Improved wave functions for large-N expansions

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Existing large-N expansions of radial wave functions $\phi_{n,l}(r)$ are only accurate near the minimum of the effective potential. Within the framework of the shifted $1/N$ expansion, we use known analytic results to motivate a simple modification so that the improved wave functions are accurate over a wide range of r and any choice of quantum numbers n and l. It is shown that these wave functions yield simple and accurate analytic expressions for certain quantities of interest in quarkonium physics.

The large-N expansion,[1] where N is the number of spatial dimensions, is a useful new technique of solving the Schrödinger equation. However, problems of physical interest are in three dimensions and one only obtains results of modest accuracy from the first few terms in the $1/N$ expansion for these cases. One way of getting accurate results is to compute many orders in perturbation theory, and indeed this has been done for low-lying states for many potentials.[1(a)] An alternative recently developed approach, called the shifted $1/N$ expansion,[2-4] consists of using a shifted expansion parameter. It provides an excellent analytic approximation to the energy eigenvalues of the radial Schrödinger equation—simpler, more accurate, and applicable to a much wider class of problems than other currently available approximation schemes. However, the shifted-$1/N$-expansion wave functions, although accurate around the minimum of the large-N effective potential, have poor behavior for both very small and large values of the radial coordinate r.[3] In this note, we modify the leading term in the wave-function expansion by (i) incorporating known results about the limiting behavior in r of solutions of the radial equation and (ii) requiring that the analytic expressions for the wave functions of the harmonic oscillator and Coulomb potentials be exactly obtained. The incorporation of known analytic results is effectively a way of including important pieces from higher-order corrections into the leading term. This new leading-order wave function is a substantial improvement and can be used to calculate improved leading-order energies for any choice of quantum numbers n,l. Finally, we show how it is possible, with the aid of previous results,[3] to obtain higher-order corrections in the shifted expansion parameter $1/\bar{k}$ (defined below) to the new zeroth-order results.

Initially, let us only consider attractive power-law potentials $V(r)=\mathrm{sgn}(\nu)Ar^{\nu}$, $A>0$. The radial Schrödinger equation in N spatial dimensions is

$$\left[-\frac{\hbar^2}{2m}\frac{d^2}{dr^2}+\frac{\hbar^2(\bar{k}+a-1)(\bar{k}+a-3)}{8mr^2}+\mathrm{sgn}(\nu)Ar^{\nu}\right]\phi_{n,l}(r)=E\phi_{n,l}(r)\ , \quad (1)$$

where $\bar{k}=k-a$, $k=N+2l$, $a=2-2(2n+1)(m\omega/\hbar)$, $\omega=\hbar\tilde{\omega}/2m$, $\tilde{\omega}=\sqrt{\nu+2}$, $n=0,1,2,\ldots$ is the radial quantum number and $l=0,1,2,\ldots$ is the orbital angular momentum quantum number.

Although Eq. (1) cannot be solved in general, there are two important special cases for which a complete analytic solution is known.

Harmonic oscillator ($\nu=2$):

$$\phi_{n,l}(r)\propto r^{(k-1)/2}\exp[-(r/r_c)^2/2]\times L_n^{(k-2)/2}((r/r_c)^2)\ , \quad (2)$$

$$E_{n,l}=E_c\bar{k}\ . \quad (3)$$

Coulomb potential ($\nu=-1$):

$$\phi_{n,l}(r)\propto r^{(k-1)/2}\exp[-(r/r_c)/\bar{k}]L_n^{k-2}(2r/(r_c\bar{k}))\ , \quad (4)$$

$$E_{n,l}=-E_c/\bar{k}^2\ , \quad (5)$$

where $r_c\equiv(\hbar^2/2mA)^{1/(\nu+2)}$, $E_c\equiv Ar_c^{\nu}$ are the characteristic length and energy involved in these problems and the L_n^m are the generalized Laguerre polynomials.[5]

