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Geometry of Kaluza-Klein theory. I. Basic setting

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Kaluza-Klein theory is derived from the hypothesis that the four-dimensional space-time is locally and isometrically embedded in a high-dimensional space which presumably originated at the big bang. For mathematical simplicity the high-dimensional space is taken to be a flat, Minkowski space with 14 dimensions assumed to be the ground state of the theory. The resulting metric is more general than the usual zero-mode metric ansatz but it reduces to the latter in the low-energy sector of the theory. The compactification of the internal space results from the existence of the second quadratic form of the embedded V_4. A simple model of spherical compact space is considered as a working example, where the spontaneous compactification is a hyperbolic function of the strength of the gravitational field. The symmetry group of the embedding is a combined symmetry which breaks into $P_4 \times SO(10)$ in the flat limit of the space-time.

I. INTRODUCTION

The modern versions of Kaluza-Klein theory assume the existence of a $(4+n)$-dimensional space with topology $V_4 \times B_n$, where V_4 is the four-dimensional space-time and B_n is an n-dimensional compact coset space G/H for a given gauge group G and a maximal subgroup H. In this construction the functions defined in $V_4 \times B_n$ depend periodically on the internal variables so that they can be harmonically expanded in these variables. The field equations are derived from the Einstein-Hilbert action for that space with a metric ansatz for the zero mode. The extreme high-energy levels associated with the theory suggests that the unification of the gravitational and Yang-Mills fields is effective near the big bang. With the expansion and cooling off of the universe the compact range of the n extra dimensions becomes sufficiently small so that they become "invisible" at low energies. The ground state of the theory is tentatively assumed to be $M_4 \times B_n$, where M_4 is the Minkowski space. The basic difficulties of the theory are the inexistence of a more fundamental principle leading to the metric ansatz, the compactification of the space B_n, the problem of vacuum stability for the ground state $M_4 \times B_n$, and the large fermion masses which appear in nonzero modes.[1-7]

The purpose of the present paper is to present a geometrical approach to the theory with the hope that some of the mentioned difficulties could be solved. The basic postulates of the theory such as the existence of the high-dimensional space and the low-energy sector are retained. However, the topology $V_4 \times B_n$ and the metric ansatz are derived from a more fundamental principle stating that the four-dimensional space-time is a subspace locally and isometrically embedded in a space V_{4+n} which presumably emerged at the big bang. In fact, V_4 is a subspace of $V_4 \times B_n$, but in this case, with the usual metric ansatz the embedding of V_4 is trivial in the sense that the second quadratic form of V_4 vanishes. In a more general embedding this quadratic form cannot be neglected, and it assumes an important physical role when regarded from the point of view of Kaluza-Klein theory.

The derived metric structure differs from the usual metric ansatz by the presence of the second-quadratic-form coefficients and as such it possibly represents not only the zero-mode metric but the full metric. The presence of the second quadratic form in the metric is also responsible for the compactification of the space of internal coordinates so that locally the structure $V_4 \times B_n$ is recovered. By use of the low-energy sector of the theory a simple model of spherical B_n is implemented whose radius varies with the curvature of the space-time. In the limit of weak gravitation the zero-mode metric is recovered.

The gauge potentials (and the gauge group) are also derived from the embedding assumption. The dynamics of these fields and of the second quadratic form is left to a subsequent paper where the analysis of fermion masses is also studied.

The geometry of the high-dimensional space is subjected to the Einstein-Hilbert action so that in principle we could start with an arbitrary pseudo-Riemannian space V_{4+n}. However the embedding of V_4 in an arbitrary curved space V_{4+n} is a problem of difficult solution. In particular the physical interpretations of the integrability conditions are not simple. For this reason we have chosen to start with a simpler case where V_{4+n} is the Minkowski space M_{4+n} for which at least some embedding results are known. Physically this choice may be explained in the following way: If an observer sitting in V_4 approaches the big bang he will become aware of the surrounding space by the impossibility of separating gauge and space-time symmetries (Sec. III). Such a situation prevails even if the observer performs experiments which are sufficiently local so as to neglect the curvature of V_{4+n} but not that of V_4 itself. The separation of gauge and space-time symmetries occurs in the flat limit of V_4. Following the example of general relativity M_{4+n} is taken as the ground state of the theory so that no problems with vacuum in-stability arise. Contrarily to what may seem, the results obtained with the flat space M_{4+n} are not trivial. To a great extent the standard Kaluza-Klein theory results from the embedding of V_4 into M_{4+n}.

