are exceedingly complex but the restricted class of infinitesimal transformations X for which

$$\operatorname{div}(\xi) = 0 \ , \tag{2.5}$$

$$\mathscr{L}_{X^1}(L) = 0 \tag{2.6}$$

is a subalgebra of infinitesimal symmetries for which the symmetry equations are evidently Eqs. (2.5) and (2.6). The one-parameter group $\{g_t\}_{t\in R}$ generated by such an X provides the most commonly used type of symmetry of L:

$$L(g_t^1(x,\psi,\psi')) = L(x,\psi,\psi') \ , \tag{2.7}$$

for every $t \in R$. While in practice it is often easy to verify that certain *groups* satisfy Eq. (2.7) and therefore constitute symmetries of L, it is however the *algebra* of infinitesimal symmetries, defined by virtue of the symmetry equations, which furnishes the most powerful technique in the theory.

III. YANG-MILLS-UTIYAMA CONSTRUCTION

Throughout the remainder of the paper $L(\psi,\psi')$ will be a given Lagrangian which does not depend on the variables x. L could be the Dirac Lagrangian for a free electron, a proton-neutron pair, etc., and we wish to review briefly Utiyama's general construction, generalizing that of Yang and Mills, for introducing the gauge fields into L. First we need some explicit assumptions abut the structure of the L we start with.

For any choice of constant matrices $C = \{C_{\mu\nu}\}, N = \{N_{ij}\}$, with $\operatorname{tr}(C) = 0$, and for any constants b_ν, let $X = X_{(C,b,N)}$ be the vector field given by

$$X(x,\psi) = \xi_\mu(x)\frac{\partial}{\partial x_\mu} + \eta^i(\psi)\frac{\partial}{\partial \psi^i} \ , \tag{3.1}$$

where

$$\xi_\mu(x) = C_{\mu\nu}x_\nu + b_\mu \ , \tag{3.2}$$

$$\eta^i(\psi) = N_{ij}\psi^j \ . \tag{3.3}$$

Assumption 1. For certain C, N, and b, with $\operatorname{tr}(C) = 0$, it is assumed that

$$\mathscr{L}_{X^1}(L) = 0 \ , \tag{3.4}$$

so that X is an infinitesimal symmetry of L. One can show that the collection of all matrices C,N for which Eq. (3.4) holds with $X = X_{(C,b,N)}$ forms a Lie algebra. For the Dirac Lagrangian L, the symmetry equation (3.4) forces C to be a matrix in the Lorentz Lie algebra, and $N = N_C$ to depend on C in such a way so as to give a representation of the Lorentz algebra as transformations on the internal space. In general, however, C and N need not be related.

Next suppose that $\{T^a\}_{a=1}^n$ is a basis for a Lie algebra $\mathscr{A}$ of $m \times m$ matrices with structure constants g^{abc}:

$$[T^a, T^b] = g^{abc}T^c \ . \tag{3.5}$$

For any choice of functions $\{W^a(x)\}_{a=1\ldots n}$ let $Y = Y_W$ be the vector field:

$$Y = \eta^i_W \frac{\partial}{\partial \psi^i} \ , \tag{3.6}$$

where

$$\eta^i_W(x,\psi) = W^a(x)T^a_{ij}\psi^j \ . \tag{3.7}$$

Assumption 2. For any choice of *constant* functions W^a, it is assumed that

$$\mathscr{L}_{Y^1}(L) = 0 \ , \tag{3.8}$$

so that Y is an infinitesimal global gauge symmetry of L.

With these assumptions then, the Yang-Mills-Utiyama construction extends L to a new Lagrangian L_U which admits local gauge symmetries. One introduces a collection of new dynamical variables $A = \{A^a_\mu\}^{a=1\ldots n}_{\mu=1\ldots p}$, representing additional fields (gauge fields), together with $A' = \{A^a_{\mu\nu}\}$ representing their partial derivatives. The Utiyama Lagrangian is then

$$L_U = L^* + M \ , \tag{3.9}$$

where

$$L^*(\psi,\psi',A) = L(\psi^i, \psi^i_\mu - A^a_\mu T^a_{ij}\psi^j) \tag{3.10}$$

and

$$M(A,A') = -\tfrac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} \ ,$$

with

$$F^a_{\mu\nu} = A^a_{\mu\nu} - A^a_{\nu\mu} - g^{abc}(A^b_\mu A^c_\nu - A^b_\nu A^c_\mu) \ . \tag{3.11}$$