For general values of ν, the behavior of $\phi_{n,l}(r)$ as $r\to 0$ and as $r\to\infty$ can be readily obtained from Eq. (1),

$$\phi_{n,l}^{r\to 0}(r)\propto r^{(k-1)/2}\ , \quad (6)$$

$$\phi_{n,l}^{r\to\infty}(r)\propto\begin{cases}\exp\left[\dfrac{-2}{(\nu+2)}\left(\dfrac{r}{r_c}\right)^{(\nu+2)/2}\right], & \nu\geq 0\\ \exp\left[-\left(\dfrac{2m|E_{n,l}|}{\hbar^2}\right)^{1/2}r\right], & -2<\nu\leq 0\ .\end{cases} \quad (7)$$

Furthermore, from our previous work on the shifted $1/N$ expansion, we know the behavior of $\phi_{n,l}(r)$ near the minimum of the effective potential at large $\bar{k}$,[2]

$$V_{\mathrm{eff}}^{\bar{k}\to\infty}(r)\propto\frac{\hbar^2}{8m\eta^2}+\mathrm{sgn}(\nu)A\eta^{\nu}\ , \quad (8)$$

where $\eta=r\bar{k}^{-2/(\nu+2)}$. The minimum is at $r=r_0\equiv(\bar{k}^2\hbar^2/4|\nu|Am)^{1/(\nu+2)}$. The shifted-$1/N$-expansion wave function is expressed as a series in powers of $(1/\bar{k})^{1/2}$. The leading term in this expansion is[3]

$$\phi_{n,l}^{(0)}(r)\propto e^{-\alpha x^2/2}H_n(\sqrt{\alpha}x)\ , \quad (9)$$

where $\alpha=m\omega/\hbar$, $x=\bar{k}^{1/2}(r-r_0)/r_0$, and the H_n are the Hermite polynomials.

To sum up, if we could find a function that reproduces Eqs. (2) and (4) when $\nu=2$, -1, respectively, that possesses the limiting behaviors shown in Eqs. (6) and (7),

and, when $\bar{k}$ is large and r near r_0, reduces to Eq. (9), it should provide an excellent approximation to the eigenfunctions of Eq. (1). We make the simple choice

$$\tilde{\phi}^{(0)}_{n,l}(r) \propto r^{(k-1)/2} e^{-\lambda (r/r_0)^{\tilde{\omega}}} L_n^{(k-2)/\tilde{\omega}}(2\lambda (r/r_0)^{\tilde{\omega}}) , \quad (10)$$

where $\lambda = \bar{k}/2\tilde{\omega}$, which satisfies all the above criteria except the $r \to \infty$ behavior [Eq. (7)]. Although Eq. (7) is not satisfied, we obtain a much improved large-r behavior over Eq. (9). Note that Eq. (10) reduces to Eq. (9) when $\bar{k}$ is large and r near r_0 by virtue of the following crucial properties:[6]

$$L_n^{\alpha}(q) \propto e^q q^{-\alpha} \frac{d^n}{dq^n}(e^{-q} q^{n+\alpha}), \quad H_n(q) \alpha e^{q^2} \frac{d^n}{dq^n}(e^{-q^2}) , \quad (11)$$

and, for q large, $(\alpha - q)$ small,

$$\frac{e^{-q} q^{\alpha}}{\alpha!} \simeq \frac{1}{\sqrt{2\pi\alpha}} \exp[-(q-\alpha)^2/2\alpha] . \quad (12)$$

The expectation value of the energy corresponding to Eq. (10) for $n=0$ can be easily calculated. The result for power-law potentials is

$$\tilde{E}^{(0)}_{0,l} = \frac{\bar{k}\hbar^2}{4mr_0^2} \frac{(2\lambda)^{2/\tilde{\omega}}}{\Gamma((\bar{k}+2-\tilde{\omega})/\tilde{\omega})} \left[\frac{\tilde{\omega}}{2} \Gamma\left(\frac{\bar{k}}{\tilde{\omega}}\right) + \frac{\bar{k}}{(\tilde{\omega}^2-2)(2\lambda)^{\tilde{\omega}}} \Gamma\left(\frac{\bar{k}}{\tilde{\omega}} + \tilde{\omega} - 1\right) \right] . \quad (13)$$