For reasons of clearness, Sec. II includes some embedding basics with the appropriate notation.

II. EMBEDDING THE SPACE-TIME

Assuming that V_4 is a pseudo-Riemannian space-time with general metric $\bar{g}_{ij}$, we can always regard it as a hypersurface locally and isometrically embedded in M_{4+n} with metric signature $r(+)+s(-)$ and with a sufficiently large dimension. In order to avoid dimension and signature ambiguities the embedding is also assumed to be minimal in the sense that no dimension is taken in excess.[9]

The embedding is realized when a point in V_4 with arbitrary coordinates x^i, $i=1,\ldots,4$, can be described by a set of $4+n$ Cartesian coordinates $X^\mu(x^i)$, $\mu=1,\ldots,4+n$ in M_{4+n} with respect to an arbitrary origin. Then the isometric condition is expressed by

$$\bar{g}_{ij}=X_{,i}{}^{\mu}X_{,j}{}^{\nu}\eta_{\mu\nu}\ , \tag{1}$$

where $\eta_{\mu\nu}$ denotes the Cartesian components of the metric tensor of M_{4+n}. Thus $\bar{g}_{ij}$ is induced by $\eta_{\mu\nu}$ via the tangent vectors $X_{,i}{}^{\mu}$ which act as generalized vierbeins.[10] When the functions $X^\mu(x^i)$ are analytic the embedding of V_4 into M_{4+n} requires that $4+n$ is at most 10. On the other hand, if the functions $X^\mu(x^i)$ are differentiable, then $4+n$ is at most 14.[11] These theorems refer to a general metric $\bar{g}_{ij}$. For some specific cases the number of dimensions required to embed V_4 is smaller than the above limits. For example, all space-times with constant curvature can be embedded in M_5 [either $M_5(3,2)$ or $M_5(4,1)$]. Since we are assuming that the space M_{4+n} emerged at the moment of the big bang, then it is reasonable to suppose that the dimension $4+n$ reached the differentiable limit of 14 dimensions. This is compatible with the idea that a space-time admitting a point singularity like the big bang would not exhibit any specific symmetry except perhaps moments after the big bang itself. As is known, the minimal embedding dimension decreases as the space-time becomes more symmetric.[11] On the other hand, the pointlike character of the singularity also suggests that the metric signature of M_{4+n} is $(3+n)(+)+1(-)$. Although there is no general proof of this result, the example of Schwarzschild space-time serves as an indication: The minimal isometric embedding space of Schwarzschild space-time is $M_6(4,2)$. However if its singularity is removed to the origin—by means of a nonisometric transformation leading to the Kruskal metric—the embedding space changes to $M_6(5,1)$.[12] From the point of view of physics, the signature $(3+n)(+)+1(-)$ is important and even if the other signatures are mathematically possible, in what follows we restrict to the above case.

The index convention is as follows. All greek indices run from 1 to $4+n$. Lower-case latin indices run from 1 to 4 and capital latin indices run from 5 to $4+n$.

In a neighborhood of the embedding point of V_4 construct n unit vector fields N_A with Cartesian components $N^\mu_A(x^i)$ orthogonal to each other and to V_4:

$$N^\mu_A N^\nu_B\eta_{\mu\nu}=\bar{g}_{AB}\,,\quad N^\mu_A X^\nu{}_{,i}\eta_{\mu\nu}=0\ , \tag{2}$$

where in the present case $\bar{g}_{AB}=\delta_{AB}$ (for different embedding signatures $g_{AB}=\epsilon_A\delta_{AB}$, $\epsilon_A=\pm1$). Equations (1) and (2) are the basic equations for the embedding. In order to derive the integrability conditions for these equations consider a point of M_{4+n} not necessarily in V_4 given by the Cartesian coordinates

$$Z^\mu(x^i,x^A)=X^\mu(x^i)+x^AN^\mu_A(x^i)\ . \tag{3}$$

Here x^A are n parameters which together with x^i define a natural V_4-based Gaussian coordinate system $\{x^\alpha\}=\{x^i,x^A\}$, in which V_4 is defined by $x^A=0$. The expression (3) is then a transformation between Gaussian and Cartesian coordinates in M_{4+n} so that tensors are related by its derivative map. Thus, for example, the metric of M_{4+n} in the V_4-based Gaussian frame is

$$\gamma_{\alpha\beta}=Z^\mu{}_{,\alpha}Z^\mu{}_{,\beta}\eta_{\mu\nu}\ . \tag{4}$$

