Cancellation of higher-order anomalies

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It is demonstrated that the gauge-anomaly coefficient in high-dimensional space-time manifolds is closely related to the indices of representations studied earlier by the authors. The cancellation in the superstring theory of Green and Schwarz based on SO(32) is discussed in detail.

The importance of gauge anomalies in high-dimensional space-time has recently been discussed by many authors.[1] In particular, Green and Schwarz described a remarkable anomaly-free superstring theory[2,3] based on the SO(32) group and another possible one with $E_8\times E_8$ as the underlying group. A crucial condition for the existence of such theories is a cancellation of higher-order anomalies. The traditional formulation of this requirement is geometric in its nature and offers only a limited answer to the question of what happens when the underlying representations and/or groups are changed. The purpose of this Brief Report is to demonstrate that the conditions imposed on the anomaly coefficients are precisely conditions on higher-order indices (6th-order index in Ref. 2) studied extensively before[4,5] and that such a formulation naturally answers the above questions.

Let L be a simple Lie algebra whose basis $t_1, t_2, \ldots, t_d$ satisfies the commutation relations

$$[t_\mu, t_\nu] = c_{\mu\nu}^{\lambda} t_\lambda \ . \tag{1}$$

It is known[6] that any simple L of rank n possesses exactly n fundamental Casimir invariants. By J_p we denote the pth-order Casimir invariant of the form

$$J_p = g^{\mu_1\mu_2\cdots\mu_p} t_{\mu_1} t_{\mu_2} \cdots t_{\mu_p} \ , \tag{2}$$

where $g^{\mu_1\cdots\mu_p}$ is completely symmetric in indices $\mu_1, \ldots, \mu_p$ and satisfies the orthogonality relations[4] such as

$$g^{\mu\nu\alpha\beta} g_{\mu\nu} g_{\alpha\beta} = 0 \ , \tag{3}$$

for the special case[7] $p=4$. An explicit form of J_p for $p \leqslant 6$ is given in Refs. 4 and 7. Note that J_p is identically zero for any value of p other than those in the following list:

$$\begin{array}{ll}
SU(n+1) & :2,3,\ldots,n+1\ ; \\
SO(2n+1) \text{ and } Sp(2n) & :2,4,\ldots,2n\ ; \\
SO(2n) & :2,4,\ldots,2n-2,n\ ; \\
G_2 & :2,6\ ; \\
F_4 & :2,6,8,12\ ; \\
E_6 & :2,5,6,8,9,12\ ; \\
E_7 & :2,6,8,10,12,14,18\ ; \\
E_8 & :2,8,12,14,18,20,24,30\ .
\end{array} \tag{4}$$

Next let $\phi(L)$ be a generic representation of L. Then in general, the pth-order index $D_p(\phi)$ of $\phi(L)$ is defined[4] as the trace

$$D_p(\phi) = \mathrm{tr}_\phi J_p \ . \tag{5}$$

Here $\mathrm{tr}_\phi Y$ denotes the trace of Y in the representation ϕ. It follows from (4) that $D_p(\phi)$ is unique up to normalization for any p except in the case of $L=SO(2n)$ with even n. Then there are two indices $D_n(\phi)$ and $\hat{D}_n(\phi)$ of the same order n. Since from $J_p \equiv 0$ it follows that

$$D_p(\phi) = 0 \ , \tag{6}$$

the index $D_p(\phi)$ can differ from zero only if it corresponds to one of the entries in the list (4). However, if J_p is not identically zero, then there exists[6] at least one irreducible representation of L, say $\square$, such that

$$D_p(\square) \neq 0 \ .$$

In general, we can choose $\square$ to be one of the lowest-dimensional representations of L except for SO($2n$) with $p=n$, where $\square$ should be the fundamental spinor representation for the index $\hat{D}_n(\square)$. The dimension of this representation is 2^{n-1}. Since the normalization of $D_p(\square)$ is not fixed (it is given by the length of simple roots of L), it is convenient to introduce the quantity

$$Q_p(\phi) = \frac{D_p(\phi)}{D_p(\square)} \ , \tag{7}$$

and to extend the definition (7) by requiring that $Q_p(\phi)=0$ whenever J_p is identically zero. The values of $Q_p(\phi)$ for SU($n+1$), SO(k), Sp($2n$) are found in Ref. 4. In particular, for the adjoint representation $\rho(L)$, one has

$$Q_p(\rho) = \begin{cases} [1+(-1)^p]N & \text{for } L=SU(N)\ , \\ N-2^{p-1} & \text{for } L=SO(N)\ , \\ N+2^{p-1} & \text{for } L=Sp(N),\ N \text{ even}\ , \end{cases} \tag{8}$$

for p listed in (4).

