Factorization of the Drell-Yan cross section in perturbation theory

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(Received 10 December 1984)

We develop, through detailed one- and two-loop examples, a procedure for expressing the leading-twist Drell-Yan cross section in terms of the factored form proposed by Collins, Soper, and Sterman. We then show that this factorization program can be implemented to all orders in perturbation theory. The factored cross section takes the form of an on-shell $q\bar{q}$ annihilation cross section (less certain collinear subtractions) convolved with structure functions for the incoming hadrons. The structure functions contain all the collinear singularities and spectator interactions. They are shown to be simply related to those that occur in deep-inelastic lepton scattering. We also demonstrate that the $q\bar{q}$ cross section minus the collinear subtractions is free of infrared singularities if one integrates over the transverse momentum of the lepton pair.

I. INTRODUCTION

The Drell-Yan model[1] for lepton-pair production in hadron-hadron collisions has as its conceptual basis a parton-model picture: a quark from one hadron annihilates against an antiquark from another hadron to produce a timelike virtual photon of momentum Q, which then decays into a lepton pair. In the very first discussion of the model, it was pointed out that, in a complete field theory, the exchange of soft partons could destroy this simple picture.[1] Subsequently, it was realized in the more specific context of quantum chromodynamics that one could make use of asymptotic freedom in order to carry out a perturbative calculation of the lepton-pair cross section only if the effects involving quanta with small invariant momentum squared could be decoupled from the perturbative calculation. Politzer[2] suggested that the collinear singularities associated with the vanishing of the active quark or antiquark mass could be absorbed into a structure-function factor. He further proposed that the collinear singularities in the Drell-Yan process and in deep-inelastic scattering have the same structure, so that one could construct singularity-free quantities by comparing the Drell-Yan cross section with the deep-inelastic cross section. Early treatments of factorization in the Drell-Yan model discussed the organization of the collinear singularities into the factored form, but did not adequately address the question of soft exchanges with both initial and final-state spectator partons.[3]

More recently, Bodwin, Brodsky, and Lepage (BBL)[4] resurrected the issue of soft exchanges with spectators by showing that in QCD there is contribution to the Drell-Yan process due to exchange of a gluon between an active quark and a spectator quark that persists even in the limit $Q^2 \to \infty$ (leading twist). Their calculation was restricted to the "semiclassical" or "Glauber" region of the gluon momentum, in which the quarks scatter from near mass shell to near mass shell.

Subsequently, Lindsay, Ross, and Sachrajda[5] carried out a calculation of spectator interactions in quark-meson scattering up to the two-loop level that included contributions from outside the Glauber region. (Their calculation omitted only the "very soft" region of momentum space. See. Sec. IV.) They found, on comparison with the deep-inelastic cross section, that the spectator contributions canceled—but only at the final step of a rather laborious calculation. This result was extended to meson-meson scattering by Lindsay[6] and both results were verified by BBL.[7]

At about the same time, Collins, Soper, and Sterman (CSS) suggested a more general approach to the problem. Their proposal consists of two parts. The first, which we call "weak factorization," is the statement that at large Q^2, $d\sigma/dQ^2Q_\perp{}^2$ is a convolution of a hard subprocess with structure functions $\mathscr{P}_{q/A}(x,\mathbf{k}_\perp)$.[8] Here x is the active quark longitudinal-momentum fraction, $\mathbf{k}_\perp$ (a two-component vector) is the momentum of the active quark transverse to the collision axis, and the subscript q/A indicates that this is the structure function for finding a quark in a hadron A. In the CSS proposal, $\mathscr{P}_{q/A}$ contains all of the collinear singularities *and* all of the spectator interactions. The second part of the CSS approach, which we call "strong factorization" is the statement that the Drell-Yan and deep-inelastic structure functions are related.[9] That is, with a suitable definition of the structure functions

$$\int_{k_\perp<Q} d^2k_\perp \mathscr{P}_{q/A}(x,\mathbf{k}_\perp)=f_{q/A}(x)\ , \tag{1.1}$$

where $f_{q/A}$ is the deep-inelastic structure function.

As was pointed out by BBL, the concept of strong factorization is somewhat counterintuitive. We know that for deep-inelastic scattering on a nucleus of mass A, the cross section is about a factor of A larger than that for protons:

$$d\sigma_{\rm DI}(A)\approx A\, d\sigma_{\rm DI}(H)\ . \tag{1.2a}$$

[Here we are neglecting corrections on the order of 20% due to the European Muon Collaboration (EMC) effect.[10]] Then for the Drell-Yan process with, say, a pion beam, strong factorization implies that

$$d\sigma_{\mathrm{DY}}(\pi - A) \approx A\, d\sigma_{\mathrm{DY}}(\pi - H)\ . \tag{1.2b}$$

Thus, it seems that quarks on the back face of a nucleus are just as likely to be annihilated as those on the front face. That is, the nucleus is "transparent" to the incoming pion beam. Of course, we know that this picture must break down for a macroscopic target, since in that case there is measurable depletion of the beam as it traverses the length of the target. This is a constraint on any proof of factorization: in the limit of a very long target it must break down so that the classical picture of depletion of the beam is restored.

In this paper we discuss the proof of factorization of leading-twist contributions to all orders in perturbation theory.[11] The remainder of the discussion is divided into three parts. In Sec. II we introduce a model for the perturbative calculation in meson-meson scattering, give the definition of the structure functions proposed by CSS, and illustrate the basic concepts with a discussion of factorization (weak and strong) at the one-loop level. We also derive the target-length condition for the validity of factorization that was written down by BBL. In Sec. III we give a demonstration of weak factorization at the two-loop level. This discussion (combined with the proof of strong factorization) allows us to recover the result of Lindsay, Ross, and Sachrajda, without the need for an explicit calculation. Finally, in Sec. IV we give the all-orders proof of weak factorization, discuss the effect of the very soft" region of Landshoff and Stirling,[12] demonstrate the cancellation of soft divergences in the Drell-Yan cross section,[13] and present an all-orders proof of strong factorization.

II. BASIC CONCEPTS AND FACTORIZATION IN THE ONE-LOOP LEVEL

We focus our discussion of factorization in the Drell-Yan process on the simplest example that contains all the essential features: meson-meson scattering. Our results can easily be generalized to the case of hadron-hadron scattering. We take as a model for the meson a scalar vertex ($M\phi^*\phi$ for scalar quarks, $M\overline{\Psi}\Psi$ for spin-$\frac{1}{2}$ quarks). The meson is assumed to be stable against decay: $2m_{\text{quark}} > m_{\text{meson}}$. In explicit discussions of power counting, propagator denominators, and numerator factors, we always have in mind scalar quarks, but all of our results are valid for spin-$\frac{1}{2}$ quarks as well. We work in the center-of-mass frame of the colliding mesons, with the z axis, the collision axis, taking meson 1 to have momentum $P_1=(P_1^+,P_1^-,\mathbf{P}_{1\perp})=(0,P,0)$ and meson 2 to have momentum $P_2=(P_2^+,P_2^-,\mathbf{P}_{2\perp})=(P,0,0)$. The diagrammatic notation for the basic Drell-Yan cross section is indicated in Fig. 1. For simplicity, we drop the meson "blobs," the photon lines, and the lepton lines in subsequent diagrams. The analysis is carried out in the Feynman gauge.

A. The CSS structure functions

The structure function proposed by CSS[8] is defined diagrammatically in Fig. 2. It has the form of the interaction of an eikonal "test charge" (double line) with the

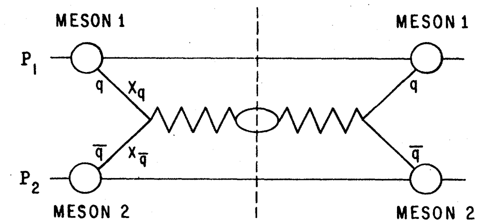


FIG. 1. The contribution of the basic Drell-Yan process to the meson + meson$\rightarrow$lepton$^+$+lepton$^-$ cross section. We use straight lines to represent leptons, wavy lines to represent gluons, and a saw-toothed line to represent the virtual photon. The incoming mesons have momenta P_1 and P_2. The active quark (antiquark) has longitudinal momentum fraction $x_q(x_{\bar{q}})$. In diagrams to follow, we drop the virtual photon- and lepton-pair lines and the meson blobs.

Yang-Mills field of the meson. The Feynman rules for the eikonal line are

$$\frac{i}{n\cdot l+i\epsilon} \tag{2.1a}$$

for an eikonal propagator carrying gluon momentum l, and

$$ign_\mu\lambda_a \tag{2.1b}$$

for an eikonal vertex. Here g is the coupling constant, λ_a is an SU(n) matrix in the adjoint representation, and n is a spacelike vector, which, for definiteness, we take to be along the collision axis:

$$n\equiv(n^+,n^-,\mathbf{n}_\perp)=(-1,1,0)\ , \tag{2.2a}$$

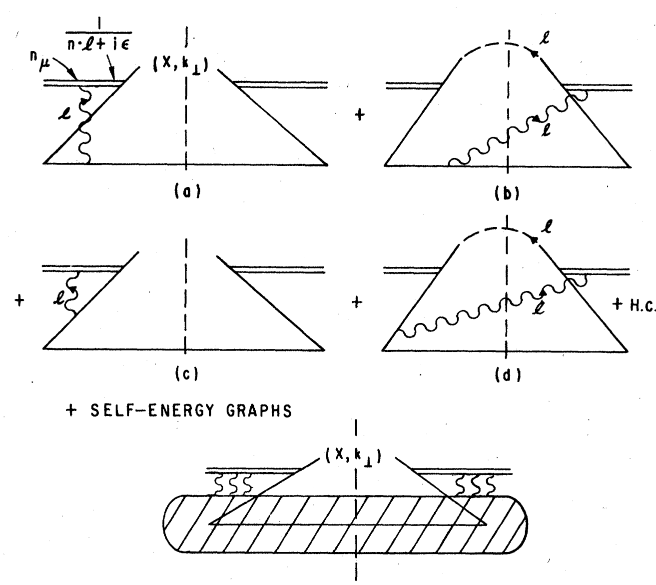


FIG. 2. Contributions to the meson structure function $\mathscr{P}(x,\mathbf{k}_\perp)$. The double line represents an eikonal, and the vertex and propagator for it are indicated in (a). (a)—(d) are $O(\alpha_s)$ contributions, while (e) is a schematic representation of a contribution of arbitrary order. The routing of real gluon momenta through the active-quark line is indicated in (b) and (d).

where

$$n^{\pm}=n^0\pm n^z\,. \tag{2.2b}$$

The incoming eikonal line carries only gluon momentum, which is conserved at the eikonal-quark vertex. There is no diagrammatic factor associated with this vertex.

The structure functions contain no interactions that are one- or two-particle reducible with respect to the eikonal lines. Such interactions would lead to a linear divergence proportional to the length of the eikonal line:[14]

$$\frac{1}{2\pi i}\int d^4l \frac{1}{(n\cdot l+i\epsilon)^2}\frac{1}{l_0{}^2-\mathbf{l}^2+i\epsilon}=-\int \frac{d^2l_\perp dl_z}{2\,|\mathbf{l}\,|\,l_z{}^2}\,. \tag{2.3}$$

They never appear in the perturbation analysis of the cross section in the Feynman gauge.

The structure functions can be written formally as an operator matrix element:[8]

$$\mathscr{P}_{q/A}(x,\mathbf{k}_\perp)=\frac{1}{2(2\pi)^2}\int dy^-d^2y_\perp \exp[-i(\tfrac{1}{2}xp_A^+y^- - \mathbf{k}_\perp\cdot\mathbf{y}_\perp)]\times\langle p_A\,|\,\overline{\Psi}_{\rm DY}(0,y^-,\mathbf{y}_\perp)\gamma^+\Psi_{\rm DY}(0)\,|\,p_A\rangle\,, \tag{2.4}$$

where

$$\Psi_{\rm DY}(x^\nu)=\text{path-ordered}\times\exp\left[-ig\int_{-\infty}^0 d\lambda\, n\cdot A(x^\nu+\lambda n^\nu)\right]\Psi(x^\nu)\,.$$

Here A is the Yang-Mills field, and the expression has been written in terms of spin-$\frac{1}{2}$ quarks. The deletion of interactions that are one- and two-particle reducible in the eikonal lines is understood. This deletion can be regarded as part of a regulator scheme for the operator matrix element (see Ref. 14). The path-ordered exponential makes the structure function manifestly gauge invariant. In terms of the diagrammatic definition, the terms $n\cdot A$ give the eikonal vertices, whereas the path-ordering gives the eikonal propagators:

$$\int e^{il_z z}\theta(z)dz=\frac{i}{l_z+i\epsilon}\,. \tag{2.5}$$

B. Regions of momentum space

It is useful to establish a terminology for certain regions of momentum space. These are as follows:

(1) Central: $l^+\sim l^-\sim l_\perp$, which contains certain special regions of particular interest:

(a) Ultraviolet: $|\,l_\mu\,|\gg P$.
(b) Hard: $l_\mu\sim P$.
(c) Soft: $m\lesssim|\,l_\mu\,|\ll P$.
(d) Very soft: $|\,l_\mu\,|\ll m$.