Making use of (1)—(3), it follows that

$$\gamma_{ij}=\bar{g}_{ij}-2x^Ab_{ijA}+x^Ax^BN^\mu_{A,i}N^\nu_{B,j}\eta_{\mu\nu}\ ,$$

$$\gamma_{iA}=x^EN^\mu_AN^\nu_{E,i}\eta_{\mu\nu},\quad \gamma_{AB}=\bar{g}_{AB}\ ,$$

where we have used the notation

$$b_{ijA}=-N^\mu_{A,i}X^\nu{}_{,j}\eta_{\mu\nu}\ , \tag{5}$$

representing the coefficient of the second quadratic form of the embedded V_4. It is also convenient to denote (g_0 is a constant to be later identified with the Yang-Mills coupling constant. For pure geometrical considerations we may take $g_0=1$ or absorb g_0 in A_{iAB})

$$A_{iAB}=g_0\bar{A}_{iAB}=-N^\mu_{A,i}N^\nu_B\eta_{\mu\nu}\ . \tag{6}$$

From (2) and (3) it follows that $b_{ijA}=b_{jiA}$ and $A_{iAB}=-A_{iBA}$. Using (5) and (6), $N_{A,i}$ can be expressed as

$$N^\mu_{A,i}=\bar{g}^{MN}A_{iMA}N^\mu_N-\bar{g}^{mn}b_{imA}X^\nu{}_{,n}\ , \tag{7}$$

where $\bar{g}^{AC}\bar{g}_{CB}=\delta^A_B$ and $\bar{g}^{ik}\bar{g}_{kj}=\delta^i_j$. Replacing (6) and (7) in (4), it follows that

$$\gamma_{ij}=\bar{g}_{ij}-2x^Eb_{ijE}+x^Ex^F(\bar{g}^{mn}b_{imE}b_{jnF}+\bar{g}^{MN}A_{iME}A_{jNF})\ ,$$

$$\gamma_{iA}=x^EA_{iAE}\ .$$

The Lorentz group of M_{4+n}, $SO(3+n,1)$, has as a noninvariant subgroup the group of rotations $SO(n)$ of the space N_n generated by N_A. The Lie-algebra generators of this subgroup can be described as

$$L^{AB}=\tfrac{1}{2}(x^A\partial/\partial x_B-x^B\partial/\partial x_A)$$

and the Killing vector field basis of the space generated by x^A are

$$k^{AB}_C=L^{AB}(x_C)=x^{[A}\delta^{B]}_C\ .$$

Therefore for the Lie-algebra-valued quantities $A_i=A_{iAB}L^{AB}$, the Killing basis components are $A_{iC}=A_{iAB}k^{AB}_C=A_{iAC}x^A$. It follows that

$$x^Ex^F\bar{g}^{MN}A_{iME}A_{jNF}=\bar{g}^{MN}A_{iM}A_{jN}$$

so that $\gamma_{\alpha\beta}$ can be expressed as

$$\gamma_{\alpha\beta}=\begin{bmatrix} g_{ij}+\bar{g}^{MN}A_{iM}A_{jN} & A_{iA} \\ A_{iB} & \bar{g}_{AB} \end{bmatrix}, \tag{8}$$

where

$$g_{ij}=\bar{g}_{ij}-2x^{E}b_{ijE}+x^{E}x^{F}\bar{g}^{mn}b_{imE}b_{jnF}\ . \tag{9}$$

As can be seen, Eq. (8) resembles the Kaluza-Klein metric ansatz where A_{iM} plays the role of the gauge potentials for SO(n) in the Killing basis. In Sec. IV we shall see that these quantities, or rather A_{iAB}, in fact, transform as gauge potentials.

III. COMPACTIFICATION

The real difference from (8) to the Kaluza-Klein metric is that g_{ij} as given by (9) is not the space-time metric. The coefficients b_{ijA} cannot be neglected in the general embedding picture and they are responsible for the compactification of the coordinate space B_n generated by x^A. In fact, Eq. (9) can also be expressed as

$$g_{ij}=\bar{g}^{mn}(\bar{g}_{im}-x^{A}b_{imA})(\bar{g}_{jn}-x^{B}b_{jnB})\ .$$

Since $\det\gamma_{\alpha\beta}=\det g_{ij}\det g_{AB}\neq 0$ it follows that $\det g_{ij}\neq 0$. Therefore x^A *cannot* be a solution of

$$\det(g_{im}-x^{A}b_{imA})=0\ . \tag{10}$$

It happens that (10) is the definition of the local curvature radii ρ_m^A of V_4 corresponding to a normal N_A and a principal direction dx^m[10] (see also the Appendix).