Let t be a generic element of L,

$$t = \xi^\mu t_\mu \ , \tag{9}$$

with real or complex parameters ξ^μ. Corresponding to (9), one has

$$X = \phi(t) = \xi^\mu \phi(t_\mu) = \xi^\mu X_\mu \ , \tag{10}$$

where X_μ is the matrix representing t_μ in $\phi(L)$. In the particular case $\phi \equiv \Box$, we write (10) as

$$x = \Box(t) = \xi^\mu x_\mu \ . \tag{11}$$

We define $f_p(\xi)$ by

$$f_p(\xi) = g^{\mu_1 \cdots \mu_p} \xi_{\mu_1} \cdots \xi_{\mu_p} \ . \tag{12}$$

Now we can express the trace $\mathrm{tr}_\phi X^p$ in terms of $\mathrm{tr}_\Box x^p$ as follows. Consider first the simplest cases $p=2$ and 3:

$$\mathrm{tr}_\phi X^2 = c_2 f_2(\xi) D_2(\phi) \ , \quad \mathrm{tr}_\phi X^3 = c_3 f_3(\xi) D_3(\phi) \ , \tag{13}$$

where c_2 and c_3 are constants independent of the representation $\phi(L)$ and the parameters ξ^μ. Using (7) and (11) in (13), one has

$$\mathrm{tr}_\phi X^2 = Q_2(\phi) \mathrm{tr}_\Box x^2 \ , \tag{14a}$$

$$\mathrm{tr}_\phi X^3 = Q_3(\phi) \mathrm{tr}_\Box x^3 \ . \tag{14b}$$

Equation (14b) is recognized[8] as the triangle-anomaly equation in four-dimensional space-time. For higher-dimensional space-time we have to consider higher-degree traces $\mathrm{tr}_\phi X^p$.

Next consider the case $p=4$. We have[4,7]

$$\mathrm{tr}_\phi X^4 = c_4 f_4(\xi) D_4(\phi) + K(\phi)(\mathrm{tr}_\phi X^2)^2 \ , \tag{15}$$

for any irreducible ϕ, where

$$K(\phi) = \frac{1}{2[2+d(\rho)]} \left[6 \frac{d(\rho)}{d(\phi)} - \frac{Q_2(\rho)}{Q_2(\phi)} \right] \ , \tag{16}$$

and $d(\omega)$ is the dimension of the representation $\omega(L)$. Note that any exceptional Lie algebra as well as SU(2) and SU(3) have no fourth-order fundamental Casimir invariant J_4 so that $D_4(\phi)=0$ for any ϕ. Therefore, (15) reduces to the quadratic trace identity[7]

$$\mathrm{tr}_\phi X^4 = K(\phi)(\mathrm{tr}_\phi X^2)^2 \ . \tag{17}$$

In terms of $R_4(\phi)$ defined in Ref. 4, we can rewrite (15) as

$$\mathrm{tr}_\phi X^4 = Q_4(\phi) \mathrm{tr}_\Box x^4 + A_4(\phi)(\mathrm{tr}_\Box x^2)^2 \ , \tag{18a}$$

$$A_4(\phi) = K(\phi)[Q_2(\phi)]^2 - K(\Box) Q_4(\phi) = 3R_4(\phi) \ , \tag{18b}$$

provided $J_4 \neq 0$. When $J_4=0$ identically, Eq. (18a) is still valid for an arbitrary value of $Q_4(\phi)$. It is natural then to define $Q_4(\phi)=0$. The method can readily be applied to higher orders. For example, we find