(2) Collinear to $+$: $|\,l^+\,|\gg|\,l^-\,|, l^+l^-\sim l_\perp{}^2$.
(3) Collinear to $-$: $|\,l^-\,|\gg|\,l^+\,|, l^+l^-\sim l_\perp{}^2$.
(4) Glauber: $|\,l^+l^-\,|\ll l_\perp{}^2$.

Here m is the hadronic mass scale that sets the size of the hadron. (In this model m is a function of $m_{\rm quark}$ and $m_{\rm meson}$.) A given propagator can, in general, carry a momentum that is a linear combination of momenta from any of these regions.

We dispense with the ultraviolet region by subtracting the appropriate counterterm graphs. Since the counterterm graphs are obtained by contracting higher-order vertex or propagator corrections to a point, the counterterm subtractions in a given order in perturbation theory always have the form of a lower-order graph. Thus, if we have proven factorization for ordinary graphs through a given order, we have proven it for the counterterm graphs. We do not discuss the counterterm graphs further, but simply keep in mind the fact that their inclusion removes the ultraviolet region.

The very soft region is the one that is responsible for the off-shell—on-shell ambiguity in the QCD corrections to quark-antiquark annihilation.[15] In order to streamline our initial discussion of the essential features of Drell-Yan factorization, we defer the treatment of this region to Sec. IV, where we find that its effect can easily be incorporated into the general analysis. We also postpone the discussion of the very soft collinear regions (those with $l_\perp{}^2\ll m^2$) until that point. Thus, in our initial analysis, one can think of the collinear regions as

$$\text{collinear to }+:\ l^+\sim P\,,\ \ l^-\sim m^2/P\,,\ \ l_\perp{}^2\sim m^2\,,$$

$$\text{collinear to }-:\ l^-\sim P\,,\ \ l^+\sim m^2/P\,,\ \ l_\perp{}^2\sim m^2\,.$$

The Glauber, collinear to $+$ or $-$, and soft regions potentially destroy factorization by driving perturbative calculations into regions of small invariant (momentum)^2 (Ref. 16). The treatment of these regions is at the heart of our demonstration of factorization. First we deal with the Glauber region of gluon momentum. This region corresponds to the kinematically allowed range of momentum transfers for the case of two opposite-moving relativistic quarks scattering from a nearly on-shell initial state to a nearly on-shell final state—that is, semiclassical scattering. We show that, by a suitable deformation of l^+ and l^- contours into their complex planes, the leading contributions from this region can be eliminated. The collinear to $+$ and $-$ contributions can then be disentangled from each other and from certain central contributions through the use of Ward identities. All the collinear contributions and spectator interactions and some central contributions are seen to reside in the meson structure functions, which depend only on the properties of the meson in question. The remaining central contributions attach only to the active quark (or antiquark) lines. For these, the soft (and very soft) contributions cancel via the Block-Nordsieck mechanism. The hard contributions are perturbatively calculable.

C. The factored form

In order to demonstrate weak factorization, we must show that the Drell-Yan cross section for meson-meson scattering takes the form

$$G=\sigma_{\text{central}}\mathscr{P}_{q/1}\mathscr{P}_{\bar{q}/2}\ , \tag{2.6a}$$

where σ_{central}, to be defined later, has connections only to the *active* quark and antiquark lines, and $\mathscr{P}_{q/1}$ and $\mathscr{P}_{\bar{q}/2}$ are the CSS meson structure functions. (The subscripts 1 and 2 indicate meson 1 and meson 2.) $\mathscr{P}_{q/1}$ and $\mathscr{P}_{\bar{q}/2}$ contain all the collinear contributions and all the spectator interactions. This weak factorization form is illustrated in Fig. 3.

The general scheme by which we organize the collinear contributions into the factored form is as follows: for each set of Feynman graphs $G^{(n)}$ of nth order in the square of the QCD coupling (g^2), we construct a set of collinear subtractions $S^{(n)}$, such that

$$G^{(n)}=S^{(n)}\equiv\sigma^{(n)}_{\text{central}}\mathscr{P}^{(0)}_{q/1}\mathscr{P}^{(0)}_{\bar{q}/2}$$

$$(\text{integrations over } x,\mathbf{k}^{\perp} \text{ implied}) \tag{2.6b}$$

has only central contributions. Equation (2.6b) is the definition of $\sigma^{(n)}_{\text{central}}$. Then we use Ward identities to show that

$$\begin{aligned} S^{(n)}&=\sigma^{(0)}_{\text{central}}\mathscr{P}^{(n)}+\cdots+\sigma^{(n-1)}_{\text{central}}\mathscr{P}^{(1)}\\ &\equiv S^{(n)}_{n}+\cdots+S^{(n)}_{1}\ , \end{aligned} \tag{2.6c}$$

where

$$\mathscr{P}^{(n)}=\mathscr{P}^{(n)}_{q/1}\mathscr{P}^{(0)}_{\bar{q}/2}+\cdots+\mathscr{P}^{(0)}_{q/1}\mathscr{P}^{(n)}_{\bar{q}/2}\ . \tag{2.6d}$$

It turns out that all the gluons that attach to the spectators have momentum collinear to the corresponding meson, so σ_{central} has connections only to the active quark and antiquark. We shall see that σ_{central} takes the form

$$\sigma^{(n)}_{\text{central}}=\sigma^{(n)}_{q\bar{q}}-S^{(n)}_{q\bar{q}}\ , \tag{2.6e}$$

where $\sigma_{q\bar{q}}$ is the nth-order contribution to the $q\bar{q}$ annihilation cross section, and $S^{(n)}_{q\bar{q}}$ is the corresponding set of collinear subtractions.

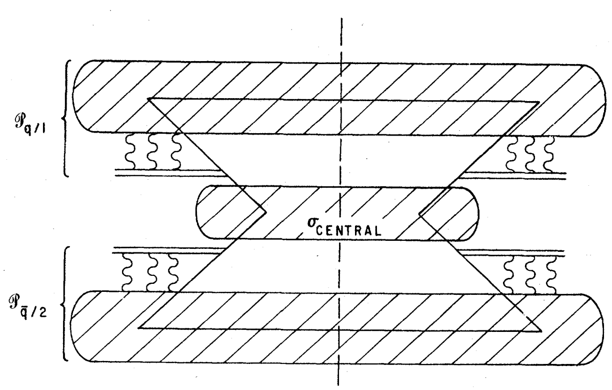


FIG. 3. Schematic representation of the factored form for a contribution to the Drell-Yan cross section of arbitrary order.

D. Ultraviolet regularization of the structure functions

The structure functions and the subtractions from which they arise of course contain ultraviolet divergences, which must be regulated. (The usual UV counterterms render the original graphs G finite. The counterterms for the subtractions S are somewhat more complicated because of the dependence of S on the vector n.) One can think of the regularization as a cutoff on the range of integration of the $\mathbf{l}_{\perp}$'s. Then, if μ is the regulator scale, the subtractions contain only those collinear contributions for which $l_{\perp}{}^2<\mu^2$, which includes the divergence at $l_{\perp}{}^2=0$. A more elegant approach is to use a covariant regulator scheme such as dimensional regularization.[17]

The regular scale μ must be chosen so that

$$m^2\ll\mu^2 \tag{2.7}$$

in order to include all contributions involving spectators in the structure functions. Condition (2.7) also ensures that the evolution of the structure function with μ is perturbatively calculable. μ can be chosen to be identical to the renormalization scale for the original graph G, but this need not be the case. The dependence of the structure functions on μ is of course compensated, in the usual renormalization-group way by the dependence of $\sigma_{\text{central}}=G/(\mathscr{P}_{q/1}\mathscr{P}_{\bar{q}/2})$, so that the cross section is independent of μ.

Suppose that G and S have both been dimensionally regulated (though not necessarily with the same regular scale) in 4-ϵ dimensions. Then the poles in ϵ corresponding to the collinear singularities (regulated by $\epsilon<0$) cancel in G-S. There remain poles corresponding to soft singularities, which ultimately cancel in the inclusive cross section through the Block-Nordsieck mechanism, and UV poles in G and S (regulated by $\epsilon>0$), which are canceled by their UV counterterms (or dropped in the case of minimal subtraction).

E. Factorization in one loop

In order to illustrate some of the basic techniques we employ in proving factorization, let us demonstrate factorization at the one-loop level.[18] As always, we are concerned only with the limit $Q^2\to\infty$ (leading twist). We begin with a subset of graphs that involves only an active quark and a meson.

(1) The active-spectator graph.

The graph involving exchange of a virtual gluon between the active quark and the spectator quark is shown in Fig. 4. The various propagator denominators labeled in Fig. 4 are given, to leading order in P by

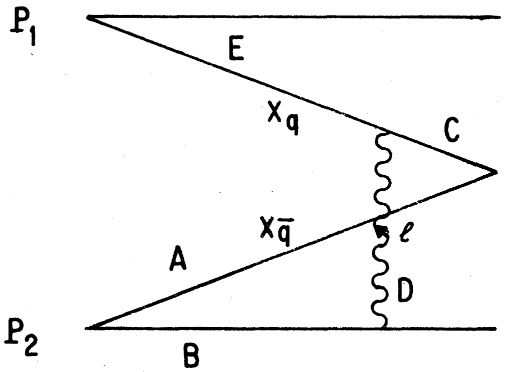


FIG. 4. The $O(\alpha_s)$ virtual active-spectator diagram.

$$
\begin{aligned}
A &\sim \frac{1}{(x_{\bar q}P-l^+)(-l^-)-\perp^2+i\epsilon}\,, \\
B &\sim \frac{1}{[(1-x_{\bar q})P+l^+](l^-)-\perp^2+i\epsilon}\,, \\
C &\sim \frac{1}{(x_qP+l^-)l^+-\perp^2+i\epsilon}\,, \\
D &= \frac{1}{l^+l^- - l_\perp{}^2+i\epsilon}\,, \\
E &\approx \perp^2\,.
\end{aligned} \tag{2.8}
$$

The symbol $\perp^2$, which has a different meaning in each line (2.8), stands for any quadratic form in transverse components of momentum plus terms proportional to (masses)^2. First we notice that the denominators A and B, corresponding in our model to the meson wave function, pinch the l^- contour at $l^-\sim\perp^2/P$. By closing the l^- contour (or more simply by power counting), we see that the result of the l^- integration is of the form

$$
\int d^2l_\perp dl^+ \frac{1}{\perp^2}\frac{1}{x_qPl^+-\perp^2+i\epsilon}\frac{1}{(l^+)(\perp^2/P)-l_\perp{}^2+i\epsilon}\,. \tag{2.9}
$$

Power counting then yields

$$
l_\perp{}^2 \sim m^2\,,\quad l^+ \lesssim P\,. \tag{2.10}
$$

In fact, it is easy to see that we can eliminate the $l^+ \ll P$ (Glauber) contribution altogether. We simply deform the l^+ contour into the upper half of the complex l plane (uhp) until $l^+\sim P$ everywhere along the contour. The only pole in the l^+ complex plane within $O(P)$ of the origin is in the denominator C, but it is in the lower half-plane (lhp), so we avoid it. Thus, we find that the entire leading contribution comes from the region l collinear to $+$.

Now,

$$
C \approx \frac{1}{x_qPl^+ + i\epsilon} \approx \frac{1}{x_qP(l^+-l^-)+i\epsilon} = \frac{1}{2x_qP(n\cdot l+i\epsilon)}\,. \tag{2.11a}
$$

Furthermore, if J_U is the current with which the upper end of the gluon is contracted and J_L the current with which the lower end is contracted $[J_U=2x_qP_1+l$, $J_L=2(1-x_{\bar q})P_2+l]$, then the leading part of the numerator factor (convection current) is given by

$$
\tfrac{1}{2}J_U^-J_L^+ \approx x_qP(J_L^+-J_L^-)=2x_qPn\cdot J_L\,. \tag{2.11b}
$$

Combining the factors (2.11a) and (2.11b) we arrive at the eikonal form

$$
\frac{n\cdot J_L}{n\cdot l+i\epsilon}\,, \tag{2.11c}
$$

which is illustrated in Fig. 2(a).