Utiyama showed that L_U possesses local gauge symmetries and in fact any Lagrangian having such local gauge symmetries must have the form of L_U in Eq. (3.9) (with M a more general type of function of $F^a_{\mu\nu}$ than we have assumed here). Thus the fundamental contribution of Yang, Mills, and Utiyama was to achieve local gauge invariance by replacing $\partial_\nu\psi^j$ by what is now called the gauge-covariant derivative: $(D_\nu\psi)^j \equiv \partial_\nu\psi^j - A^a_\nu T^a_{jk}\psi^k$.

The notation with the asterisk above is designed for use in the sequel where, for example,

$$\left(\frac{\partial L}{\partial \psi_\mu^i}\right)^*(\psi,\psi',A) \equiv \frac{\partial L}{\partial \psi_\mu^i}(\psi^j,\psi_\nu^j - A_\nu^a T_{jk}^a \psi^k) \ . \tag{3.12}$$

IV. EXTERNAL GAUGE FIELDS

Having reviewed the construction of the Utiyama Lagrangian we now consider the Lagrangian L_{UA} obtained from it by substituting a given collection of gauge fields $\{A_\nu^a(x)\}$ into it. These gauge fields, considered as external fields, will be fixed throughout the remainder of the paper. For the sake of clarity and distinction it is perhaps best to introduce the following notation for the new Lagrangians that arise by fixing A.

Definition.

$$L_A^*(x,\psi,\psi') = L^*(\psi,\psi',A(x)) \ , \tag{4.1}$$

$$M_A(x) = M(A(x),A'(x)) \ , \tag{4.2}$$

$$L_{UA} = L_A^* + M_A \ . \tag{4.3}$$

Comment. Since M_A depends only on x, it is a trivial Lagrangian (has identically vanishing Euler-Lagrange equations). This being the case one can dispense with M_A altogether in the determination of the symmetries and conservation laws for L_{UA}. More precisely, it can be shown that since $L_{UA} = L_A^* + M_A$ and M_A is trivial, the symmetries, infinitesimal symmetries, and conservation laws for L_{UA} and L_A^* are identical. Consequently, one can work simply with L_A^*.

V. INFINITESIMAL COORDINATE-GAUGE SYMMETRIES OF L_A^*

It will be convenient at this point to introduce some notation commonly used in Yang-Mills gauge theory. Excellent accounts of the modern formulation are the recent paper by Daniel and Viallet[11] and the works of Hermann.[12] One regards W and A_μ as the following Lie-algebra-valued functions (values in $\mathscr{A} = span\{T^a\}$):

$$W(x) = W^a(x) T^a, \quad A_\mu(x) = A_\mu^a(x) T^a \ .$$

Then A, the connection form, is the $\mathscr{A}$-valued differential one-form given by

$$A(x) = A_\mu(x) dx_\mu \ .$$

Also DW and $[N,A]$ are the $\mathscr{A}$-valued one-forms with components

$$(DW)_\mu = \frac{\partial W}{\partial x_\mu} + [W,A_\mu] = \left(\frac{\partial W^c}{\partial x_\mu} + g^{abc} W^a A_\mu^b\right) T^c \ , \tag{5.1}$$

$$[N,A]_\mu = [N,A_\mu] = (DN)_\mu \ . \tag{5.2}$$

With the preliminary discussion out of the way we are now in a position to prove the main result.

Theorem. Suppose that assumptions 1 and 2 on L hold and that $\{A_\mu^a(x)\}$ are given functions. Suppose further that $X = X_{(C,b,N)}$ is as in Eqs. (3.1)–(3.3) with $\mathrm{tr}(C) = 0$, and that $Y = Y_W$ is as in Eqs. (3.6) and (3.7) with $\{W^a(x)\}$ a given collection of functions. Then

$$\mathscr{L}_{(X+Y)^1}(L_A^*) = B_{ij\mu}\psi^j \left(\frac{\partial L}{\partial \psi_\mu^i}\right)_A^* \ , \tag{5.3}$$

where

$$B_{ij\mu} = (DW - \mathscr{L}_\xi A + [N,A])_{ij\mu} \tag{5.4}$$

$$= \left[\frac{\partial W^c}{\partial x_\mu} + g^{abc} W^a A_\mu^b - \left(A_\nu^c \frac{\partial \xi^\nu}{\partial x_\mu} + \xi^\nu \frac{\partial A_\mu^c}{\partial x_\nu}\right)\right] T_{ij}^c + [N,A_\mu^c T^c]_{ij} \ . \tag{5.5}$$