$$\mathrm{tr}_\phi X^6 = Q_6(\phi) \mathrm{tr}_\Box x^6 + A_6(\phi) \mathrm{tr}_\Box x^2 \mathrm{tr}_\Box x^4 + B_6(\phi)(\mathrm{tr}_\Box x^2)^3 + C_6(\phi)(\mathrm{tr}_\Box x^3)^2, \tag{19}$$

where $A_6(\phi)$, $B_6(\phi)$, and $C_6(\phi)$ are functions of $Q_2(\phi)$, $Q_3(\phi)$, $Q_4(\phi)$, $Q_6(\phi)$, and of the dimension $d(\phi)$ but *not* of $Q_p(\phi)$ $(p>6)$. Explicit expressions for these functions are found in Ref. 4 for all L except SU(N), $N \geq 3$. The general formula for $\mathrm{tr}_\phi X^p$ in terms of $\mathrm{tr}_\Box x^p$ could also be derived. Indeed, following Ref. 4, one has

$$\mathrm{tr}_\phi X^p = Q_p(\phi) \mathrm{tr}_\Box x^p + A_p(\phi) \mathrm{tr}_\Box x^2 \mathrm{tr}_\Box x^{p-2} + \cdots \ . \tag{20}$$

Let us emphasize that (20) is valid for any representation $\phi(L)$ which does not have to be irreducible. Since the dominant term $\mathrm{tr}_\Box x^p$ in (20) cannot be canceled by another mechanism in general, the requirement that the anomaly is zero implies that $D_p(\phi)=0$ and thus

$$Q_p(\phi) = 0 \tag{21}$$

for the appropriate value of p given in Ref. 1 corresponding to a given dimension of space-time. As we have already noted, Eq. (21) is automatically satisfied whenever we have $J_p=0$. For example, consider $p=6$. Then we have $D_6(\phi)=0$ and consequently also $Q_6(\phi)=0$ for any $\phi(L)$, $L=$ SU(2), SU(3), SU(4), SU(5) as well as $L=$ Sp(4) and E_8. However, for other L one can also have $Q_p(\phi)=0$ for some particular representations ϕ.

Let us now restrict our attention to the superstring theory of Ref. 2. In this case ϕ is the adjoint representations ρ of L and $p=6$ and the trace identities (14a), (18), and (19) with coefficients given in Eq. (8) coinciding with those of Ref. 2:

$$\mathrm{tr}_\rho X^2 = (N \pm 2) \mathrm{tr}_\Box x^2 \ , \quad \mathrm{tr}_\rho X^4 = (N \pm 8) \mathrm{tr}_\Box x^4 + 3(\mathrm{tr}_\Box x^2)^2 \ , \quad \mathrm{tr}_\rho X^6 = (N \pm 32) \mathrm{tr}_\Box x^6 + 15\, \mathrm{tr}_\Box x^2 \mathrm{tr}_\Box x^4 \ , \tag{22}$$

for L being Sp(N) (upper sign) and SO(N) (lower sign). For $L=$ SU(N), one has

$$\mathrm{tr}_\rho X^2 = 2N\, \mathrm{tr}_\Box x^2 \ , \quad \mathrm{tr}_\rho X^4 = 2N\, \mathrm{tr}_\Box x^4 + 6(\mathrm{tr}_\Box x^2)^2 \ , \quad \mathrm{tr}_\rho X^6 = 2N\, \mathrm{tr}_\Box x^6 + 30\, \mathrm{tr}_\Box x^2 \mathrm{tr}_\Box x^4 - 20(\mathrm{tr}_\Box x^3)^2 \ . \tag{23}$$

Let us point out that similar trace identities were also discussed by Cvitanovic.[9] It is clear from (8) that $Q_6(\rho)=0$ for SO(32) although in general $Q_6(\phi) \neq 0$ for that group. For the exceptional Lie algebras G_2, F_4, E_6, and E_7 we have computed $Q_6(\rho)$ to be equal to -26, -7, -6, and -2, respectively. For $L=E_8$, we have $\rho=\Box$ and $J_4=J_6=0$ so that the trace identity[4,10] is

$$\mathrm{tr}_\phi X^6 = D(\phi)(\mathrm{tr}_\phi X^2)^3 \ ,$$

$$D(\phi) = \frac{15}{[d(\rho)+2][d(\rho)+4]} \times \left[\left(\frac{d(\rho)}{d(\phi)} \right)^2 - \frac{1}{2} \frac{d(\rho)}{d(\phi)} \frac{Q_2(\rho)}{Q_2(\phi)} + \frac{1}{12} \left(\frac{Q_2(\rho)}{Q_2(\phi)} \right)^2 \right] \ . \tag{24}$$

