Polar gyroscopic tests of general relativity

John R. Urani and Ronald W. Carlson
Physics Department, University of Missouri-Kansas City, Kansas City, Missouri 64110
(Received 26 December 1984)

Gyroscopic precession at the terrestrial poles is treated using the original Kerr metric in Cartesian coordinates and an isometry method for treating accelerating observers. The results corroborate predictions for gyroscopic precession obtained by gravitomagnetic methods and reinforce recent suggestions that ground-based precession experiments may be viable.

Gyroscopic precession as a test of general relativity (GR) has been an active theoretical topic for a number of years.[1–4] The well-known results[1] of weak-field GR predict that "geodetic precession" would be modified by the rotation of the Earth, giving a small "motional precession," a gravitomagnetic effect due to "frame dragging."

Since satellite gyro experiments will soon be attempted,[4] these GR tests are of substantial current interest. Very recently, several authors[5–7] have renewed the discussion of ground-based gravitomagnetic experiments, some suggesting terrestrial gyro[6] and Foucault pendulum[7] tests of GR. The frame-dragging effects in such environments are predicted to be a half an order of magnitude larger than in orbital experiments, which suggests that ground-based experiments be considered in spite of their extreme technical difficulty.[5]

Here we report precession results obtained using an exact isometry method[8] for accelerating observers. The Cartesian Kerr[9] metric avoids coordinate singularities at the poles. Applied to Earth's exterior geometry, the Kerr solution is accurate to first order in $\hat{a}=|\mathbf{J}|/mc$. Higher multipoles make negligible contributions in this precession problem. We calculate exact precession frequencies at the poles for the Kerr portion of Earth's exterior geometry which are in excellent agreement with gravitomagnetic treatments of gyroscopic[6] and Foucault[7] precession.

The isometry method[8] uses an orthornormal (ON) tetrad *field* rather than a local ON tetrad[10] Fermi-Walker transported along an observer world line $x(\tau)$. This permits simple computations of connection coefficients and "boosted" values for relevant tensors (a^μ, v^μ, and S^μ) for the accelerating observer. The ON tetrad field $K_b^\mu(x)$, and its dual one-form field $K^{-1}{}_\mu^b(x)$, play the role of local rest frames along the observer path $x(\tau)$ for the construction of boost Jacobians for tensors. One first obtains the tetrad components $u_b=K_b^\mu(x)v_\mu$ of the covariant velocity and then uses the spatial components of u_b to form a Lorentz boost matrix $\Lambda^c{}_d(u)$. The tangent-space boost Jacobians are then formed by similarity transforming into and out of the local ON tetrad (Lorentz) frame, that is, by constructing the inverse Jacobian,

$$J^{-1}{}_\nu^\mu(x)=K_c^\mu(x)\Lambda^c{}_d(u)K^{-1}{}_\nu^d(x) \ ,$$

with Λ^{-1} replacing Λ in J. These maps are isometric because $J^TgJ=g$ due to the orthonormality relation $K^TgK=\eta=(+---)$ and the O(3,1) isometery $\Lambda^T\eta\Lambda=\eta$. Tensors are then boosted on their contravariant/covariant indices using J^{-1}/J. No change has been made in local coordinate charts but rather a Lorentz-frame change expressed in local coordinate bases.[8] The boost is easily shown to be independent of the choice of tetrad field. The Levi-Civita coefficients transform, as does any affine connection, via

$$\bar{\Gamma}^\mu_{\alpha\beta}=J^{-1}{}_\theta^\mu J_\alpha^\lambda J_\beta^\sigma\Gamma^\theta_{\lambda\sigma}+J^{-1}{}_\theta^\mu J_\alpha^\lambda\partial_\lambda J_\beta^\theta \ .$$

For the polar gyroscope problem, we chose Kerr's original right-handed Cartesian metric[9] with $\mathbf{J}$ along the $+z$ axis, thus avoiding the coordinate singularities in Boyer-Lindquist[11] and isotropic coordinates.[12] At the poles, $|z|=+\rho$ (Boyer-Lindquist) $=+r_0$(polar radius). A gyro fixed at either pole has covariant velocity

$$v^\mu=(c(\rho^2+\hat{a}^2)^{1/2}/(\rho^2+\hat{a}^2-2m^*\rho)^{1/2},0,0,0)$$

with

$$\frac{dt}{d\tau}=(\rho^2+\hat{a}^2)^{1/2}/(\rho^2+\hat{a}^2-2m^*\rho)^{1/2} \ ,$$

$m^*=Gm/c^2$, and $\hat{a}=|\mathbf{J}|/mc$.

Polar observers have covariant acceleration

$$a^\mu=(0,0,0,\Gamma^3_{00}v^0v^0)$$
$$=(0,0,0,m^*(\rho^2-\hat{a}^2)c^2/(\rho^2+\hat{a}^2)^2)$$

because $\Gamma^0_{00}=\Gamma^1_{00}=\Gamma^2_{00}=0$. Note too that $\rho>>\hat{a}=329$ cm gives $a^3\simeq Gm/\rho^2$ as the correct Newtonian limit.