Proof. The proof is quite straightforward. Since

$$\mathscr{L}_{(X+Y)^1}(L_A^*) = \mathscr{L}_{X^1}(L_A^*) + \mathscr{L}_{Y^1}(L_A^*) \ ,$$

one calculates each part separately and using the chain rule together with $\mathscr{L}_{X^1}(L) = 0$, $\mathscr{L}_{Y^1}(L) = 0$ for constant W in $Y = Y_W$ one arrives at the asserted identity. The details are as follows.

(a) In computing the components of the prolongation of X, one finds the extra components in Eq. (2.3) are

$$X_\mu^i = N_{ij}\psi_\mu^j - C_{\nu\mu}\psi_\nu^i \ .$$

Consequently the relation $\mathscr{L}_{X^1}(L) = 0$ written out is

$$\eta^i \frac{\partial L}{\partial \psi^i} + (N_{ij}\psi_\mu^j - C_{\nu\mu}\psi_\nu^i)\frac{\partial L}{\partial \psi_\mu^i} = 0 \ .$$

Since this holds for all choices of ψ^i and ψ_μ^i, replacing everywhere the ψ_μ^i by $\psi_\mu^i - A_\mu^a T_{ij}^a \psi^j$ gives [cf. Eq. (3.12)]

$$\eta^i \left(\frac{\partial L}{\partial \psi^i}\right)_A^* + (N_{ij}\psi_\mu^j - C_{\nu\mu}\psi_\nu^i)\left(\frac{\partial L}{\partial \psi_\mu^i}\right)_A^* = (N_{ij}A_\mu^a T_{jk}^a \psi^k - C_{\nu\mu} A_\nu^a T_{ij}^a \psi^j)\left(\frac{\partial L}{\partial \psi_\mu^i}\right)_A^* \ . \tag{5.6}$$

Using this identity and calculating the Lie derivative according to Eq. (2.4), one finds

$$\mathscr{L}_{X^1}(L_A^*) = \xi_\mu \left[\left(\frac{\partial L}{\partial \psi_\nu^i}\right)_A^* \left(\frac{-\partial A_\nu^a}{\partial x_\mu} T_{ij}^a \psi^j\right)\right] + \eta^l \left[\left(\frac{\partial L}{\partial \psi_\mu^k}\right)_A^* (-A_\mu^a T_{kl}^a) + \left(\frac{\partial L}{\partial \psi^l}\right)_A^*\right] + (N_{ij}\psi_\mu^j - C_{\nu\mu}\psi_\nu^j)\left(\frac{\partial L}{\partial \psi_\mu^i}\right)_A^*$$

$$= \left(-\xi_\nu \frac{\partial A_\mu^a}{\partial x_\nu} T_{ij}^a \psi^j - \eta^k A_\mu^a T_{ik}^a + N_{ij} A_\mu^a T_{jk}^a \psi^k - C_{\nu\mu} A_\nu^a T_{ij}^a \psi^j\right)\left(\frac{\partial L}{\partial \psi_\mu^i}\right)_A^* \ . \tag{5.7}$$

(b) In computing the components of the prolongation of Y, one finds the extra components are

$$Y_\mu^i=\frac{\partial W^a}{\partial x_\mu}T_{ij}^a\psi^j+W^aT_{ij}^a\psi_\mu^j \ . \tag{5.8}$$

Now for any choice of constants $V=\{V^a\}$ the relation $\mathscr{L}_{Y_V^1}(L)=0$ written out gives

$$V^aT_{ij}^a\psi^j\frac{\partial L}{\partial\psi^i}+V^aT_{ij}^a\psi_\mu^j\frac{\partial L}{\partial\psi_\mu^i}=0 \ . \tag{5.9}$$