Another way to arrive at this result, which can be generalized in order to discuss more complex examples, is as follows. First, we define a collinear to $+$ subtraction, a contribution to $S_1^{(1)}$, by making the Grammer-Yennie[19] replacement

$$
g_{\mu\nu} \to \frac{l_\mu n_\nu}{l\cdot n+i\epsilon} \tag{2.12}
$$

in the original graph. Here, the index μ is on the active-quark line and the index ν is on the spectator line. We call a gluon in which a replacement like (2.2) has been made a "subtraction gluon." The subtraction is indicated diagrammatically as shown in Fig. 5(a). The arrowhead of the broken line symbolizes the factor l_μ, and the n symbolizes the factor n_ν. (In subsequent diagrams we often suppress the label n.) The replacement (2.12) gives the correct leading contribution provided that

$$
(J_U\cdot l)(n\cdot J_L) \approx (J_U\cdot J_L)(l\cdot n)\,. \tag{2.13}
$$

Since the active-quark momentum has only a large $-$ component and the spectator-quark momentum has only a large $+$ component, (2.13) is satisfied if l is collinear to the spectator momentum:

$$
l^+ \gg l^-, l_\perp\,. \tag{2.14}
$$

That is, in this case, the original graph (Fig. 2) less the subtraction (Fig. 4) vanishes to leading order as $P\to\infty$. Next we apply the quark-gluon vertex Ward identity to the subtraction. This Ward identity is written out diagrammatically in Fig. 6. On the right-hand side, the circle on the $q-g$ vertex stands for a scalar coupling $(A_\mu J^\mu \to 1)$ and the hash marks on a propagator mean that the propagator factor has been canceled. Application of the Ward identity to the subtraction leads to the two terms shown in Fig. 5(b). The first term gives a contribution that is suppressed by powers of $1/P$ relative to the second term. That is, the canceled propagator E is off-

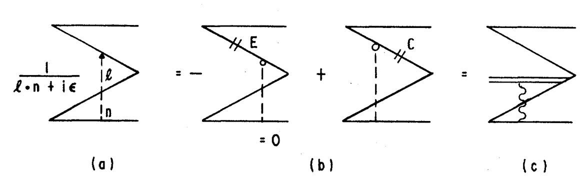


FIG. 5. Transformation of the active-spectator collinear subtraction into the eikonal form.

mass-shell by $O(\perp^2)$, whereas the canceled propagator C is off-shell by $O(P^2)$. The incoming active quark is "effectively on-shell." The second term in Fig. 6 is equal to the structure-function contribution shown in Fig. 5(c).

(2) The active-active graph

Next let us examine the graph containing a virtual gluon exchanged between the active quark and active antiquark (Fig. 7). The relevant propagator denominators are

$$A \sim \frac{1}{(x_q P + l^-)l^+ - \perp^2 + i\epsilon}\,,$$
$$B \sim \frac{1}{(x_{\bar q} P - l^+)(l^-) - \perp^2 + i\epsilon}\,, \tag{2.15}$$
$$C = \frac{1}{l^+ l^- - l_\perp{}^2 + i\epsilon}\,.$$

Now there is no "wave-function pinch" to limit the component of momentum l^- to be small. However, we can still deform contours to eliminate the Glauber contribution. We deform the l^+ contour into the uhp and the l^- contour into the lhp. If $l^+ l^- \ll l_\perp{}^2$ (i.e., if we are in the Glauber region), we can drop the terms $l^+ l^-$ in A and B, so that the pole in A is in the lh of the l^+ plane and the pole in B is in the uh of the l^- plane. The contour deformation avoids these poles. The pole in C is of course at

$$l^+ l^- \sim l_\perp{}^2\,. \tag{2.16}$$

It blocks the contour deformation, but only at a point outside the Glauber region.

Now l can be collinear to $+$ or $-$ or central. We remove the collinear to $+$ contributions by making a collinear subtraction, which is obtained from the original graph by making the replacement

$$g_{\mu\nu} \to \frac{l_\mu n_\nu}{l\cdot n + i\epsilon} \tag{2.17a}$$

in the gluon propagator. Here μ is the index corresponding to the attachment of the gluon to the upper quark line in Fig. 7. From (2.13) we see that the replacement (2.17a) gives the correct leading contribution for $l^+ \gg l^-, l_\perp$. Similarly, we construct a subtraction that accounts for the collinear to $-$ contribution by making the replacement

$$g_{\mu\nu} \to \frac{l_\nu n_\mu}{l\cdot n + i\epsilon}\,, \tag{2.17b}$$

which gives the correct leading contribution for

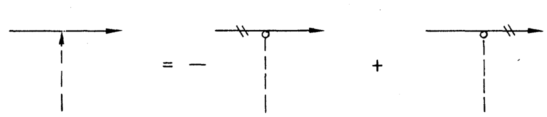


FIG. 6. The q-g Ward identity. Here, in anticipation of the introduction of subtraction gluons, we represent the gluon by a dashed line. The arrow indicates that a factor of the gluon momentum is contracted into the gluon's Lorentz index at the vertex; the open circle indicates a scalar coupling; hash marks indicate that a propagator has been canceled.

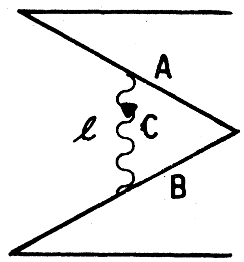


FIG. 7. The $O(\alpha_s)$ virtual active-spectator diagram.

$l^- \gg l^+, l_\perp$. At this point we can see the motivation for choosing n spacelike. In

$$\frac{1}{l\cdot n + i\epsilon} = \frac{2}{l^+ - l^- + i\epsilon} \tag{2.18}$$

the term l^- suppresses the collinear to $+$ subtraction in the region $l^- \gg l^+$ and the term l^+ suppresses the collinear to $-$ subtraction in the region $l^+ \gg l^-$. That is, the subtractions have been constructed so as to avoid double counting of the collinear contributions.

Note that the collinear subtractions also contain some of the leading central contributions. In fact, in the soft central region $(l^+ \sim l^- \sim l_\perp \sim m)$, $J_U \approx (0, J_U^-, 0)$ and $J_L \approx (J_L^+, 0, 0)$, so

$$\frac{(l_\mu n_\nu + l_\nu n_\mu)}{l\cdot n + i\epsilon} J_U^\mu J_L^\nu \approx \frac{\frac{1}{4}(l^+ J_U^- J_L^+ - l^- J_U^- J_L^+)}{l\cdot n + i\epsilon}$$
$$\approx g_{\mu\nu} J_U^\mu J_L^\nu\,. \tag{2.19}$$

That is, because the currents J_U and J_L are collinear to $+$ or $-$, the subtractions account completely for the soft central contribution. This is not the case in general. For l in the hard central region, J_U and J_L are not purely collinear; and for more complicated graphs J_U and J_L are not necessarily collinear, even if some of the gluon momenta are soft. Then the difference between the original graph and the collinear subtractions contains a central remainder. We call the sum of all such remainders $\sigma_{\rm central}$ [see Fig. 8(a)].

As a final step, we use the q-g Ward identity to show that the collinear subtractions can be put into the CSS form. The procedure is identical to that employed in the active-spectator diagram (see Fig. 9). For the collinear to $+$ subtraction, l is collinear to $+$ or central, so the term in Fig. 9(a) in which the propagator D is canceled is suppressed by powers of $1/P$ relative to the term in which

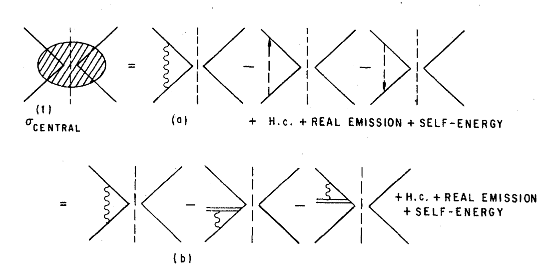


FIG. 8. $O(\alpha_s)$ contributions to $\sigma_{\rm central}$ (a) in terms of the collinear subtractions and (b) in terms of the eikonal form.

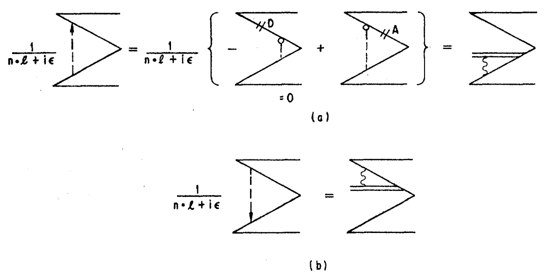


FIG. 9. Transformation of the $O(\alpha_s)$ active-active collinear subtractions into the eikonal form.

the propagator A is canceled. That is, the active quark is effectively on-shell. Then we immediately obtain a contribution to the lower meson's structure function. Similarly, for the collinear to $-$ subtraction, the active antiquark is effectively on-shell, and we obtain a contribution to the upper meson's structure function [Fig. 9(b)].

(3) The spectator-spectator graph

The graphs in which a virtual gluon is exchanged between the spectators are shown in Fig. 10. In Fig. 10(a) the propagators C and D pinch l^- at

$$l^- \sim \perp^2/P\,, \tag{2.20a}$$

and the denominators A and B pinch l^+ at

$$l^+ \sim \perp^2/P\,. \tag{2.20b}$$

That is, l^+ and l^- are trapped in the Glauber region. The gluon propagator then is given by

$$E = \frac{1}{l^+l^- - l_\perp{}^2 + i\epsilon} \approx \frac{-1}{l_\perp{}^2}\,. \tag{2.21}$$

We carry out the l^+ and l^- integrations, closing the l^+ contour in the uhp and the l^- contour in the lhp. Then the graph has a factor, relative to the basic process

$$\int d^2 l_\perp \frac{(i)^5}{l_\perp{}^2} \frac{\perp_1{}^2(0)}{\perp_1{}^2(\mathbf{l}_\perp)} \frac{\perp_2{}^2(0)}{\perp_2{}^2(-\mathbf{l}_\perp)}\,, \tag{2.22}$$

where $\perp_1{}^2(\mathbf{l}_\perp)$ is the quantity obtained by evaluating the propagator denominator B at the pole in A, and $\perp_2{}^2(\mathbf{l}_\perp)$ is the corresponding quantity obtained by evaluating the denominator C at the pole in D. Since the mesons are stable against decay, $\perp_1{}^2$ and $\perp_2{}^2$ are negative definite. Thus, we see that the contribution of Fig. 10(a) is purely imaginary and it is canceled by the contribution of Fig. 10(b).

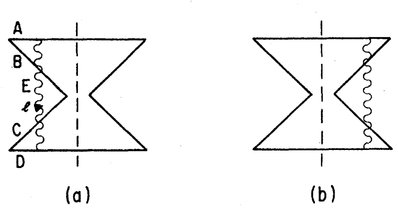


FIG. 10. The $O(\alpha_s)$ virtual spectator-spectator contributions to the Drell-Yan cross section.

In evaluating Fig. 10(a), we have picked up the residues at both spectator poles. That is, we have cut the spectator lines. In terms of time-ordered perturbation theory, this is equivalent to picking up the residue at the pole in the energy denominator X for the final-state interaction shown in Fig. 11. The cancellation between Fig. 10(a) and Fig. 10(b) is then just a manifestation of the fact that the sum over all cuts of the final-state interaction ($q\bar{q}$ scattering) is zero.[20] As we shall see, this sort of final-state unitarity cancellation can be used quite generally to eliminate contributions in which momentum contours are trapped in the Glauber region. It is, in essence, the cancellation noted by Cardy and Winbow, and DeTar, Ellis, and Landshoff (in Ref. 1) in their pre-QCD discussions of spectator interactions.

(4) Real emission graphs

So far, we have discussed only virtual gluon contributions to the one-loop Drell-Yan cross section. However, the treatment of each real emission graph follows closely that given for the corresponding virtual graph. One important difference is that the on-shell condition for the real gluon gives $l^+l^- = l_\perp{}^2$. Thus, we need not deform contours in order to eliminate the Glauber contribution. In the case of spectator interactions, the on-shell condition combined with the lower meson wave-function pinch [Eq. (2.20a)] yields $l^+ \sim P$, $l^- \sim \perp^2/P$, so we see without having to make a contour deformation that the real gluon is collinear to $+$. The real gluon graphs and their contributions to the structure functions are shown in Fig. 12.

(5) The factorized form in one loop

At this stage we have succeeded in putting the one-loop contributions into the factorized form (2.6a). That is, we have shown that

$$G^{(1)} - S^{(1)} = \sigma^{(1)}_{\text{central}} \mathcal{P}^{(0)}_{q/1} \mathcal{P}^{(0)}_{\bar{q}/2} \tag{2.23a}$$

and

$$S^{(1)} = \sigma^{(0)}_{\text{central}} (\mathcal{P}^{(1)}_{q/1} \mathcal{P}^{(0)}_{\bar{q}/2} + \mathcal{P}^{(0)}_{q/1} \mathcal{P}^{(1)}_{\bar{q}/2})\,. \tag{2.23b}$$

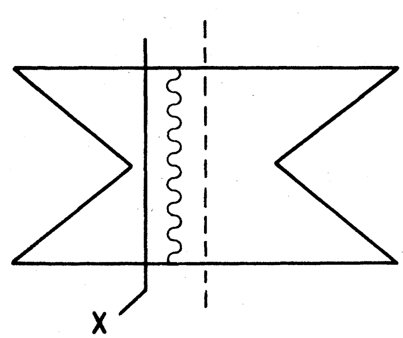


FIG. 11. The final-state interaction associated with the diagram of Fig. 10(a). X indicates the final-state energy denominator.

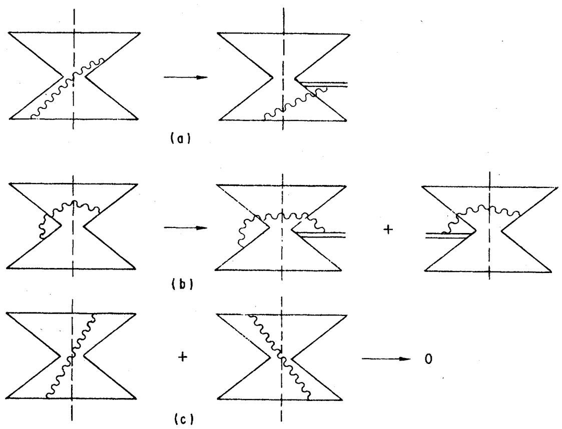


FIG. 12. $O(\alpha_s)$ real emission diagrams and their corresponding contributions to the meson structure functions.