Choosing the timelike tetrad vector as $K_0^\mu=v^\mu/c$ at either pole yields $g(K_b,v)=u_b=0$ for $b=1,2,3$. Thus the boost is parametrized by $\mathbf{u}=0$ giving $J=J^{-1}=I$ for *any* coordinate charts at the poles. Thus tensors are trivially boosted by the identity and the Γ coefficients are unaltered.

The gyroscope spin S^μ satisfies $g(S,v)=0$ and the Fermi transport law $\nabla S/\partial\tau=dS/d\tau+\Gamma(v,S)=-g(S,a)v/c^2$ in the absence of torques. With $S^0=0$ and only $v^0\neq0$ we have $dS^i/d_\tau+\Gamma^i_{0j}v^0S^j=0$ along with

$$\nabla S^0/\partial\tau=\Gamma^0_{0j}v^0S^j=-(v^0/c^2)(S^3a_3) \ . \quad (1)$$

The pertinent Γ coefficients which are nonzero at the poles are

$$\Gamma^0_{03}=m^*(\rho^2+\hat{a}^2+2m^*\rho)(\rho^2-\hat{a}^2)/(\rho^2+\hat{a}^2)^3$$

and

$$\Gamma^2_{01}=-\Gamma^1_{02}=2m^*\rho\hat{a}/(\rho^2+\hat{a}^2)^2 \ . \quad (2)$$

Equation (1) holds since $\Gamma^0_{03}=-a^3g_{33}/c^2$ and $g_{33}=-(\rho^2+\hat{a}^2+2m^*\rho)/(\rho^2+\hat{a}^2)$ at the poles. We also have $dS^3/d\tau=0$ as expected from axial symmetry since all $\Gamma^3_{0j}=0$. From Eq. (2) the precession equations for S^1 and S^2 are $dS^1/d\tau=+\Gamma^2_{01}v^0S^2$ and $dS^2/d\tau=-\Gamma^2_{01}v^0S^1$, which yield the proper-time precession frequency

$$\omega_p = 2m^*\rho\hat{a}c/(\rho^2+\hat{a}^2)^{3/2}(\rho^2+\hat{a}^2-2m^*\rho)^{1/2}$$

relative to the global right-handed (x^0,x,y,z) coordinate chart (the "fixed stars"). In vector form, with $\rho=|z| >> \hat{a} >> m^*$, we have $d\mathbf{S}/d\tau = 2G(\mathbf{S}\times\mathbf{J})/c^2\rho^3$, which reproduces the "motional precession" result of Schiff.[1] Relative to rotating Cartesian Earth axes (right-handed) at the poles, the proper-time precession frequency is $\tilde{\omega}_p=\omega_p-\omega_E$, where the proper-time rotation frequency ω_E of Earth could be determined astrometrically at the south polar station.[7]

It has been suggested recently[6,7] that ω_p may be large enough to be measurable, given certain technological improvements. We echo this optimism because, upon numerical evaluation of ω_p using $|\mathbf{J}|=5.9\times10^{40}$, $m=5.98\times10^{27}$, and $r_0=6.37\times10^8$ in cgs units, we obtain the value $\omega_p=0.22''/\text{yr}$, in excellent agreement with the gravitomagnetic value $(0.20''/\text{yr})$ of Ref. 6 and the Foucault pendulum result $(0.218''/\text{yr})$ of Ref. 7.

While the difficulties[5] of the terrestrial gyroscopic and Foucault experiments are clearly formidable (if not prohibitive) at present, the terrestrial south polar gravitomagnetic effects are five times larger than expected orbital effects with no competing geodetic effects. Thus serious consideration should be given to the future viability of these delicate and exacting experiments using present and expected technological advances.

[1] L. Schiff, Proc. Nat. Acad. Sci. **46** 871 (1960); Phys. Rev. Lett. **4**, 216 (1960).

[2] R. Van Patten and C. Everitt, Phys. Rev. Lett. **36**, 629 (1976); V. Braginsky, C. Caves, and K. Thorne, Phys. Rev. D **15**, 2047 (1977).

[3] J. Schwinger, Am. J. Phys. **42**, 507 (1974).

[4] J. Anderson *et al.*, in *Proceedings of the Second Marcel Grossman Meeting on General Relativity,* edited by R. Ruffini (North-Holland, New York, 1982), p. 939.

[5] C. Everitt, in *Experimental Gravitation,* edited by B. Bertotti (Academic, New York, 1974), p. 331; J. Lipa, W. Fairbank, and C. Everitt, in *ibid.*, p. 361.

[6] H. Peng and C. Qin, Phys. Lett. **103A**, 197 (1984).

[7] V. Braginsky, A. Polnarov, and K. Thorne, Phys. Rev. Lett. **53**, 863 (1984).

[8] J. Urani and M. Kemp, J. Math. Phys. **24**, 2061 (1983).

[9] R. Kerr, Phys. Rev. Lett. **11**, 237 (1963).

[10] C. Misner, K. Thorne, and J. Wheeler, *Gravitation* (Freeman, San Francisco, 1973).

[11] R. Boyer and R. Lindquist, J. Math. Phys. **8**, 265 (1967).

[12] R. Adler, M. Bazin, and M. Schiffer, *Introduction to General Relativity* (McGraw-Hill, New York, 1975), p. 258.