Since this equation holds for all V^a, ψ^i, and ψ_μ^j, in particular, it holds when ψ_μ^j is everywhere replaced by $\psi_\mu^j-A_\mu^b(x)T_{jk}^b\psi^k$ and V^a replaced by $W^a(x)$ where x is some fixed value. With these replacements Eq. (5.9) becomes (suppressing the x)

$$W^aT_{ij}^a\psi^j\left(\frac{\partial L}{\partial\psi^i}\right)_A^*+W^aT_{ij}^a\psi_\mu^j\left(\frac{\partial L}{\partial\psi_\mu^i}\right)_A^*=W^aT_{ij}^aT_{jk}^bA_\mu^b\psi^k\left(\frac{\partial L}{\partial\psi_\mu^i}\right)_A^* \ . \tag{5.10}$$

Using this identity, one finds that

$$\begin{aligned}\mathscr{L}_{Y^1}(L_A^*)&=(W^aT_{ij}^a\psi^j)\left[\left(\frac{\partial L}{\partial\psi^i}\right)_A^*-\left(\frac{\partial L}{\partial\psi_\mu^k}\right)_A^*A_\mu^bT_{ki}^b\right]\\&\quad+\left(\frac{\partial W^a}{\partial x_\mu}T_{ij}^a\psi^j+W^aT_{ij}^a\psi_\mu^j\right)\left(\frac{\partial L}{\partial\psi_\mu^i}\right)_A^*\\&=\left(W^aT_{ij}^aT_{jk}^bA_\mu^b\psi^k-W^aT_{kj}^aT_{ik}^bA_\mu^b\psi^j+\frac{\partial W^a}{\partial x_\mu}T_{ij}^a\psi^j\right)\left(\frac{\partial L}{\partial\psi_\mu^i}\right)_A^*\\&=\left(\frac{\partial W^a}{\partial x_\mu}+g^{abc}W^aA_\mu^b\right)T_{ij}^c\psi^j\left(\frac{\partial L}{\partial\psi_\mu^i}\right)_A^* \ .\end{aligned} \tag{5.11}$$

The theorem now follows by combining Eqs. (5.7) and (5.11).

Corollary 1. $X+Y$ is an infinitesimal symmetry of L_A^* if and ony if

$$B_{ij\mu}\psi^j\left(\frac{\partial L}{\partial\psi_\mu^i}\right)_A^* \tag{5.12}$$

is a trivial Lagrangian.

Corollary 2. A sufficient condition for $X+Y$ to be an infinitesimal symmetry of L_A^* is that ξ, W, and N satisfy the modified JMF equation:

$$B_{ij\mu}=0 \ , \tag{5.13}$$

where $B_{ij\mu}$ is the expression given in Eq. (5.4). Additionally, if N commutes with every element in the gauge Lie algebra $[N,T^a]=0$, $a=1,\ldots,n$, then Eq. (5.13) clearly reduces to the JMF equation:

$$\frac{\partial W^c}{\partial x_\mu}+g^{abc}W^aA_\mu^b-\left(A_\nu^c\frac{\partial\xi^\nu}{\partial x_\mu}+\xi^\nu\frac{\partial A_\mu^c}{\partial x_\nu}\right)=0 \ . \tag{5.14}$$

Furthermore, when Eq. (5.13) holds, the standard expression[13] for the conserved vector $[\mathrm{div}(V)=0]$ corresponding to $X+Y$ is easily computed to be

$$V_\mu=\left(\frac{\partial L}{\partial\psi_\mu^i}\right)_A^*\left(\xi_\nu\frac{\partial\psi_i}{\partial x_\nu}-[(N+W)\psi]^i\right)-\xi_\mu L_A^* \ . \tag{5.15}$$

Comment. The conditions for a Lagrangian to be trivial have been generally described in Refs. 8 and 14. This can be used to determine precisely when the particular Lagrangian (5.12) is trivial. However, the details of this are not especially illuminating and so will not be presented. Instead, we offer the following example.