$\sigma^{(0)}_{\text{central}}$ is, of course, just the bare $q\bar{q}$ cross section and $\mathscr{P}^{(0)}_{q/\text{meson}}$ is the bare-meson structure function (square of the meson wave function). The virtual gluon contributions to

$$\sigma^{(1)}_{\text{central}}=\sigma^{(1)}_{q\bar{q}}-S^{(1)}_{q\bar{q}} \tag{2.23c}$$

are shown in Fig. 8, and the virtual gluon contributions to $\mathscr{P}^{(1)}_{q/\text{meson}}$ (excluding self-energy graphs) are shown in Figs. 5(c), 9(a), and 9(b). Note that the subtractions $S^{(1)}_{q\bar{q}}$ can be written in the eikonal form [Fig. 8(b)].

F. Breakdown of factorization

As was mentioned in Sec. I, we expect factorization in Drell-Yan models to break down for a sufficiently long target. In order to see this breakdown at the one-loop level, let us reexamine the discussion for the active-spectator graph (Sec. II E). There, in order to arrive at the eikonal form, it was necessary to deform contours so that the quantity $l\cdot n$ lay in the uhp and was everywhere $O(P)$. This deformation relied on the fact that, in our model, the meson wave function is insensitive to the quantity $l\cdot n$. For a target of length L (in the target rest frame), we expect the target wave function to be sensitive to $l\cdot n$ for

$$l\cdot n\sim[L(M/P)]^{-1}\ , \tag{2.24}$$

where M is the target mass. $L(M/P)$ is the Lorentz-contracted length of the target in the center-of-mass frame. For L large enough, then, l is trapped in the Glauber region, and we observe semiclassical scattering, as expected. In order to obtain the eikonal (factored) form, we must have

$$(x_qP)(l\cdot n)\gg l_\perp{}^2\ . \tag{2.25}$$

Substituting (2.24), we find

$$x_qP^2\gg LMl_\perp{}^2\ ,$$

or (2.26)

$$Q^2\gg x_{\bar{q}}LMl_\perp{}^2\ ,$$

which is the BBL target-length condition.[4]

G. Identity of the Drell-Yan and deep-inelastic structure functions

Finally, let us demonstrate that the Drell-Yan and deep-inelastic structure functions are identical at the one-loop level. Contributions to the deep-inelastic structure function (Fig. 13), differ from the corresponding contributions to the Drell-Yan structure function only in that the eikonal factor

$$\frac{1}{l^+-l^-+i\epsilon}\rightarrow\frac{-1}{-l^+-l^-+i\epsilon}\ . \tag{2.27}$$

The relative signs in (2.27) reflect the fact that one must replace an incoming quark with an outgoing antiquark in going from the Drell-Yan case to the deep-inelastic case.

For the active-spectator graphs, l is collinear to $+$ $(l^+\gg l^-)$. Thus, the leading contribution to the difference between the Drell-Yan and deep-inelastic structure functions is proportional to

$$\frac{1}{l^+}-\left[\frac{-1}{-l^+}\right]=0\ . \tag{2.28}$$

The analysis of the active-active contributions is slightly more involved. In the region l collinear to $+$, the difference between structure functions is nonleading, as in the preceding example. The region l collinear to $-$ is power suppressed in each graph. In the region l central there is, for each virtual graph, only one pole in the l^- complex plane—namely, the gluon pole. We carry out the l^- integration for the region l central by picking up the residue at that pole. It is precisely canceled by the contribution of the corresponding real emission graph, provided that we can route the momentum l from one eikonal-antiquark vertex to the other. [This routing is shown for the deep-inelastic case in Figs. 13(b) and 13(d) and for the Drell-Yan case in Figs. 2(b) and 2(d).] Now, $l^+\sim l^-\sim l_\perp$, so the momentum l has no significant effect on the longitudinal components of the antiquark's momentum (i.e., l affects the virtual gluon mass by relative order $1/P$) if

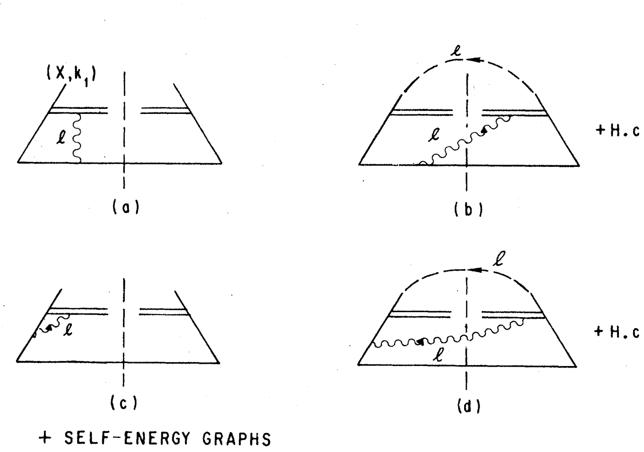


FIG. 13. $O(\alpha_s)$ contributions to the deep-inelastic meson structure functions.

$$Q^2 \gg l_\perp^2 ,$$

or equivalently (2.29)

$$Q^2 \gg \mu^2 .$$

$\mathbf{l}_\perp$ affects the antiquark's transverse momentum $\mathbf{k}_\perp$, but this is not observable, provided that we consider the integrated structure function

$$\int_{|\mathbf{k}_\perp|<Q} \mathscr{P}(x,\mathbf{k}_\perp) d^2 k_\perp , \tag{2.30a}$$

and

$$Q^2 > l_\perp^2$$

or (2.30b)

$$Q^2 \gg \mu^2 .$$

[Note that the definition of $f_{q/A}(x)$ contains a similar integration over $\mathbf{k}_\perp$.] Therefore, we have demonstrated that, in one-loop, the Drell-Yan and deep-inelastic structure functions are related by (1.1).

III. FACTORIZATION IN TWO LOOPS

In this section we demonstrate that the two-loop contributions to the Drell-Yan cross section can be put into the factorized form (2.6). The procedure we outline here is somewhat more elaborate than is necessary for this example, but our aim is to give a discussion that generalizes directly to higher orders. The basic strategy is the same as in the one-loop case. First we make use of the analyticity properties of the amplitudes, power counting, and contour-deformation arguments to eliminate the Glauber contributions and to show that gluons that attach to spectators have momentum collinear to the corresponding meson. Then we define a set of collinear subtractions $S^{(2)}$ that removes the collinear contributions from the original graphs $G^{(2)}$. Finally, we use Ward identities to put the subtraction into the factored form.

Our analysis begins with the graphs that appear in q-meson scattering, which we discuss in Secs. III A, III B, and III C. In Sec. III D we discuss the additional graphs that appear in meson-meson scattering.

A. The double active-spectator graphs

The easiest examples to analyze are the graphs involving a double gluon exchange between an active quark and a spectator quark shown in Fig. 14. The analysis of the graph of Fig. 14(c) follows precisely (with $l \to l_1 + l_2$) the analysis of the one-loop active-spectator graph given in Sec. II. Let us concentrate, then, on the analysis of the graphs of Figs. 14(a) and 14(b). We will show that the leading contribution as $P^- \to \infty$ comes from the region where both gluons are collinear with the lower meson, i.e., $l_i^- \sim \perp^2/P$, $l_i^+ \sim P$. First note that the lower wave-function pinch puts a restriction on the sum of the minus components of the gluon's momenta:

$$l_1^- + l_2^- \sim \perp^2/P . \tag{3.1}$$

Now let us show that l_1^- and l_2^- are individually small. Suppose one of l_1^-, l_2^- is large ($\sim \perp$). Then, from (3.1) we

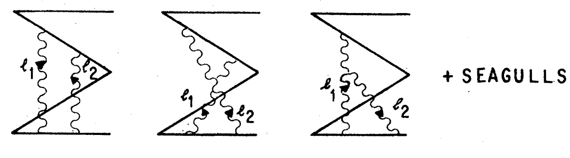


FIG. 14. $O(\alpha_s^2)$ double virtual active-spectator contributions to the Drell-Yan cross section.

see that one of them, l_i^-, is large and positive. This means that the pole in the l_i gluon propagator is in the l_i^+ lhp, so we can deform the l_i^+ contour into the uhp out to $l_i^+ \sim P$ to obtain a contribution that vanishes as $1/P$. (The poles in the active-quark propagators within P of the origin are in the lhp, as in the one-loop example; there are no poles within P of the origin in the spectator propagators or active antiquark propagator.) We conclude that l_1^- and l_2^- must both be small if we are to obtain a leading contribution. Then the l_1^+ and l_2^+ contours can be deformed into the uhp out to $l_i^+ \sim P$, so that the l_i are collinear to $+$.

The collinear subtraction that we use to remove these collinear to $+$ contributions is obtained by making the substitution

$$g^{\mu\nu} \to \frac{l_1^\mu n^\nu}{l_1 \cdot n + i\epsilon} \quad \text{(gluon 1)} , \tag{3.2a}$$

$$g^{\mu\nu} \to \frac{(l_1+l_2)^\mu n^\nu}{(l_1+l_2)\cdot n + i\epsilon} \quad \text{(gluon 2)} \tag{3.2b}$$

for the gluons' polarization sums [see Fig. 15(a)], where μ is the index corresponding to the attachment of the gluon to the upper quark line in Fig. 15.

Implicit in the substitution (3.2) is a choice of ordering of the two collinear gluons. With the momentum routing that we have chosen, it is convenient to order the gluons from the outside in along the quark line. By construction the numerators that result from these substitutions cancel the active-quark denominators and give the eikonal form illustrated in Fig. 15(b), which contains a contribution to $\mathscr{P}^{(2)}_{q/1}$. More generally, this procedure can be understood as an application of the q-g Ward identity (Fig. 6). In particular, for the first diagram of Fig. 15(a) one can apply the Ward identity first to gluon 1 to obtain the result shown in Fig. 16(b). Then one can apply the Ward identity to the combination of gluons 1 and 2 in Fig. 16(b), treating them as one gluon with momentum $l_1 + l_2$. The numerator structure of (3.2b) is, of course, exactly the

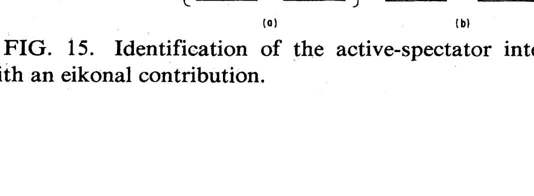


FIG. 15. Identification of the active-spectator interactions with an eikonal contribution.

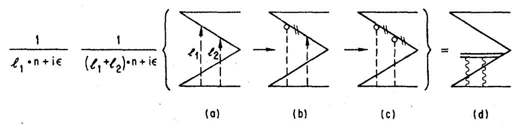


FIG. 16. The use of the q-g Ward identity to transform the double active-spectator collinear subtractions into the eikonal form.

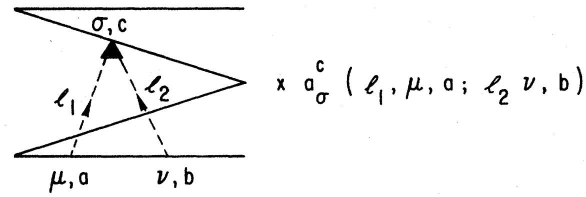


FIG. 17. A two-gluon auxiliary subtraction. l_1 and l_2 are the gluon momenta, μ, γ, σ are Lorentz indices, and a, b, c are color indices. $A_\sigma^c(l_1,a;l_2,\nu,b)$ is defined in the text [Eq. (3.4)].

form required for this application of the Ward identity. The result, Fig. 16(c), is equal to the eikonal form, Fig. 16(d).

The concept of "ordering" of gluons is rather awkward to implement in higher orders. Therefore, let us describe an alternative to the procedure given in the preceding paragraph, which can be generalized more easily. Rather than introduce an asymmetry between the gluons by making the substitution (3.2), we construct the main collinear subtraction, denoted by M, by making the substitution

$$g_{\mu\nu} \rightarrow \frac{l_{i\mu} n_\nu}{l_i \cdot n + i\epsilon} \tag{3.3}$$

for both gluons. In order to apply the Ward identities to obtain the eikonal result, we also need to make an auxiliary subtraction, denoted by A, which is shown in Fig. 17. The auxiliary subtraction is proportional to

$$a_\sigma^c(l_1,\mu,a;l_2,\nu,b) = -\frac{l_{2\sigma} n_\mu n_\nu [\lambda_a,\lambda_b]_c}{(l_1\cdot n+i\epsilon)(l_2\cdot n+i\epsilon)} + \frac{(l_1+l_2)_\sigma n_\mu n_\nu [\lambda_a,\lambda_b]_c}{(l_1\cdot n+i\epsilon)[(l_1+l_2)\cdot n+i\epsilon]}$$

$$\equiv a'(l_1,\mu,a;l_2,\nu,b) + a''(l_1,\mu,a;l_2,\nu,b)\ , \tag{3.4a}$$

where the λ_i are the generators for the color group in the fundamental representation, and the Lorentz indices μ, ν, σ, and the color indices a, b, c are specified in Fig. 17. It might appear from (3.4a) that the auxiliary subtraction destroys the symmetry between gluons 1 and 2. However, by combining the terms a' and a'', we obtain the form

$$a_\sigma^c = \frac{(l_{1\sigma} l_2\cdot n - l_{2\sigma} l_1\cdot n)[\lambda_a,\lambda_b]_c n_\mu n_\nu}{(l_1\cdot n+i\epsilon)(l_2\cdot n+i\epsilon)[(l_1+l_2)\cdot n+i\epsilon]}\ , \tag{3.4b}$$

which is symmetric under interchange of gluons 1 and 2. From (3.4b) we see that the auxiliary subtraction vanishes in the region l_1 and l_2 both collinear to $+$ (and of course for l_1 and l_2 collinear to $-$) so that there is no overcounting of the collinear to $+$ contribution: the main subtraction accounts for all of it. In (3.4a), the term a' has the form of a commutator between subtraction gluons of the type (3.3), as shown in Fig. 18(a). The term a'', has the form of a commutator between a subtraction gluon with momentum l_1 and a subtraction gluon with momentum l_1+l_2. It can be written in terms of the trigluon vertex, as shown in Fig. 18(b).