Since we have started with an observer in V_4 for which $x^A=0$, then the coordinates x^A are limited to an interval $x^A\in[0,\alpha\rho_m^A]$, $\alpha<1$. Therefore at each point of V_4 there is a bounded coordinate space B_n generated by x^A. It would be desirable to eliminate the dependence on the principal directions. This can be achieved by introducing a curvature radius for each direction N_A defined by

$$\rho^{A^2}=\bar{g}^{mn}\rho_m^A\rho_m^A$$

which is an invariant with respect to the coordinate transformations of V_4. In this case the bounded interval becomes $x^A\in[0,2\pi\alpha\rho^A]$, where α is chosen so that $2\pi\alpha\rho^A<\rho_m^A$ for all values of A and m. Since the various directions N_A are mapped into one another by the action of SO(n), the space B_n can also be constructed in a more invariant fashion by use of the invariant curvature radius ρ (see the Appendix) and the interval $[0,\alpha(\rho)]$, with $\alpha(\rho)=2\pi\alpha\rho<\rho_m^A$. Following the Einstein tube construction, we may assoicate to each point P of V_4 a closed space B_n generated by x^A. In such a construction the physical space considered is the local slab limited by $V_4(x^A=0)$ and $V_4'(x^A=\alpha(\rho))$ and such that P coincides with certain P' of V_4' (see Fig. 1).[3] In particular, B_n may be taken to be the n-sphere

$$S_n=\mathrm{SO}(n)/\mathrm{SO}(n-1)\ ,$$

with radius $\alpha(\rho)/2\pi$ and equation

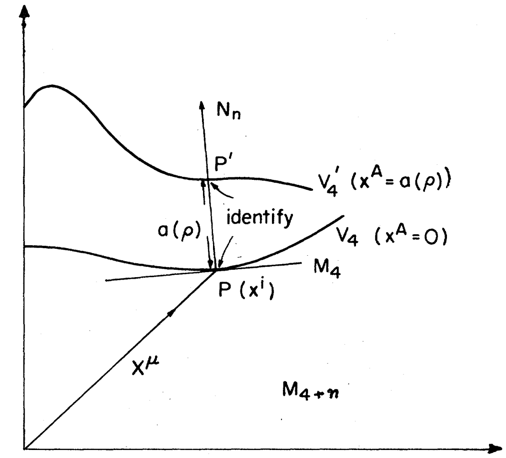

FIG. 1. The local Einstein tube.

$$\sum(x^A)^2-4\pi\alpha\sum x^A\rho^A=0\ .$$

Therefore even though M_{4+n} is not compact a local structure line $V_4\times B_n$ is specified at each neighborhood of V_4. Consequently, as in the usual Kaluza-Klein formulations, functions defined in the local slab may be expanded harmonically in terms of x^A.

In order to achieve the size of Planck's length $R_0\approx 10^{-33}$ cm for $a(\rho)$ one has to appeal to the low-energy sector of Kaluza-Klein theory which requires that in the limit of weak gravitation (that is, for large values of ρ), $a(\rho)$ tends to R_0.[13] This suggests that the factor α cannot be constant, but a function of the space-time curvature. Thus, $a(\rho)$ may be represented by a truncated asymptotic expansion $\sum_{k=0}^{N}C_k/\rho^k$ with $C_0=R_0$. The remaining coefficients may be adjusted to meet other conditions of the low-energy sector. A simple example of compactification may be obtained with two terms in the asymptotic expansion of $a(\rho)$ with $C_1=R_0{}^2$. In this case B_n compactifies hyperbolically

$$(a(\rho)-R_0)\rho=R_0{}^2\ ,$$

where the diameter of S_n becomes of the order of R_0 in the very early universe when $\rho=R_0$. Now the compact local coordinate space B_n can be identified with the internal space corresponding to the internal symmetry SO(n). The energy required to observe or excite the internal states of radius $a(\rho)$ depends on the strength of the gravitational field represented by ρ: $E(\rho)=hc/a(\rho)$. Physically this means that the gravitational environment contributes to the excitation of the internal states. In the simple hyperbolic example the excitation energy is

$$E(\rho)=\frac{\rho}{\rho+R_0}\frac{hc}{R_0} \tag{11}$$

so that when $\rho=R_0$ (a very strong gravitational field) only half of the energy is required as compared to an observer in Minkowski space-time.