Equation (10) is easily generalized to any sufficiently well-behaved spherically symmetric potential $V(r)$. The only changes are that

$$\tilde{\omega} = \left[3 + \frac{r_0 V''(r_0)}{V'(r_0)} \right]^{1/2} \quad (14)$$

and r_0 is determined from the equation

$$N + 2l - 2 + (2n+1) \left[3 + \frac{r_0 V''(r_0)}{V'(r_0)} \right]^{1/2} = \left[\frac{4mr_0^3 V'(r_0)}{\hbar^2} \right]^{1/2} . \quad (15)$$

It is possible to calculate higher-order corrections in $1/\bar{k}$ to Eq. (10). For $n=0$, Eqs. (A19), (13), and (14) in Ref. 3 give corrections to order $1/\bar{k}^2$ to Eq. (9). Our modified wave function Eq. (10) has taken into account some of these terms into the new leading-order expression, the leftover terms can be obtained after some straightforward algebra. We display the result for power-law potentials:

$$\phi_{0,l}(r) \propto \tilde{\phi}^{(0)}_{0,l}(r) \left\{ 1 - \frac{(\tilde{\omega}-1)(\tilde{\omega}-2)}{6\bar{k}^{1/2}} \left[\left(x + \frac{\tilde{\omega} x^3}{6} \right) - \frac{(\tilde{\omega}-2)}{24\bar{k}^{1/2}} \left((\tilde{\omega}-7)x^2 + \frac{\tilde{\omega}(\tilde{\omega}-19)x^4}{6} + \frac{\tilde{\omega}^2(\tilde{\omega}-1)x^6}{18} \right) + O\left(\frac{1}{\bar{k}}\right) \right] \right\} . \quad (16)$$

A similar technique can be applied to Eq. (3) in Ref. 3 to obtain corrections to the energy in Eq. (13). The result is

$$E_{0,l} = \tilde{E}^{(0)}_{0,l} \left\{ 1 - \frac{(\tilde{\omega}-1)^2(\tilde{\omega}-2)^2(\tilde{\omega}^2-2)}{36(\bar{k}\tilde{\omega})^2} \left[11 + \frac{(8\tilde{\omega}^4 - 69\tilde{\omega}^3 + 59\tilde{\omega}^2 + 96\tilde{\omega} - 76)}{6\bar{k}\tilde{\omega}} + O\left(\frac{1}{\bar{k}^2}\right) \right] \right\} . \quad (17)$$

Of course, by construction, Eqs. (16) and (17) yield the exact results for the Coulomb and harmonic-oscillator potentials.

We have examined the accuracy of Eqs. (10), (16), and (17) for various potentials and found them to yield excellent results. The energies computed from Eq. (17) are compared with those obtained from an accurate numerical integration of the Schrödinger equation[7] in Fig. 1 for various values of ν $(n=l=0)$. The new leading term $\tilde{E}^{(0)}_{0,l}$ [Eq. (13)] is always substantially better than the corresponding leading term from Eq. (3) in Ref. 3. However, when corrections are included in both expressions complete to the same order in $1/\bar{k}$, Eq. (17) does only slightly better for the range of physically interesting potentials $-1 \lesssim \nu \lesssim 4$. The accuracy of the ground-state $(n=l=0)$ wave function for $V(r) = r^3$, calculated from Eq. (16), is apparent from Fig. 2. Note the substantial improvement of our new leading term, Eq. (10), over the old, Eq. (9). For states with $n > 0$, we find that the leading-order wave function, Eq. (10), yields results which typically agree within 10% with "exact" numerical results. Also, as before,[3,4] the accuracy of the energies and wave functions increases as l increases.