Now we can apply the Ward identities to obtain the eikonal result. First we apply the q-g Ward identity to gluon 1 in the main subtractions [Fig. 15(a), but with denominators $(l_1\cdot n+i\epsilon)$ and $(l_2\cdot n+i\epsilon)$]. Of the three leading-twist terms that appear, two give just the negative of the commutator term a' in the auxiliary subtraction. Thus, if A' is the part of the auxiliary subtraction corresponding to a', then $M+A'$ gives the result shown in Fig. 19(a). Application of the Ward identity to gluon 2 leads to the result shown in Fig. 19(b). We can also apply the q-g Ward identity to the part A'' of the auxiliary subtraction that corresponds to the term a''. The result is shown in Fig. 19(c). Now, the contributions of Figs. 19(b) and 19(c) combine as

$$\frac{\lambda_b\lambda_a}{(l_1\cdot n+i\epsilon)(l_2\cdot n+i\epsilon)} + \frac{[\lambda_a,\lambda_b]}{(l_1\cdot n+i\epsilon)[(l_1+l_2)\cdot n+i\epsilon]} = \frac{\lambda_a\lambda_b}{(l_1\cdot n+i\epsilon)[(l_1+l_2)\cdot n+i\epsilon]} + \frac{\lambda_b\lambda_a}{(l_2\cdot n+i\epsilon)[(l_1+l_2)\cdot n+i\epsilon]}\ . \tag{3.5}$$

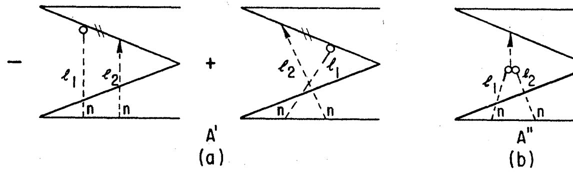


FIG. 18. Decomposition of the auxiliary subtraction into (a) a commutator term, which we call A' and (b) a term involving the sum of the gluons' momenta, which we call A''.

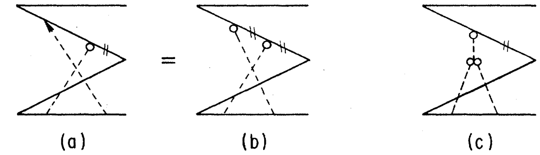


FIG. 19. Application of the q-g Ward identity to one of the active-spectator main subtractions and the corresponding auxiliary subtraction.

This last expression is in the eikonal form [Fig. 15(b)].

B. The active-active—active-spectator graphs

The two-loop graphs involving one virtual active-active exchange and one virtual active-spectator exchange are shown in Fig. 20. Their analysis is somewhat more complicated than the one for the double active-spectator case, since it turns out that the active-active gluon momentum l_1 can be collinear to $+$, $-$, or central.

Again it is easy to see that the lower meson wave-function pinch gives $l_2^- \sim \perp^2/P$. We wish to show that the l_2^+ contour can be deformed into the upper half-plane so that $|l_2^+|$ is of order P everywhere on the contour. In order to do this, we must address the possibility of encountering poles in the complex l_2^+ plane due to the l_2^+ dependence of the various propagators. Since $l_2^- \sim \perp^2/P$, the l_2-gluon propagator has a pole only at $l_2^+ \sim P$. The propagators along the active-quark line present no problem since these are of the form

$$\frac{1}{(P^- + l_1^-)(l_1^+ + l_2^+) - \perp^2 + i\epsilon}, \tag{3.6}$$

so that the poles are in the lower half of the complex l_2^+ plane. [In the case $P^- + l_1^- < 0$ we can deform the l_1^+ contour into the lower half-plane—avoiding the pole in the l_1-gluon propagator

$$\frac{1}{l_1^+ l_1^- - l_{1\perp}{}^2 + i\epsilon}, \tag{3.7}$$

so that $|l_1^+|$ is $O(P)$ along the entire contour. Then, the amplitude is suppressed by powers of P.] The propagators along the active-antiquark line are of the form

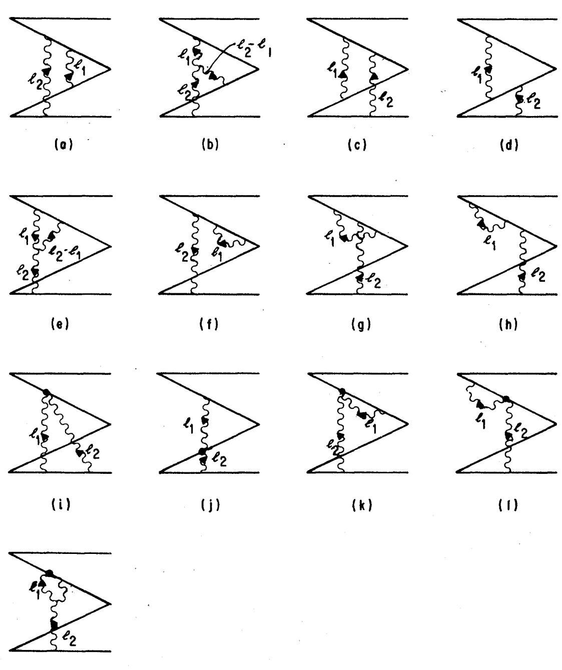


FIG. 20. The $O(\alpha^2)$ virtual active-active—active-spectator contributions to the Drell-Yan cross section.

$$\frac{1}{(x_q P^+ - l_1^+ - l_2^+)(-l_1^-) - \perp^2 + i\epsilon}, \tag{3.8}$$

so that the l_2^+ contour can be defomed out to $|l_2^+| \sim P$ before the l_2^+ dependence becomes important $[(x_q P^+ - l^+) \sim \perp$ is too small a region to matter]. The propagator

$$\frac{1}{(l_1 - l_2)^2 + i\epsilon} = \frac{1}{(l_1^+ - l_2^+) l_1^- - (\mathbf{l}_{1\perp} - \mathbf{l}_{2\perp})^2 + i\epsilon} \tag{3.9}$$

in the trigluon graphs has a pole in the upper half-plane only for $l_1^- > 0$. Furthermore, this pole presents an obstacle to the l_2^+ contour deformation only if $|l_1^-|$ is $\gtrsim |\perp|$. In that case we can deform the l_1^+ contour into the upper half-plane—avoiding the pole in the l_1-gluon propagator and the active-quark propagator—until $|l^+|$ is $O(P)$ along the entire contour. The result is then suppressed by powers of P (since $|l_1^-| \gtrsim |\perp|$). In summary, we are able to carry out the deformation of the l_2^+ contour without picking up any pole contributions. Thus, l_2 is collinear to $+$.

Suppose we attempt a similar deformation of the l_1^+ contour into the uhp. Then we encounter poles in the gluon propagators carrying the momentum l_1 at $l_1^+ = \perp^2 / l_1^-$. Thus, we conclude that we can deform the l_1^+ (or l_1^- contours) such that $l_1^+ l_1^- \sim \perp^2$ everywhere on the l_1^+ and l_1^- contours. We have eliminated the Glauber region, so l_1 can be collinear to $\pm$ or central. This deformation of contours out of the Glauber region is essential, since the collinear subtractions we are about to describe would not be a good approximation to the original graph in the collinear region if a subtraction gluon were to attach to a gluon with momentum in the Glauber region.

Our goal now is to account for the various collinear contributions by constructing the appropriate subtractions. We begin with the subtractions for the regions of momentum space in which l_1 and l_2 are collinear (l_2 collinear to $+$ and l_1 collinear to $\pm$). This set of subtractions is denoted by $S_2^{(2)}$, where the subscript indicates the number of independent gluon momenta that are collinear to $+$ or $-$.

Consider first the contribution to $S_2^{(2)}$ for the case in which both l_1 and l_2 are collinear to $+$. The main subtractions (contributing to $M_2^{(2)}$) are shown in Fig. 21 and the corresponding auxiliary subtractions (contributing to $A_2^{(2)}$) in Fig. 22. In constructing these subtractions, we have observed two general principles. The first is that there are no subtraction gluons internal to a subgraph that is potentially entirely collinear to $+$ (or entirely collinear to $-$). That is, the subtraction gluons always run from the boundary of a potentially collinear subgraph to its exterior. Thus, there is no subtraction of the type shown in Fig. 23(a). Note that the substitution (2.12) is a bad approximation even for l in the collinear region if both ends of the gluon in question attach to lines collinear to the same direction. The second important principle is that all the subtraction gluons have their n_ν ends attached to a line that is purely collinear to $+$. Thus, there is no subtraction of the type shown in Fig. 23(b). It is easy to see that for l_1 and l_2 both collinear to $+$, all three denominators on the active-quark line are $\sim 1/P^2$, and the contri-

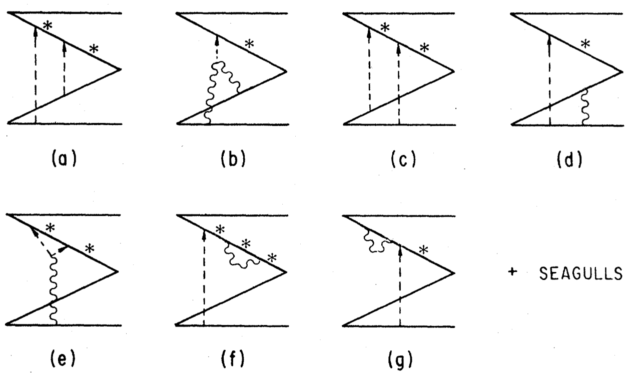


FIG. 21. The main subtractions for the active-active—active-spectator diagrams for the case of both gluons collinear to $+$. The asterisks indicate potentially large propagator denominators, as explained in the text.

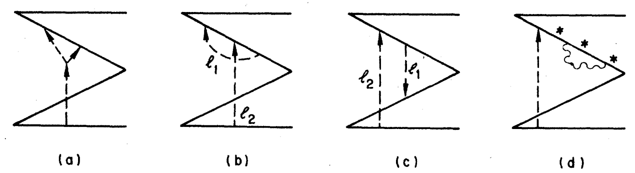


FIG. 23. Examples of diagrams that are not allowed according to the rules for constructing collinear subtractions. (a) contains a subtraction gluon on the interior of a potentially collinear subgraph. In (b) and (c) a subtraction gluon is not connected to a potentially collinear subgraph. (d) is power suppressed.

bution is power suppressed. In general, if such a power suppression is to be avoided, the region of the graph that is collinear to $+$ (or $-$) must be *connected.*

Now it is a straightforward matter to apply the q-g Ward identity to the main and auxiliary subtractions in Figs. 21 and 22. The analysis is identical to that for the double active-spectator case. The resulting contribution to $\mathscr{P}^{(2)}_{q/1}$ is shown in Fig. 24.

Next let us take up the case l_2 collinear to $+$, l_1 collinear to $-$. The subtractions for this case (also contributions to $S_2^{(2)}$) are shown in Fig. 25. Again, in constructing the subtractions we have observed the principles that there are no subtraction gluons on the interior of a potentially collinear subgraph, and each collinear subgraph is connected. We can refine this notion of a connected collinear subgraph further. To say that a collinear to $+$ (or $-$) subgraph is connected means that there is a path from a point in the subgraph [say, the meson moving in the corresponding collinear to $+$ (or $-$) direction] to every other point in the subgraph, along which every propagator's momentum is collinear to $+$ (or $-$). For example, there is no subtraction of the type shown in Fig. 23(c), because the momentum l_2 interrupts the flow of collinear to $-$ momentum from the upper meson to the collinear to $-$ subtraction gluon with momentum l_1. It is easy to see that the two active-quark propagators and the active-antiquark propagator are all $\sim 1/P^2$, so such a contribution is power suppressed. Similarly, in the diagram of Fig. 20(f), the self-energy correction on the active-quark line is completely disconnected from either the collinear to $+$ or $-$ subgraphs, so there is no subtraction of the type shown in Fig. 23(d).