Notice that the metric given by (8) is exact in the sense that the dependence on x^A is complete. If we wish we may expand $\gamma_{\alpha\beta}(x^i,x^A)$ harmonically in x^A in the periodic Einstein tube construction of B_n. However this is not necessary and (8) possibly represents the full Kaluza-Klein metric. To recover the zero-mode metric we remember that there is a coupling constant g_0 absorbed in the definition of A_{iAB} and that x^A is limited to $a(\rho)$. Therefore if the gravitational field is sufficiently weak so as to neglect the terms containing x^A in (9) but not neglect the factor $(g_0 a(\rho))^2$ in (8), then the metric (8) reduces to the usual Kaluza-Klein metric ansatz. It is now clear why the compactification of B_n has to be postulated in the usual Kaluza-Klein theory formulations: The zero-mode metric ansatz does not give any information concerning the existence of the second quadratic form which responds for the compactification of B_n.

IV. THE EMBEDDING SYMMETRY

Previously we have described a compactification of the space of internal coordinates x^A to a sphere $S^n = SO(n)/SO(n-1)$, by examining the conditions for the metric (8) to be invertible. In the limit of vanishing gravitation the radius of the compact space tends to R_0. Such compactification can also be associated to a symmetry breaking which is triggered by the limit of vanishing gravitation. We shall see that in this limit the group of invariance of the embedding as seen from V_4, breaks into $P_4 \times \bar{G}$ where P_4 is the Poincaré group, and $\bar{G}$ is SO(n).

The embedding coordinates $X^\mu(x^i)$ are determined up to an isometry of M_{4+n}. Therefore the local isometric embedding is invariant under the homogeneous group of isometries SO$(3+n,1)$ of M_{4+n} (besides the manifold mapping group of V_4). In particular the infinitesimal transformations of this group relating two V_4-based Gaussian systems is

$$x'^\alpha = x^\alpha + \xi^\alpha(x^i, x^A), \quad \xi^{(\alpha;\beta)} = 0 \text{ (fixed origin)}, \tag{12}$$

where the covariant derivative is calculated with respect to $\gamma_{\alpha\beta}$. An observer in V_4 will read the Killing equations as

$$\xi^{(i;j)} \mid_{V_4} = 0, \quad \xi^{(i;A)} \mid_{V_4} = 0, \quad \xi^{(A;B)} \mid_{V_4} = 0, \tag{13}$$

where $|_{V_4}$ means the restriction to V_4: $x^A = 0$. The corresponding group is denoted by $\overline{\mathrm{SO}(3+n,1)}$. Explicitly, the first equation (13) reads

$$\bar{\xi}^{(i;j)} = \bar{g}^{k(i} \bar{g}^{j)n} b_{knM} \bar{\xi}^M, \tag{14}$$

where $\bar{\xi}^{(i;j)}$ denotes the covariant derivative of $\bar{\xi}^i = \xi^i \mid_{V_4}$ with respect to $\bar{g}_{ij}$ and where $\bar{\xi}^A = \xi^A \mid_{V_A}$. The second equation (13) is equivalent to

$$\tfrac{1}{2}\bar{g}^{ki}\xi^A{}_{,k} \mid_{V_4} + \tfrac{1}{2}\bar{g}^{MA}\xi^i{}_{,M} \mid_{V_4} + \tfrac{1}{2}\bar{g}^{ki}\bar{g}^{AD} A_{kDM}\bar{\xi}^M = 0 .$$

Therefore

$$\xi^i{}_{,M} \mid_{V_4} = -\bar{g}_{MA}\bar{g}^{ki}(\bar{\xi}^A{}_{,k} + \bar{g}^{AD} A_{kDN}\bar{\xi}^N) . \tag{15}$$

Finally the last equation (13) gives

$$(\gamma^{\alpha(A}\xi^{B)}{}_{,\alpha} + \gamma^{\alpha(A}\Gamma^{B)}_{\alpha\gamma}\xi^\gamma) \mid_{V_4} = 0$$

so that

$$g^{M(A}\xi^{B)}{}_{,M} \mid_{V_4} = \xi^{(A,B)} \mid_{V_4} = 0 . \tag{16}$$

Notice that the particular transformations which send space-time points to space-time points require that $\xi^A \mid_{V_4} = \bar{\xi}^A = 0$. From (14) and (15) it follows that these particular transformations correspond to isometries of V_4 (if they exist). In the general case, however, the transformations generated by ξ^i and ξ^A are not independent.