We now give an application of our improved wave functions in quarkonium physics. The shifted $1/N$ expansion has already been shown to yield superior results for the energy eigenvalues of potentials of interest in this area.[3] In addition, one can calculate the leptonic decay width of a heavy neutral vector meson $V = (Q\bar{Q})$ using the familiar Van-Royen–Weisskopf formula[8]

$$\Gamma(V \to l^+ l^-) = 16\pi\hbar^3 \alpha e_Q^2 M_V^{-2} |\psi(0)|^2 [1 + O(\alpha_s)] , \quad (18)$$

where M_V is the mass of the meson and $\psi(0)$ is the total wave function of the composite system evaluated at the origin. $\psi(r,\theta,\phi)$ is related to $\phi(r)$ in Eq. (1) by $\psi(r,\theta,\phi) = r^{(1-N)/2} \phi(r) Y_l^m(\theta,\phi)$, where the $Y_l^m(\theta,\phi)$ are the spherical harmonics.

It is customary to eliminate the uncertainties about the nature of QCD corrections and quark charges by considering the ratio of leptonic widths of two mesons with the same quark content:

$$\frac{\Gamma(V' \to l^+ l^-)}{\Gamma(V \to l^+ l^-)} = \frac{M_{V'}^2}{M_V^2} \frac{|\psi_{V'}(0)|^2}{|\psi_V(0)|^2} . \quad (19)$$

The leading-order improved total wave function in the

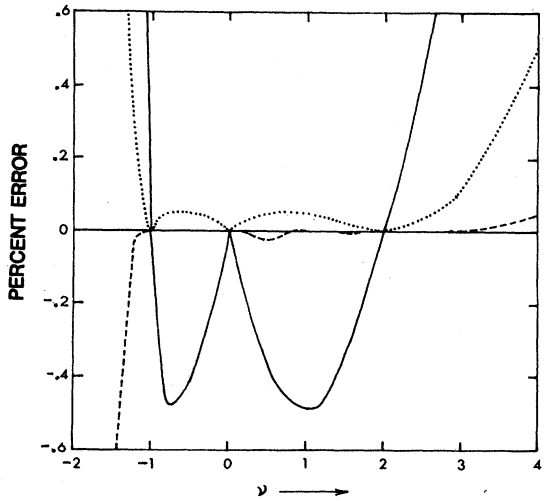

FIG. 1. Plots of the percentage error in the calculated value of the ground-state energy of power-law potentials $V(r)=\mathrm{Ar}^{\nu}$ vs the power ν. The solid line comes from energy calculations using the leading term in Eq. (3) of Ref. 3. The dotted line is drawn using Eq. (13), and corresponds to our improved leading-order wave function Eq. (10). After incorporating corrections up to $O(1/\bar{k}^3)$, one gets Eq. (17), which yields the dashed-line curve.

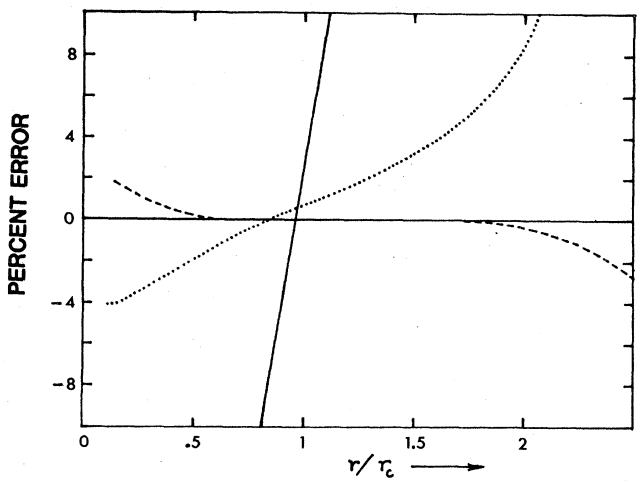

FIG. 2. Plots of the percentage error in the calculated wave function for the ground state of the potential $V(r)=r^3$ as a function of r/r_c. The solid line corresponds to Eq. (9) and is clearly not satisfactory. The error is much reduced when our improved leading-order wave function Eq. (10) is used, as can be seen from the dotted curve. When further corrections in $1/\bar{k}$ are made, one gets the dashed-line curve corresponding to Eq. (16), which is remarkably accurate throughout the region where the wave function is large. On this scale, the classical turning point is at approximately 1.5.