We may notice that (24) is valid also for SU(2). For the particular case $L=E_8$ and ϕ being the adjoint representation, $\phi \equiv \rho$, one can simplify (24) and (17) to give

$$\mathrm{tr}_\rho X^4 = \tfrac{1}{100} (\mathrm{tr}_\rho X^2)^2 \ , \quad \mathrm{tr}_\rho X^6 = \tfrac{1}{7200} (\mathrm{tr}_\rho X^2)^3 \ , \tag{25}$$

in agreement with Ref. 2.

Summarizing our results, we conclude that the dominant anomaly coefficient $Q_6(\rho)$ for $p=6$ is absent only when the Lie algebra consists of the simple one of types SU(N), $N=2,3,4$, and 5, Sp(4) $\approx$ SO(5), SO(32), and E_8. Now, adoption of SU(N) ($N \gtrsim 3$) is known to lead to an inconsistency in quantized theory. Moreover, cancellation of the mixed anomaly requires[2] the validity of

$$\mathrm{Tr}_\rho X^6 = \tfrac{1}{48} \mathrm{Tr}_\rho X^2 \mathrm{Tr}_\rho X^4 - \tfrac{1}{14400} (\mathrm{Tr}_\rho X^2)^3 \ , \tag{26}$$

which is satisfied by E_8 and SO(32) in view of Eqs. (22) and (25) but not by SU(2) and Sp(4). In a realistic superstring theory, one has to cancel also the gravitational anomaly.

The condition requires[2] that the dimension $d(\rho)$ of L is 496. These conditions are satisfied only by SO(32) and $E_8 \times E_8$ as has already been noted in Ref. 2.

In conclusion, we see that the anomaly coefficient is nothing but the general index $Q_p(\phi)$ defined in Ref. 4, and that various trace identities are intimately related to these indices.

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[1]P. H. Frampton and T. W. Kephart, Phys. Rev. Lett. **50**, 1343 (1983); **50**, 1347 (1983); P. K. Townsend and G. Sierra, Nucl. Phys. **B222**, 493 (1983); L. Alvarze-Gaumé and E. Witten, *ibid.* **B234**, 269 (1983); B. Zumino, Y. S. Wu, and A. Zee, *ibid.* **B239**, 477 (1984); L. Baulieu, *ibid.* **B241**, 557 (1984).

[2]M. B. Green and J. H. Schwarz, California Institute of Technology Report No. CALT-68-1182, 1984 (unpublished); California Institute of Technology Report No. CALT-68-1194, 1984 (unpublished).

[3]E. Witten, Princeton University Report, 1984 (unpublished).

[4]S. Okubo and J. Patera, J. Math. Phys. **24**, 2722 (1983); **25**, 219 (1984).

[5]J. Patera, R. T. Sharp, and P. Winternitz, J. Math. Phys. **17**, 1972 (1976); J. Patera and R. T. Sharp, *ibid.* **22**, 2352 (1981); J. McKay, J. Patera, and R. T. Sharp, *ibid.* **22**, 2770 (1981).

[6]J. Dixmier, *Enveloping Algebras* (North-Holland, Amsterdam, 1977).

[7]S. Okubo, J. Math. Phys. **23**, 8 (1982).

[8]J. Banks and H. Georgi, Phys. Rev. D **14**, 1159 (1976); S. Okubo, Phys. Rev. D **16**, 3528 (1977).

[9]P. Cvitanovic, *Group Theory* (Nordita, Copenhagen, Denmark, 1984).

[10]Y. Tosa, R. E. Marshak, and S. Okubo, Phys. Rev. D **27**, 444 (1983).