The Lie algebra of $\overline{\mathrm{SO}(3+\mathrm{n},1)}$ can be obtained from the projection of that of SO$(3+n,1)$ given by

$$[l_{\mu\nu}, l_{\rho\sigma}] = \eta_{\mu\rho} l_{\nu\sigma} + \eta_{\nu\sigma} l_{\mu\rho} - \eta_{\mu\sigma} l_{\nu\rho} - \eta_{\nu\rho} l_{\mu\sigma} .$$

In the Gaussian frame the generators of SO$(3+n,1)$ are

$$L_{\alpha\beta} = Z^\mu{}_{,\alpha} Z^\nu{}_{,\beta} l_{\mu\nu}$$

and a straightforward calculation shows that

$$[L_{\alpha\beta}, L_{\gamma\delta}] = Z^\mu{}_{,\alpha} Z^\nu{}_{,\beta} Z^\rho{}_{,\gamma} Z^\sigma{}_{,\delta} [l_{\mu\nu}, l_{\rho\sigma}] = C^{\epsilon\phi}_{\alpha\beta\gamma\delta} L_{\epsilon\phi} , \tag{17}$$

where the structure constants are

$$C^{\epsilon\phi}_{\alpha\beta\gamma\delta} = 4\delta^\epsilon_{[\beta}\gamma_{\alpha][\gamma}\delta^\phi_{\delta]} . \tag{18}$$

Using (8) and denoting $\bar{L}_{\alpha\beta} = L_{\alpha\beta} \mid_{V_4}$, the projected Lie algebra is

$$\begin{aligned} &[\bar{L}_{ij}, \bar{L}_{kl}] = \bar{C}^{mn}_{ijkl} \bar{L}_{mn}, \quad [\bar{L}_{AB}, \bar{L}_{CD}] = \bar{C}^{EF}_{ABCD} \bar{L}_{EF} , \\ &[\bar{L}_{ij}, \bar{L}_{kA}] = \bar{g}_{ik} \bar{L}_{jA} - \bar{g}_{jk} \bar{L}_{iA}, \quad [\bar{L}_{ij}, \bar{L}_{AB}] = 0 , \\ &[\bar{L}_{iA}, \bar{L}_{jB}] = \bar{g}_{ij} \bar{L}_{AB} + \bar{g}_{AB} \bar{L}_{ij} , \\ &[\bar{L}_{iA}, \bar{L}_{BC}] = \bar{g}_{AC} \bar{L}_{iB} - \bar{g}_{AB} \bar{L}_{iC} , \end{aligned} \tag{19}$$

where $\bar{C}^{mn}_{ijkl}$ and $\bar{C}^{EF}_{ABCD}$ are obtained with the projection of (18).

In complete analogy with the de Sitter group the above algebra can be contracted into another algebra in the flat limit of V_4.[16,17] For that purpose modify the basis of (19) with the introduction of the operators

$$\bar{\pi}_i = \alpha^A \bar{L}_{iA} , \tag{20}$$

where $\alpha^A = 1/\rho^A$. In terms of $\bar{\pi}_i$, (19) reads as

$$\begin{aligned} &[\bar{L}_{ij}, \bar{L}_{kl}] = \bar{C}^{mn}_{ijkl} \bar{L}_{mn}, \quad [\bar{L}_{AB}, \bar{L}_{CD}] = \bar{C}^{EF}_{ABCD} \bar{L}_{EF} , \\ &[\bar{L}_{ij}, \bar{\pi}_k] = \bar{g}_{ik} \bar{\pi}_j - \bar{g}_{jk} \bar{\pi}_i, \quad [\bar{\pi}_i, \bar{\pi}_j] = \bar{g}_{AB} \alpha^A \alpha^B \bar{L}_{ij} , \\ &[\bar{\pi}_i, \bar{L}_{BC}] = \alpha^A \bar{g}_{AC} \bar{L}_{iB} - \alpha^A \bar{g}_{AB} \bar{L}_{iC} . \end{aligned} \tag{21}$$

Notice the existence of two noninvariant subgroups: The 10-parameter subgroup $\bar{E}$ generated by $\bar{L}_{ij}$, $\bar{\pi}_k$, and the $n(n-1)/2$-parameter subgroup $\bar{G}$ generated by $\bar{L}_{AB}$:

$$[\bar{E}, \bar{E}] = \bar{E}, \quad [\bar{E}, \bar{G}] \in \overline{\mathrm{SO}(3+n,1)}, \quad [\bar{G}, \bar{G}] = \bar{G} .$$