shifted $1/N$ expansion can be obtained from Eq. (10). In $N=3$ dimensions it is

$$\psi^{(0)}_{n,l,m}(r,\theta,\phi)\propto r^{l}e^{-\lambda(r/r_0)^{\tilde{\omega}}}Y_l^m(\theta,\phi)L_n^{(k-2)/\tilde{\omega}}(2\lambda(r/r_0)^{\tilde{\omega}}) \ . \tag{20}$$

For power-law potentials, $V(r)=Ar^{\nu}$, which are commonly used in quarkonium studies[8,9] we compute $|\psi^{(0)}_{n,l,m}(0)|^2$ using the identity[8,10] ($l=0$)

$$|\psi(0)|^2=\frac{m}{2\pi\hbar^2}\left\langle\frac{\partial V}{\partial r}\right\rangle \ . \tag{21}$$

For small values of n, the expectation value in Eq. (21) is readily calculated. For example, one finds the simple result

$$R^{(0)}\equiv\frac{|\psi^{(0)}_{100}(0)|^2}{|\psi^{(0)}_{000}(0)|^2}=\left(\frac{\bar{k}_0}{\bar{k}_1}\right)^{(\nu-1)(\tilde{\omega}-2)/(\nu+2)}\frac{(\tilde{\omega}^4-\tilde{\omega}^3-\tilde{\omega}^2+2\tilde{\omega}+1)}{(\tilde{\omega}^2-\tilde{\omega}+4)} \tag{22}$$

which only depends on the power ν. Here $\bar{k}_1\equiv N-2+3\tilde{\omega}$, $\bar{k}_0\equiv N-2+\tilde{\omega}$. Equation (22) yields the exact known analytic results for $\nu=-1,1,2$ (this will be true for ratios of the wave functions at the origin for any choice of quantum numbers). Below we give the results from Eq. (22) and exact numerical results[8] for two other potentials ($N=3$):

$$\begin{aligned} &V(r)=Ar^{-0.5}, \quad R^{(0)}=0.311, \quad R^{\mathrm{exact}}=0.297 \ ; \\ &V(r)=A\ln(r/b), \quad R^{(0)}=0.521, \quad R^{\mathrm{exact}}=0.510 \ . \end{aligned} \tag{23}$$

These results are more accurate than those obtained using the WKB approximation which is the method most commonly used to calculate quantities of interest in quarkonium physics. Also, one can include higher-order terms in $1/\bar{k}$ in our improved wave-function expansion [see, e.g., Eq. (16)], so there exists a systematic way of obtaining corrections to quantities such as Eq. (22). This feature does not exist in the WKB scheme.

We have also tested these improved wave functions for the Yukawa (screened Coulomb) potential $V(r)=Ae^{-\mu r}/r$, which is useful in essentially all branches of physics. Details can be found in Ref. 4.

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[6]Equation (2) is familiar from probability theory as the Gaussian approximation to a Poisson distribution near its maximum. The properties in Eq. (11) can be found in Ref. 5.
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[10]It is more straightforward to simply normalize ψ in Eq. (20) and evaluate it at the origin. However, we employ Eq. (21), since even though Eq. (20) yields the exact analytic results for $\nu = -1, 2$, it does not give the exact known $\psi(0)$ for $\nu = 1$. The use of Eq. (21) trivially assures this result, thereby improving the accuracy of results for potentials with ν near 1.