Example 1 [SU(l) gauge fields]. With some minor modifications in the customary notation[15] we write our Lagrangian as follows

$$\begin{aligned}L&=\tfrac{1}{2}i\bar q\cdot\gamma_\mu\partial_\mu q-\tfrac{1}{2}i\partial_\mu\bar q\cdot\gamma_\mu q-m\bar q\cdot q\\&=\tfrac{1}{2}i\bar q^{k\alpha}\gamma_\mu^{\alpha\beta}\partial_\mu q^{k\beta}-\tfrac{1}{2}i\partial_\mu\bar q^{k\alpha}\gamma_\mu^{\alpha\beta}q^{k\beta}-m\bar q^{k\alpha}q^{k\alpha} \ .\end{aligned} \tag{5.16}$$

Here, the $\gamma_\mu=\{\gamma_\mu^{\alpha\beta}\}$ are the Dirac matrices and $q=(q^1,\ldots,q^l)$ where each $q^k=\{q^{k\alpha}\}$ consists of 4 scalar fields. This is similar for $\bar q$. Then $\psi=(q,\bar q)$ represents the total collection of $8l$ scalar fields. Applying the preceding analysis to this Lagrangian yields two results.

First, the extra term in the modified JMF equation vanishes and so the regular JMF equation suffices for the analysis. To see this, note that L has coordinate-induced infinitesimal symmetries of the form $X=(Cx,\hat N\psi)$ where C is a traceless matrix in the Lorentz Lie algebra and $\hat N$ is a representation of certain 4×4 matrices N_1,N_2 (depending on C) whose action is

$$\hat N(q,\bar q)=(N_1q^1,\ldots,N_1q^l,N_2\bar q^1,\ldots,N_1\bar q^l) \ .$$

Also the Lie algebra su(l) of traceless $l\times l$ skew-Hermitian matrices $[\mathrm{tr}(T)=0,\bar T^t=-T]$ gets represented as infinitesimal gauge transformations $Y=(0,\tilde T\psi)$ where $\tilde T(q,\bar q)=(Tq,\overline{Tq})$. Then it is easy to see that

$$[\hat N,\tilde T]=0 \ .$$

Second, for such a Lagrangian L the JMF equation is both a *necessary* and sufficient condition for $X+Y$ to be an infinitesimal symmetry of L_A^*. Here, $X=(Cx,\hat N\psi)$ and $Y=(0,\tilde W(x)\psi)$ with $W(x)=W^a(x)T^a$. In fact, a short computation gives

$$\mathscr{L}_{(X+Y)^1}(L_A^*)=iB_{jk\mu}[\bar q^j\cdot\gamma_\mu q^k] \ ,$$

and it is easy to see that this is a trivial Lagrangian if and only if $B_{jk\mu}=0$ for all j, k, and μ.

Example 2. The hypotheses of the theorem also include the special case when L has infinitesimal symmetries which are not induced by coordinate transformations: $C=0=b$, $N=$ a transformation of the internal space, $X=(0,N\psi)$. The modified JMF equation reduces to $DW+[N,A]=0$ and solutions W of it yield infinitesimal symmetries $X+Y_W$ of L_A^*. The necessity of an extra term like $[N,A]$ seems clear since the compensating gauge transformation Y_W should *a priori* depend in some way on N. The Lagrangian in the previous example provides an illustration of this. For simplicity we consider the case $l=2$ and change notation slightly writing (ψ,ϕ) for $q=(q^1,q^2)$. Then

$$\begin{aligned}L&=\tfrac{1}{2}i(\bar\psi\cdot\gamma_\mu\partial_\mu\psi-\partial_\mu\bar\psi\cdot\gamma_\mu\psi)-m\bar\psi\cdot\psi\\&\quad+\tfrac{1}{2}i(\bar\phi\cdot\gamma_\mu\partial_\mu\phi-\partial_\mu\bar\phi\cdot\gamma_\mu\phi)-m\bar\phi\cdot\phi \ .\end{aligned}$$