Now taking the flat limit of V_4, $\rho^A \to \infty$ then $\alpha^A \to 0$ so that (21) gives

$$\begin{aligned} &[{}^0L_{ij}, {}^0L_{kl}] = {}^0C^{mn}_{ijkl} {}^0L_{mn} , \\ &[{}^0L_{ij}, {}^0\pi_k] = {}^0g_{ik} {}^0\pi_j - {}^0g_{jk} {}^0\pi_i , \\ &[{}^0\pi_i, {}^0\pi_j] = 0 , \end{aligned}$$

$$[{}^0\pi_i, {}^0L_{BC}]=0 \ ,$$

$$[{}^0L_{AB}, {}^0L_{CD}]=[\bar{L}_{AB}, \bar{L}_{CD}]=\bar{C}^{\,EF}_{ABCD}\bar{L}_{EF} \ ,$$

where the presuperscript zero indicates the flat-limit situation. It follows that the flat limit of $\bar{E}$ is isomorphic to the Poincaré group P_4 which now becomes completely disconnected from $\bar{G}$:

$$\overline{\mathrm{SO}(3+n,1)}\,|_{\mathrm{flat}} \to P_4 \times \bar{G} \ .$$

The above symmetry breaking suggests an alternative explanation for the nondirect observability of the internal states at low energies and weak gravitational field. In the presence of gravitation the orbits of the subgroups $\bar{E}$ and $\bar{G}$ belong to the same space M_{4+n} so that an observer sitting in V_4 under the action of $\overline{\mathrm{SO}(3+n,1)}$ does not necessarily remain "confined" to V_4 provided he can use probes with sufficient energy $E(\rho)$ as, for example, given by (11) in our hyperbolic model. In the case where a very strong gravitational field is present the awareness of the extra dimensions would require probes with less energy. On the other hand, in the limit of vanishing gravitation the orbits of P_4 and $\bar{G}$ belong entirely to separate subspaces M_4 and B_n of M_{4+n} so that in theory an observer of M_4 could not "look" into B_n.[8]

Now we are in position to formally identify A_{iAB} (or A_{iA}) as the gauge potentials for the group $\bar{G}=\mathrm{SO}(n)$. The general solution of Killing's Eq. (16) is

$$\xi^A=\theta_i^A(x^j)+\theta_B^A(x^j)x^B \ , \tag{22}$$

where the parameters $\theta_\alpha^A(x^i)$ depend only on x^i [so that $\mathrm{SO}(n)$ is a local gauge group] and are such that $\theta^{(AB)}=\gamma^{\alpha(B}\theta^{A)}{}_\alpha|_{v_4}=0$. Under an infinitesimal transformation $x'^\alpha=x^\alpha+\xi^\alpha$ of $\mathrm{SO}(3+n,1)$, $A_{iAB}=\partial\gamma_{iA}/\partial x^B$ transforms as

$$A'_{iAB}=A_{iCB}\xi^C{}_{,A}-A_{iAC}\xi^C{}_{,B}-A_{kAB}\xi^k{}_{,i}-\xi^k{}_{,A}\gamma_{ik,B}$$
$$-\gamma_{kA}\xi^k{}_{,iB}-\gamma_{AC}\xi^C{}_{,iB}-\gamma_{ik}\xi^k{}_{,AB}-\xi^C{}_{,AB}\gamma_{iC} \ .$$

Using (22) and introducing the structure constants of the subgroup $\bar{G}$,

$$A_{iBC}\theta_A^C-A_{iAC}\theta_B^C=-A_{iEF}\bar{C}^{\,EF}_{ABCD}\theta^{CD} \ .$$

It follows that

$$A'_{iAB}=A_{iAB}+\bar{C}^{\,EF}_{ABCD}\theta^{CD}A_{iEF}-\theta_{AB,i}-A_{kAB}\xi^k{}_{,i}$$
$$-\gamma_{ik,B}\xi^k{}_{,A}-\gamma_{ik}\xi^k{}_{,AB}-\gamma_{kA}\xi^k{}_{,iB} \ . \tag{23}$$

The first three terms in (23) correspond to a local gauge transformation of A_{iAB} where the gauge group is $\bar{G}=\mathrm{SO}(n)$. The remaining terms appear because we have used the full combined symmetry $\mathrm{SO}(3+n,1)$ and in general $\bar{G}$ is not an invariant subgroup of the latter. To obtain a pure gauge transformation either take $\xi^k=0$ (no space-time transformations) or take the flat limit of V_4 where, as we have seen, the Lie algebra of $\bar{G}$ decouples from that of P_4. Therefore the functions A_{iAB} (or A_{iM} in the Killing basis) which appear in (8) in fact transform as gauge potentials for the gauge group $\bar{G}=\mathrm{SO}(n)$.