In order to reorganize the subtractions of Fig. 25, we need an additional Ward identity—the one for the triple-gluon vertex—which is shown in Fig. 26. This identity follows directly from the Feynman rules for the vertex. Its right-hand side consists of two parts. The two terms (a) and (b) we call the convection parts. They arise from the convection current of the ordinary gluon as seen by the gluon with the arrowhead. The terms (c) and (d) are called the Λ-line parts.[21] (Λ-line refers to the broken line in the diagrams.) These contributions arise from the Λ-line—ordinary gluon convection current, as seen by the other half of the ordinary gluon. The Feynman rules for a Λ-line-gluon vertex are the same as those for a ghost-gluon vertex. Note that the Λ-line parts vanish for gluons with physical polarization, since in that case the Λ-line arrowhead symbolizes the contraction of the gluon's momentum into its polarization.

In applying the Ward identities to the subtractions, we find that there arise terms in which certain propagators are canceled. If use of a given identity yields several terms, the leading terms are those in which the denominators with the greatest power of P are canceled. These large denominators correspond to propagators that have a mixed momentum (collinear to $+$ and collinear to $-$ or collinear to $\pm$ and central). They are marked with an asterisk in Fig. 25. In applying the Ward identities, we retain only the leading terms. This means that the propagators marked with an asterisk are effectively off-shell, whereas the other propagators are effectively on-shell.

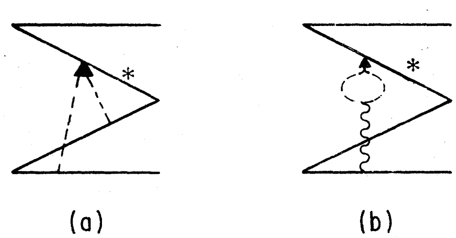


FIG. 22. Auxiliary subtractions corresponding to the main subtractions of Fig. 21.

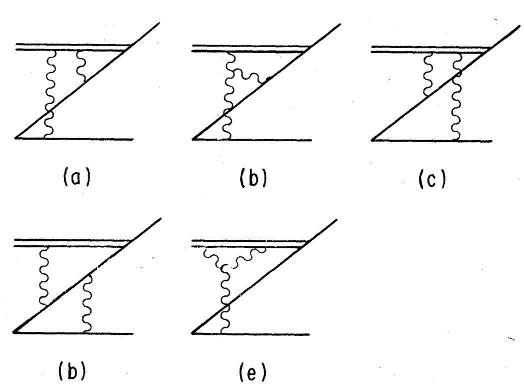


FIG. 24. Contribution of the subtractions of Figs. 21 and 22 to the meson structure function.

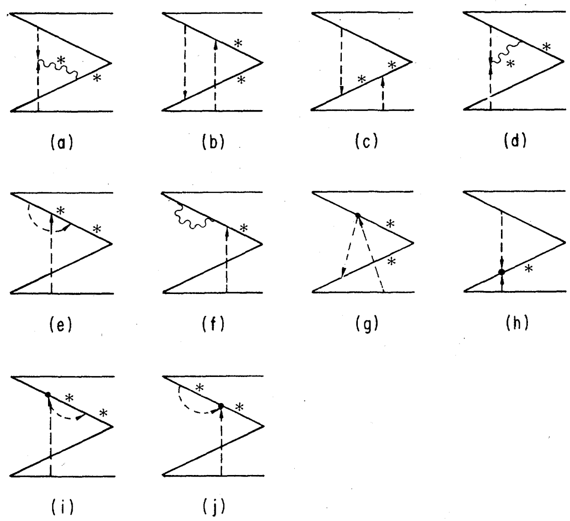


FIG. 25. The subtractions for the active-spectator—active-active diagrams for the case of one gluon collinear to + and one gluon collinear to −.

Also, the contributions in which the end of a Λ-line emerges from the vertex along an effectively on-shell line are nonleading, so we drop them.

Now the application of the Ward identities is easily worked out. First we apply the identities using the arrowhead of the subtraction gluon with momentum l_2. Then, the contributions of Figs. 25(b), 25(c), 25(g), 25(h), and the convection part of Fig. 25(a) sum to give the contribution shown in Fig. 27(c). Figure 25(f) yields the contribution in Fig. 27(b). The contributions of Figs. 25(e), 25(i), 25(j), and the convection part of Fig. 25(d) sum to give the contribution shown in Fig. 27(a).

There are also Λ-line contributions that arise from Figs. 25(a) and 25(d). It turns out that these cancel among themselves. In general, if a Λ-line arrowhead connects to an effectively on-shell region of a graph, the contribution is nonleading. Thus, in Figs. 25(a) and 25(d), the terms in which the gluon with momentum l_1 becomes a Λ-line are nonleading. The terms in which the gluon with momentum $l_2 - l_1$ becomes a Λ-line cancel between the graphs of Figs. 25(a) and 25(d) upon application of the q-g Ward identity.

The next step is to manipulate the results (Fig. 27) from the preceding applications of Ward identities. The contribution of Fig. 27(a) is nonleading since the canceled denominator is small compared to the one marked with an asterisk. In general, if application of Ward identities to one subtraction gluon (in this case gluon 2) results in a term in which a second subtraction gluon (in this case gluon 1) is on the interior of the second gluon's collinear subgraph, then the term is nonleading. Figure 27(b) is already in the form of a contribution to $\mathscr{P}^{(1)}_{q/1}\mathscr{P}^{(2)}_{\bar{q}/2}\sigma^{(0)}_{\text{central}}$, as shown in Fig. 28(a). In order to put the contribution of Fig. 27(c) into a factored form, we apply the q-g Ward identity, using the arrowhead from the subtraction gluon of momentum l_1. The incoming antiquark is effectively on-shell, so we cancel only the denominator marked with an asterisk. Then, we obtain the contribution to $\mathscr{P}^{(1)}_{q/1}\mathscr{P}^{(1)}_{\bar{q}/2}\sigma^{(0)}_{\text{central}}$ shown in Fig. 28(b).

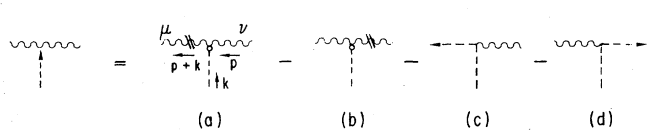


FIG. 26. The trigluon Ward identity. The dashed line is called a Λ-line. Its Feynman rules are the same as for a ghost line except that there is no factor 1 for closed loops.

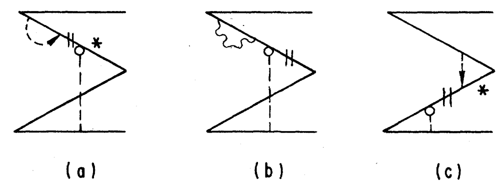


FIG. 27. Results of application of Ward identities to the subtractions of Fig. 25.

At this stage, we have accounted for all the contributions in which l_1 and l_2 are both collinear. That is, $G^{(2)} - S_2^{(2)}$ is free of such contributions. There remain collinear contributions of the type l_1 central, l_2 collinear to +. In order to account for these, we construct collinear subtractions $S_1^{(2)}$ on the set of graphs $G^{(2)} - S_2^{(2)}$. In contrast with the subtractions $S_2^{(2)}$, the collinear to + region of each graph now contains only meson 2 and the gluon with momentum l_2. Thus, the subtractions $S_1^{(2)}$ are obtained from G (Fig. 20) $-S_2$ (Figs. 21 and 25) by making the replacement (3.3) in the propagator of the gluon with momentum l_2. We call this gluon the "active subtraction gluon" for $S_1^{(2)}$. Of course, $S_1^{(2)}$ contains other subtraction gluons that arise from $S_2^{(2)}$, which we call "inactive subtraction gluons." The Ward identities are applied to the active subtraction gluon in order to put $S_1^{(2)}$ into the factored form. Again, only those propagators that carry a "mixed" (collinear plus central) momentum are effectively off-shell. The leading contributions come only from those terms in which an effectively off-shell denominator is canceled or a Λ-line emerges from a vertex along an effectively off-shell line. In applying the Ward identities to graphs with inactive as well as active subtraction gluons, it is useful to recall that two Λ-lines end to end can be written as a single Λ-line as shown in Fig. 29. The contribution to $S_1^{(2)}$ that comes from G yields the re-

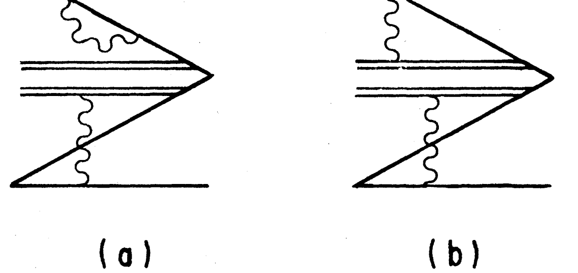


FIG. 28. Contributions of the subtractions of Fig. 25 to the factored meson structure functions.

FIG. 29. A useful diagrammatic identity for Λ-lines.

sult shown in Fig. 30(a) and the contribution that comes from $-S_2^{(2)}$ is shown in Fig. 30(b). These give the contributions to $\mathscr{P}_{q/1}^{(0)}\mathscr{P}_{\bar{q}/2}^{(1)}\sigma_{\text{central}}^{(1)}$ shown in Fig. 31 (with $\sigma_{\text{central}}^{(1)}$ given in Fig. 8).

The quantity $G^{(2)}-S_2^{(2)}-S_1^{(2)}$, by construction, contains only central contributions. Since, in the present case, we must always have l_2 collinear to $+$ in order to obtain a leading contribution, $G^{(2)}-S_2^{(2)}-S_1^{(2)}$ is power suppressed.

C. Double active-active graphs

We do not give a detailed analysis of the graphs containing two gluons that connect only to the active-quark or antiquark lines (or to each other), since it follows closely the analysis given in the preceding subsection for the active-spectator—active-active case. The essential new ingredient is that in the double active-active case, both l_1 and l_2 can be collinear to $+$ or $-$ or central. Thus, the remainder $G-S_2^{(2)}-S_1^{(2)}$ gives a leading-twist contribution to $\sigma_{\text{central}}^{(2)}$.

The fact that l_2 can be collinear to $-$ leads to a slight technical difficulty. In certain subtractions, like the one shown in Fig. 32(a), which are designed to account for contributions in which both l_1 and l_2 are collinear to $+$, there is also a leading contribution from the region l_1 and l_2 collinear to $-$. These collinear to $-$ contributions could wreck the collinear subtraction procedure since they come from a region of momentum space that is supposed to be accounted for by another subtraction. That is, they could lead to overcounting of the collinear contributions. One way to eliminate such unwanted collinear to $-$ contributions would be to introduce a cutoff into the definition of the subtraction. For example, one could introduce a factor $l_i^+/(l_i^+-l_i^-+i\epsilon)$ for each subtraction gluon; or replace the numerator factor n_ν with $\tilde{n}_\nu$, where $\tilde{n}=(0,1,0)$; or simply restrict the range of integration. All of these procedures lead to structure functions that differ slightly from the CSS form. (They give the same contribution when all the gluons are collinear to the meson, but the central contributions are somewhat different.) In general, such structure functions are more complicated than the standard one, and they are not gauge invariant.

Fortunately, there is a more elegant solution to this problem.[22] For l_1 and l_2 collinear to $+$, that is, in the region for which the subtraction is supposed to account,

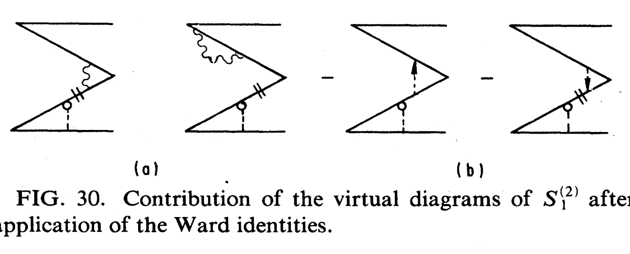


FIG. 30. Contribution of the virtual diagrams of $S_1^{(2)}$ after application of the Ward identities.

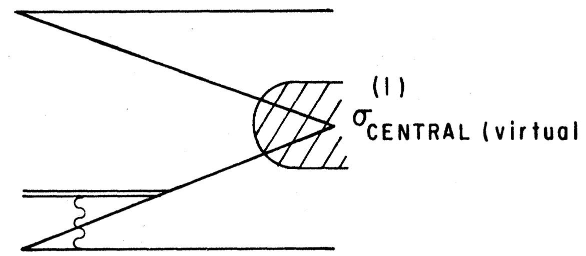


FIG. 31. The contribution of the subtractions of Fig. 30 to the factored form.

certain propagators, marked with asterisks in Fig. 32(a), are effectively off-shell, and others are effectively on-shell. If we apply the Ward identities in this momentum region, then some of the resulting terms are nonleading. These are the terms in which an effectively on-shell denominator is canceled or a Λ-line emerges from a vertex along an effectively on-shell line. We make a new definition of the collinear subtraction by dropping these terms—even when l_1 and l_2 are outside the collinear to $+$ momentum region.