Now suppose that N is the 16×16 matrix defining an infinitesimal transformation on the internal space according to $N(\psi,\phi,\bar\psi,\bar\phi)=(E\bar\psi,0,0,0)$ where $E\bar\psi=(\bar\psi^4,-\bar\psi^3,\bar\psi^2,-\bar\psi^1)$. The one-parameter group generated by N is $N_s=\exp(sN)=I+sN$, where I is the identity matrix. It is easy to check that these transformations,

$$N_s(\psi,\phi,\bar\psi,\bar\phi)=(\psi^1+s\bar\psi^4,\psi^2-s\bar\psi^3,\psi^3+s\bar\psi^2,\psi^4-s\bar\psi^1,\phi,\bar\psi,\bar\phi)\ ,$$

leave L invariant and in fact transform solutions of the field equations into new solutions. A straightforward computation of DW and $[N,A]$ yields the following 16×16 matrices:

$$(DW)_\mu=\begin{pmatrix} iW_\mu^3 I & (iW_\mu^1+W_\mu^2)I & 0 & 0\\ (iW_\mu^1-W_\mu^2)I & -iW_\mu^3 I & 0 & 0\\ 0 & 0 & -iW_\mu^3 I & (-iW_\mu^1+W_\mu^2)I\\ 0 & 0 & -(iW_\mu^1+W_\mu^2)I & iW_\mu^3 I\end{pmatrix},$$

$$[N,A]_\mu=\begin{pmatrix} 0 & 0 & -2iA_\mu^3E & (-iA_\mu^1+A_\mu^2)E\\ 0 & 0 & (-iA_\mu^1+A_\mu^2)E & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}.$$

Here $W_\mu^c\equiv(DW)_\mu^c=\partial W^c/\partial x_\mu+g^{abc}W^aA_\mu^b$ and I is the 4×4 identity matrix. Thus, in general one sees that there is no solution of the JMF equation: $DW+[N,A]=0$ for this example. However, for the special choice of external gauge field A with $A_\mu^3=0$, $A_\mu^2=iA_\mu^1$, and $\partial A_\mu^1/\partial x_\nu-\partial A_\nu^1/\partial x_\mu=0$ (so that the free field equations for A are satisfied), one has $[N,A]=0$. Then a particular solution of $DW=0$ with $W_\mu^c=0$ can be derived by taking $W^3=1$, $W^2=iW^1$, where W^1 is chosen to satisfy $\partial W^1/\partial x_\mu=-2A_\mu^1$. Then $X+Y_W$ is an infinitesimal symmetry of L_A^* and the corresponding conserved vector is

$$V_\mu=\bar\phi\cdot\gamma_\mu\phi-\bar\psi\cdot\gamma_\mu\psi-2W^1\bar\psi\cdot\gamma_\mu\phi+\frac{i}{2}\bar\psi\cdot\gamma_\mu E\bar\psi\ .$$

Thus, one sees the necessity of the new term $[N,A]$. It either forces a condition on the external gauge A, like the one imposed above, or forces an absence of symmetry (in this particular example).

VI. CONCLUSION

This paper has derived, from first principles, the formula

$$\mathcal{L}_{(X+Y)^1}(L_A^*)=(DW-\mathcal{L}_\xi A+[N,A])_{ij\mu}\psi^j\left(\frac{\partial L}{\partial\psi_\mu^i}\right)_A^*,\qquad(6.1)$$

which shows how the modified JMF equation

$$DW-\mathcal{L}_\xi A+[N,A]=0\qquad(6.2)$$

arises naturally as a sufficient condition for $X+Y$ to be an infinitesimal symmetry of L_A^* (and hence also of $L_{UA}=L_A^*+M_A$). The situation can be interpreted roughly as saying that the symmetries are altered when the background gauge field A is switched on: if the particles ψ in the absence of background gauge fields have Lagrangian L with a certain Lie algebra $\mathcal{B}$ of infinitesimal symmetries $X=X_{(C,b,N)}$, then turning on the gauge fields gives a Lagrangian L_A^* with altered Lie algebra of infinitesimal symmetries of the form $X+Y$, with $X\in\mathcal{B}$, $Y=Y_W$, and W a solution of the modified JMF equation. The Forgács-Manton paper[3] shows how to construct solutions of this equation when the term $[N,A]$ is absent, which is a commonly occurring case for many Lagrangians.

One final remark is that for the sake of simplicity the paper has been formulated along the lines of the classical Yang-Mills-Utiyama papers, rather than in terms of the modern principal fiber-bundle approach to gauge theories.[11,12] The sacrifice has been that our arguments are local rather than global and that certain details connected with the underlying spacetime manifold have been implicitly simplified. The global and intrinsic approach has pretty much the same development as that presented here.

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