V. CONCLUSIONS

We have shown that if the space-time V_4 is locally and isometrically embedded in a high-dimensional space V_{4+n} which originated at the beginning of the universe, then a geometric form of Kaluza-Klein theory is derived even when a simpler case of flat-space M_{4+n} is considered. The space B_n generated by the n extra variables is naturally compact as a consequence of the second quadratic form. Various forms of compactification of B_n can be implemented with the help of the low-energy sector of Kaluza-Klein theory. The resulting metric is more general than the traditional zero-mode metric ansatz. More interesting and puzzling however is the fact that the gauge potentials are geometric entities derived from the embedding. In order to understand the physical idea behind this we need to work out the integrability conditions for the embedding, the Gauss-Codazz-Ricci and Einstein's equations, to be dealt with in a subsequent paper.

It should be mentioned that the present analysis would also apply to local gravitational fields embedded in flat spaces with lower dimensions and possibly with noncompact gauge groups. The theory does not give gauge potentials in the five-dimensional case ($A_{i55}=0$) so that they have to be postulated. In the present case $4+n=14$ was taken to be the limit of differentiable embedding so as to accommodate the very irregular gravitational field near the big bang. This choice leads to the gauge group SO(10) which makes the theory compatible with more recent grand unification schemes. On the other hand it exceeds the dimensionality limit of supergravity.

APPENDIX: THE CURVATURE RADII OF V_4

Consider an infinitesimal displacement dx^i in V_4. The variation of the coordinates $Z^\mu(x^i,x^A)$ given by (3) corresponding to that displacement is

$$\Delta Z=(X^\mu{}_{,i}+x^A N^\mu_{A,i})dx^i \quad (\text{no sum on } A) \ .$$

The points of M_{4+n} which remain fixed, $\Delta Z^\mu=0$, are the centers of curvature of V_4 with respect to the displacement dx^i:

$$(X^\mu{}_{,i}+x^A N^\mu_{A,i})dx^i=0 \ .$$

Contraction of this equation with $X^\mu{}_{,j}\eta_{\mu\nu}$ gives

$$(\bar{g}_{ij}-x^A b_{ijA})dx^i=0 \tag{A1}$$

and contraction with $N^\mu_B\eta_{\mu\nu}$ gives

$$A_{iAB}dx^i=0 \ . \tag{A2}$$

Equation (A2) only tells that in order to have $\Delta Z^\mu=0$ the variation of N_A must be orthogonal to N_A. On the other hand, Eq. (A1) determines the conditions for the existence of the displacement dx^i. Equation (A1) admits a nontrivial solution dx^i, called a curvature line of V_4 when,

$$\det(\bar{g}_{ij}-x^A b_{ijA})=0 \ , \tag{A3}$$

which is a polynomial equation in x^A. Its solutions $x^A=\rho_i^A$ are the curvature radii of V_4 corresponding to each curvature line dx^i and to each normal N_A. There are at most four independent orthogonal curvature lines through each point in V_4, corresponding to the nondegenerate solutions of (A3). These solutions are invariant under the manifold mapping group of V_4.[10] With these solutions we may construct the 10 quantities

$$\rho_{mn}{}^2 = \bar{g}_{AB}\rho_m^A\rho_n^B .$$

The diagonal terms $\rho_m = \rho_{mm}$ are the inverses of the four eigenvalues of the Weyl tensor and in a sense they are the fundamental observables of the pure gravitational field. An invariant quantity called the gravitational length (or curvature radius) may be defined by

$$\rho^2 = \bar{g}^{mn}\rho_{mn}{}^2 .$$

On the other hand, for each normal direction N_A a curvature radius may be defined by

$$\rho^{A^2} = \bar{g}^{mn}\rho_m^A\rho_n^A .$$

Therefore, when $\bar{g}_{AB} = \delta_{AB}$,

$$\rho^2 = \bar{g}_{AB}\bar{g}^{mn}\rho_m^A\rho_n^B = \bar{g}_{AB}\rho^A\rho^B .$$

The behavior of ρ near a space-time singularity depends on the topological nature of that singularity. In the case of a pointlike singularity all values of ρ_m^A tend to zero so that ρ also tends to zero. On the other hand, in the flat limit of V_4 at least one of the quantities ρ_m^A tend to ∞ so that the flat limit of V_4 implies that $\rho \to \infty$ and $\rho^A \to \infty$. Reciprocally when $\rho^A \to \infty$, then V_4 is flat.

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