The subtraction, with this new definition is, of course, a good approximation to the original graph in the region for which it was designed (l_1,l_2 collinear to $+$). It is also easy to see that it does not contribute in the regions l_1 or l_2 collinear to $-$. After applying Ward identities to Fig. 32(a) and the corresponding auxiliary subtraction, we arrive at the eikonal form shown in Fig. 32(b). Now suppose that one (or both) of l_1 and l_2 is collinear to $-$. One of the l_i has the largest $|l^-|$. Then we deform the l_i^+ contour out to $l_i^+\sim P$ (into the uhp for $l_i^->0$ and into the lhp for $l_i^-<0$) to show that the contribution is $O(1/P)$.

In general, we make a new definition for the collinear subtractions by decomposing each vertex (trigluon or q-g) involving an active subtraction gluon arrowhead according to the Ward identities and dropping all the resulting terms that are nonleading in the momentum region for which the subtraction was designed. That is, we drop all terms in which an effectively on-shell propagator is canceled or a Λ-line arrowhead connects to an effectively on-shell region of the graph. Then, after we sum graphs to obtain the eikonal form, it is easy to see that the collinear to $+$ (meson 2) structure function has no leading contributions in which one of its gluons is collinear to $-$. We

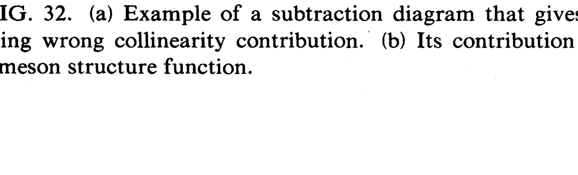


FIG. 32. (a) Example of a subtraction diagram that gives a leading wrong collinearity contribution. (b) Its contribution to the meson structure function.

simply deform the l_i^+ contour for the loop momentum with the largest $|l^-|$ out to $l_i^+ \sim P$. (Of course a similar argument holds for the collinear to $-$ structure function.)

D. The double spectator graphs

Next let us discuss the graphs in the meson-meson Drell-Yan process that involve an interaction with each spectator. We do not describe the analysis of graphs in which a gluon is exchanged between a quark and antiquark from the same meson, since these either give a correction to the meson's wave function or else are very similar in structure to one of the quark-meson examples discussed previously. However, we do consider all other graphs, beyond those treated in the quark-meson case, that are needed to complete the meson-meson analysis. The virtual gluon graphs in this set are shown in Fig. 33. Here, and in the remainder of this paper, the presence of associated seagull graphs is to be understood in the case of scalar quarks.

The essential new ingredient that appears in the analysis of these graphs is that one must add certain real emission graphs to the virtual graphs in order to remove all obstacles to the usual contour deformations. Consider, for example, the contribution corresponding to Fig. 33(d). Suppose we were to deform the l_2^+ contour into the uhp. As usual, the active-quark propagator $1/[(x_q P + l_2^-)l_2^+ - \perp^2 + i\epsilon]$ presents no obstacle until $l_2^+ \sim P$. Owing to the wave-function pinch in the propagators that emerge from the lower meson, $l_2^- \sim \perp^2/P$. Thus, the pole in the l_2 gluon propagator is also at $l_2^+ \sim P$. However, the propagator of the gluon with momentum label $l_2 - l_1$ blocks the contour deformation. Since $l_1^+ \sim \perp^2/P$ by virtue of the upper meson wave-function pinch, we can write the $l_2 - l_1$ gluon propagator in the approximate form $1/[l_2^+ l_1^- - (\mathbf{l}_{2\perp} - \mathbf{l}_{1\perp})^2 + i\epsilon]$. Thus, we see that for $l_1^- < 0$, the l_2^+ contour has a pole in the uhp. In terms of a decomposition of the original Feynman graph into time-ordered graphs, this pole is due to the final-state interaction shown in Fig. 34(a). According to the Cutkosky rules,[20] the sum over all cuts of the final-state interaction graph must be zero. Indeed, the contribution of Fig. 34(a) is canceled by the real emission (time-ordered) graph of Fig. 34(b). One can see that this cancellation removes the pole in the l_2^+ uhp directly in terms of the Feynman expressions. The contribution of the real emission graph is identical to that of the virtual graph except that the $l_1 - l_2$ gluon propagator is replaced by

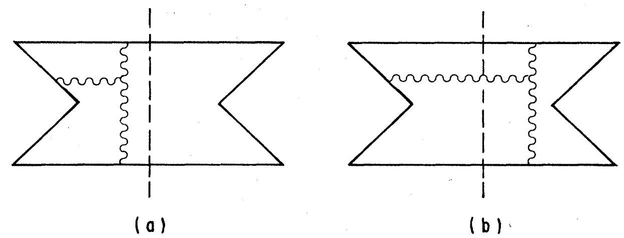


FIG. 34. Two final-state cuts of the double-spectator trigluon graph.

$$-\frac{1}{2|l_1^-|}(-2\pi i)\delta\left[l_2^+ - \frac{(\mathbf{l}_{2\perp}-\mathbf{l}_{1\perp})^2}{l_1^-}\right]\theta(-l_1^-)\ . \tag{3.10}$$

Thus, the sum of amplitudes is proportional to

$$\begin{aligned}
&\frac{1}{l_2^+ l_1^- - (\mathbf{l}_{2\perp}-\mathbf{l}_{1\perp})^2 + i\epsilon} - (-\pi i)\delta(l_2^+ l_1^- - (\mathbf{l}_{2\perp}-\mathbf{l}_{1\perp})^2)\theta(-l_1^-) \\
&= \frac{\frac{1}{2}[\theta(l_1^-)+\theta(-l_1^-)]}{l_2^+ l_1^- - (\mathbf{l}_{2\perp}-\mathbf{l}_{1\perp})^2 + i\epsilon} - \frac{1}{2}\left[\frac{1}{l_2^+ l_1^- - (\mathbf{l}_{2\perp}-\mathbf{l}_{1\perp})^2 + i\epsilon} + \frac{1}{-l_2^+ l_1^- + (\mathbf{l}_{2\perp}-\mathbf{l}_{1\perp})^2 + i\epsilon}\right] \\
&= \frac{\frac{1}{2}\theta(l_1^-)}{l_2^+ l_1^- - (\mathbf{l}_{2\perp}-\mathbf{l}_{1\perp})^2 + i\epsilon} - \frac{\frac{1}{2}\theta(-l_1^-)}{-l_2^+ l_1^- - (\mathbf{l}_{2\perp}-\mathbf{l}_{1\perp})^2 + i\epsilon}\ ,
\end{aligned} \tag{3.11}$$

which has no pole in the l_2^+ uhp.

For the contribution of Fig. 33(b), if we deform the l_2^+ contour into the uhp, there is the possibility of encountering a pole in the l_2 gluon propagator. This pole, at $l_2^+ = (l_{2\perp}^2 - i\epsilon)/l_2^-$, is in the uhp for $l_2^- < 0$, and corresponds to the final-state interaction shown in Fig. 35(a). It is removed when we add in the contribution of the real emission cut shown in Fig. 35(b). Thus, in the sum over cuts, we can always deform the l_2^+ contour into the uhp out to $l_2^+ \sim P$. We conclude then that $l_2^- \sim \perp^2/P$, since if l_2^- were $O(\perp)$, the l_2^+ integration would be power suppressed. The lower meson wave-function pinch gives the constraint $l_1^- + l_2^- \sim \perp^2/P$, so l_1^- must also be $O(\perp^2/P)$ if we are to obtain a leading contribution. But the upper wave-function pinch gives $l_1^+ \sim \perp^2/P$, so we can deform the l_1^- contour into the lhp out to $l_1^- \sim P$, provided that we observe the constraint $l_1^- + l_2^- \sim \perp^2/P$ and simultaneously deform the l_2^- contour into the uhp. (The pole in the l_2 gluon propagator poses no obstacle, since if it is in the uhp it cancels in the sum over cuts.) Thus, we

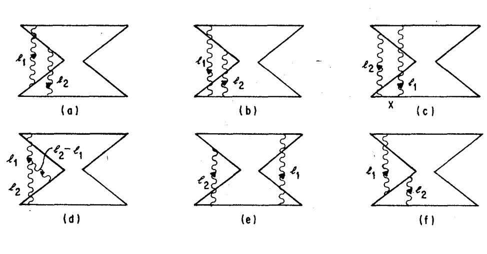


FIG. 33. Virtual double-spectator contributions to the Drell-Yan cross section.

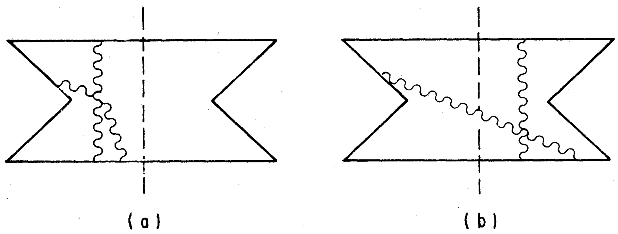


FIG. 35. Two final-state cuts of the graph of Fig. 33(b).

conclude that the contribution of Fig. 33(b) plus the contribution of Fig. 35(b) (taken as a Feynman graph) is power suppressed.

The contribution of Fig. 33(c) can be treated in a similar fashion. First we deform the l_2^+ contour into the uhp out to $l_2^+ \sim P$. The pole in the l_2 gluon propagator in the uhp cancels when we add the contribution of the real emission graph of Fig. 36. Then we conclude, as in the preceding example that l_1^-, l_2^- are $O(\perp^2/P)$. Next we deform the l_1^- contour into the lhp, observing the constraint $l_1^- + l_2^- \sim \perp^2/P$. Here we encounter a pole in the spectator propagator marked (x), but this pole is canceled when we add the contribution of the graph of Fig. 33(e). Thus, we conclude that the sum of contributions of Figs. 33(c), 36, and 33(e) is power suppressed. [This is a general property of graphs in which a gluon connects two spectator lines directly. After summing over cuts, one can always deform the $+$ momentum of that gluon into the uhp or the $-$ momentum into the lhp out to $O(P)$. But one can argue, using the wave-function pinch combined with deformations of the remaining gluon momentum contours, that leading contributions come only from the region in which all minus components of momenta flowing into the lower spectator are $O(\perp^2/P)$ and all plus components of momenta flowing into the upper spectator are $O(\perp^2/P)$.]

In the case of Figs. 33(a) and 33(f) the wave-function pinches require $l_2^- \sim \perp^2/P$ and $l_1^+ \sim \perp^2/P$. Thus we can immediately deform the l_2^+ contour into the lhp and the l_1^- contour into the lhp out to $O(P)$.

At this stage, we need to consider only the contributions of graphs of Figs. 33(a), 33(d), and 33(f), with l_2^+ and l_1^- everywhere $O(P)$. We make a collinear to $+$ subtraction on the l_2 gluon and collinear to $-$ subtraction on the l_1 gluon (Fig. 37). Since leading contributions re-

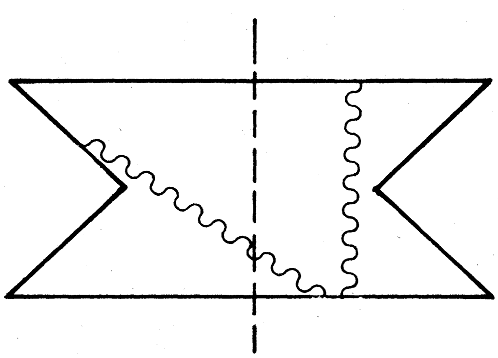


FIG. 36. A final-state cut involving real emission for the graph of Fig. 33(c).

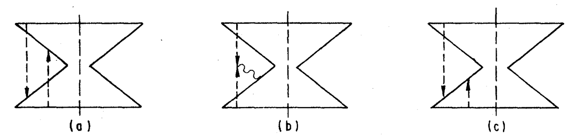


FIG. 37. Collinear subtractions for the double-spectator graphs.

sult only when l_2 is collinear to $+$ and l_1 is collinear to $-$, these subtractions account completely for the contributions of the original graphs. Application of the Ward identities to the l_2 gluon leads immediately to the result shown in Fig. 38(a). [The convection part of the Ward identity for the triple-gluon vertex graph of Fig. 33(d) supplies the commutator term required to disentangle the non-Abelian color factors. The Λ-line part from Fig. 33(d) cancels the Λ-line part from its mirror image (meson $1 \leftrightarrow$ meson 2, $l_1 \leftrightarrow l_2$). The convection part from the mirror image of Fig. 33(d) cancels the contribution of the mirror image of Fig. 33(f).] Subsequent application of the Ward identities to the l_1 gluon in Fig. 38(a) yields the standard factored form shown in Fig. 38(b), which gives a contribution to $\mathscr{P}^{(1)}_{q/1}\mathscr{P}^{(1)}_{\bar{q}/2}\sigma^{(0)}_{\text{central}}$.

E. Real emission

The treatment at the two-loop level of graphs containing real gluons differs from the treatment of the virtual gluon case only in the contour deformation arguments. The on-shell condition

$$l_i^+ l_i^- = l_{i\perp}^2 \tag{3.12}$$

eliminates Glauber contributions from the real gluons, without contour deformations. The proof that all gluons that attach to a spectator are collinear to the spectator is slightly more involved. If one gluon attaches to a spectator line, the argument proceeds as in the one-loop case. If two real gluons attach to a spectator, then the wave-function pinch gives (3.1); and the requirement that real gluons carry positive energy, combined with the on-shell condition (3.12), gives

$$l_i^+, l_i^- > 0 \, . \tag{3.13}$$

Thus,

$$l_i^- \sim \perp^2/P \, , \tag{3.14a}$$

which implies that

$$l_i^+ \sim P \, . \tag{3.14b}$$

If one real gluon and one virtual gluon attach to a specta-

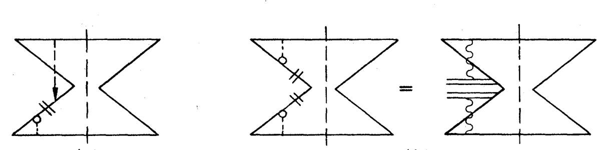


FIG. 38. Application of the Ward identities to the double-spectator subtractions of Fig. 37 to obtain a factored contribution to the meson's structure functions.

tor line, we give the virtual gluon the momentum label p-q, and the real gluon the momentum label q. The wave-function pinch gives

$$p^- \sim \perp^2/P \ , \tag{3.15}$$

so we deform the p^+ contour into the uhp. There is the possibility of encountering a pole in a virtual gluon or spectator propagator—because of the terms p^+q^-—but such a pole corresponds to a final-state interaction, and it cancels in the sum over final-state cuts. Having made the p^+ contour deformation, we are presented with two possibilities. The first is that $|q^-| \gtrsim \perp$, in which case there are at least two large denominators, and the entire contribution is power suppressed. (We are excluding the possibility of gluons that merely give a correction to the meson's wave function.) The second possibility is $q^- \sim \perp^2/P$, which implies that $q^+ \sim P$, so that we arrive at the desired result.

IV. THE ALL-ORDERS CROSS SECTION

In this section we give the procedure for putting a graph of arbitrary order into the weak factorization form. As always, we are concerned only with the leading contributions in the limit $Q^2 \to \infty$ (leading twist). We do not consider explicitly graphs in which the annihilating quark or antiquark is not a constituent of one of the initial mesons (for example, graphs involving gluonic constituents of the mesons). However, the techniques we present can easily be extended to include such cases. The procedure we follow is a slight generalization of the one outlined in the two-loop case. As before we deform contours, using the sum over final-state cuts to cancel unwanted pinches, and thereby eliminate Glauber contributions and show that all gluons that attach to a spectator are collinear to it. We define a set of subtractions that removes the collinear contributions and use Ward identities to put the subtractions into the factored form. Finally, we discuss the very soft region, give an all-orders proof of strong factorization, and demonstrate the cancellation of soft contributions in the inclusive cross section.

A. Weak factorization

(1) Contour deformation

By contour deformation arguments we can arrive at several useful results about the regions of momentum space from which the leading contributions come. We state each result and sketch its proof below.

Lemma. If we sum over all final-state cuts of a given graph, make a suitable choice of momentum labels, and, for a particular loop momentum l_i that flows only through the part of the graph to the left of the final-state cut, drop all terms of the $l_i^+l_i^-$ in propagator denominators, then we can deform the l_i^+ contour into the uhp and the l_i^- contour into the lhp so that at least one of l_i^+, l_i^- is $O(P)$. (A similar result holds for loop momenta on the right side of the graph, but, the contour deformations are in directions opposite to those for the left side.)

Proof. Let us begin with graphs that are planar (aside from the active-quark and antiquark lines) and label the momenta as shown in Fig. 39. We choose all loop momenta to flow counterclockwise, with the exception of momenta on lines that cross the final-state cut, which we always take to flow into the final state, and momenta on the incoming fermion lines, which we always take to flow parallel to the quark momentum and antiparallel to the antiquark momentum. We label momenta so that there is only one loop momentum flowing along each real gluon line.

Now let us suppose that for some internal loop with momentum l_i we attempt to deform the l_i^+ contour into the uhp out to $l_i^+ \sim P$. Suppose the contour deformation is blocked by a propagator pole in one of the propagators along the loop. Then there must be a momentum circulating in an adjacent loop whose minus component is positive and $O(\perp)$. Let us call the adjacent loop momentum with the largest minus component l_j. If in deforming the l_i^+ contour we pick up the residue at the pole in the propagator through which both l_i and l_j flow, we say that we have "broken" that propagator. Now we attempt to deform the l_j^+ contour into the uhp out to $O(P)$. If we succeed, then the contribution is power suppressed (since $l_j^- \sim \perp$). If the deformation is blocked, since $l_j^- > 0$, there must be an adjacent loop momentum with minus component greater than l_j^-. We call the one whose minus component is largest l_k, and attempt to deform the l_k^+ contour into the uhp. Proceeding in this fashion, we succeed eventually in deforming a contour to show that the contribution is nonleading, or we break a line because it carries some momentum l_r with $l_r^- > 0$, $|l_r^-| \gtrsim \perp$, where l_r is the momentum of a real final-state particle. Since l_r is constrained to be on the mass shell, we are not able to deform the l_r^+ contour into the uhp as we would for a virtual loop momentum. Thus the process of successive breaking of propagators ends at this line. Note that, because of the ordering $l_j^- < l_k^- < \cdots$, the sequence of broken propagators can never lead back to the original loop. Note also that, because of the choice of momentum routing, the active-quark line can never be broken.

Suppose we have broken a line because of the minus component of momentum of a real final-state particle. Then we go back to the original loop and attempt to deform the l_i^- contour into the lhp. Proceeding as in the l_i^+ case, we have the following possibilities: (1) we succeed in

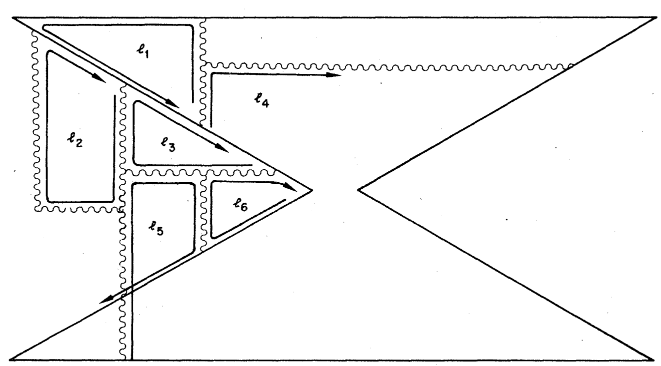


FIG. 39. Example of the choice of momentum routing used in the contour-deformation arguments.

deforming the l_i^- contour out to $l_i^- \sim P$, (2) we succeed in deforming some other l^- contour out to $l^- \sim P$ to show that the contribution is nonleading, or (3) we break a line because of the plus component of momentum of a final-state particle. Owing to our choice of momentum routing, we never cut the active-antiquark line. Furthermore, in carrying out the l^- contour deformations, we never break into a loop that had been broken in carrying out the l^+ contour deformations, since in that loop the previous break constrains l^+ to be $O(\perp^2/\tilde{l}^-)$ where $\tilde{l}^-$ is the largest l^- momentum along the loop.

If possibility (3) holds, then the contribution must correspond to a final-state interaction. In order to see this, let us consider the constraints imposed on the variables l^+, l^- in time-ordered perturbation theory. We obtain the time-ordered expression from the covariant one by integrating out one of the components, say, l^-, of each loop momentum. This gives a set of constraints on all the l^- variables and some θ-function constraints on the l^+ variables, which do not preclude the deformation of the l^+ contours into the uhp. The constraints on the l^+ contour deformations come from the poles in the intermediate state (light-cone) energy denominators. These poles correspond to the situation in which every particle in the intermediate state is on-mass shell. But, for an initial-state interaction, the intermediate state always involves the active-quark line, and, with our choice of momentum routing, this line never goes on-shell as we deform the l^+ variables into the uhp. Thus, we conclude that possibility (3) occurs only for a final-state interaction. We know from the cutting rules[20] that the sum over all final-state cuts for such an interaction is zero.

The generalization to the case of nonplanar graphs is simple. One must label momenta so that, around a given loop, the loop momenta flowing through each propagator are the loop momentum of the given loop and possibly one loop momentum of another loop. The momentum of the other loop is labeled so that it flows through the common propagator in the direction opposite to the flow of the momentum of the given loop.

It is also necessary to rule out the possibility of a loop momentum whose minus component is so large as to reverse the direction of the active quark. Note that at least one virtual gluon must have a large minus component of momentum, since conservation of energy implies that real gluon emission alone cannot reverse the direction of the active quark. If the virtual loop momentum is l_i, then $l_i^- < 0$, so one attempts to deform the l_i^+ contour into the lhp. Proceeding as in the previous discussion of the l^+ contour deformations (but now deforming into the lhp) one eventually succeeds in deforming a contour out to $O(P)$ to show that the contribution is nonleading. It is not possible to break a line going into the final state, since for such a line $l^- > 0$. Obviously, a similar argument holds for the reversal of direction of the active antiquark.

Theorem. For each l_i, after a suitable deformation of the l_i^+ and l_i^- contours (if necessary), $l_i^+ l_i^- \sim \perp^2$.

Proof. Consider first the case in which l_i does not flow across the final-state cut. The result follows almost immediately from the lemma. If we restore the terms $l_i^+ l_i^+$ to the propagator denominators, then there are two possibilities. The first is that $l_i^+ l_i^-$ is small enough so that we can still deform contours as in the lemma. Then one of l_i^+, l_i^- is $O(P)$ and the other is $O(\perp^2/P)$ or greater. [Since P is the largest momentum scale, we can always deform contours so that l^+ and l^- are at least $O(\perp^2/P)$.] The second possibility is that the terms $l_i^+ l_i^-$ block the contour deformation with a pole at $l_i^+ l^- \sim P$. In either case, the desired result is obtained.

We always label momenta so that each real gluon carries exactly one-loop momentum. Thus, if l_i crosses the final-state cut, then the gluon on-shell condition gives $l_i^+ l_i^- = l_{i\perp}^2$.

Theorem. If l_i is the momentum flowing through a gluon that attaches to the lower spectator (excluding gluons that give a correction to the meson wave function), then the leading contribution to the Drell-Yan cross section (after summing over final-state cuts) comes from the region $l_i^- \sim \perp^2/P$, $l_i^+ \sim P$. (A similar result holds for the upper spectator.)

Proof. Suppose there are r virtual gluons and s real gluons attached to the spectator line to the left of the final-state cut. We label the loop momenta in a manner consistent with the previous lemma. The momenta $p_1, \ldots, p_r$ flow along the spectator line from left to right, with at most one such momentum flowing through each spectator propagator. If a given spectator propagator has the momentum p_i flowing through it, then the adjacent spectator propagator to its left has momentum p_{i-1} flowing through it if the propagators are separated by a virtual gluon attachment and p_i flowing through it if the propagators are separated by a real gluon attachment. The left-most spectator propagator has the momentum p_1 flowing through it. In addition, each real gluon carries exactly one of the momenta $q_1, \ldots, q_s$. These momenta flow along the spectator line from the final-state cut to the corresponding real gluon and out along the gluon line. Thus, a given spectator line can carry a momentum from the set $p_1, \ldots, p_r$ and one or more momenta from the set $q_1, \ldots, q_s$. Now, none of the momenta p_i flows across the final-state cut. Furthermore, for each p_i, the deformation of the p_i^- contour into the lhp is blocked by a pole in a spectator propagator. Thus, from the lemma we know that for each p_i, the deformation of the p_i^+ contour into the uhp out to $p_i^+ \sim P$ can be blocked only by virtue of the $p_i^+ p_i^-$ terms in gluon propagators. The meson wave-function pinch gives $p_i^- \sim \perp^2/P$, so we can deform the p_1^+ contour until $p_1^+ \sim P$. Then there exist two possibilities: (1) $|p_2^-| \gtrsim \perp$, in which case there are at least two large denominators, and the contribution is power suppressed. (Here we are excluding the case in which some of the gluons simply give corrections to the meson wave function.) (2) $p_2^- \sim \perp^2/P$, in which case we can deform the p_2^+ contour into the uhp out to $p_2^+ \sim P$. Proceeding in this fashion, we eventually show that all the p_i^- are $O(\perp^2/P)$ and all the p_i^+ are $O(P)$, or else the contribution is nonleading. Since all the p_i^- are small, the on-shell condition for the final-state spectator gives

$$\sum_{i=1}^{s} q_i^- \sim \perp^2/P \ . \tag{4.1a}$$

Furthermore, the real gluons must carry positive energy, so

$$q_i^- > 0 \text{ for all } i \ . \tag{4.1b}$$

Thus, for each q_i, $q_i^- \sim \perp^2/P$. The on-shell condition for the real gluons then gives $q_i^+ \sim P$.

In summary, we have arrived, by contour deformation arguments, at two crucial results. The first is that we can eliminate the Glauber region of momentum space in the leading contributions. This guarantees that the subtractions we define are always a good approximation to the original graph in the collinear region. The second result is that, in order to obtain a leading contribution, we must take all gluons that attach to a spectator to be collinear to the spectator's meson. This means that in each collinear subtraction the spectator interactions are interior to the collinear subgraphs, so that they ultimately reside in the structure functions.

(2) Construction of the collinear subtractions

The main subtractions. The first step in constructing the main collinear subtractions is to identify the collinear regions in each graph. There is a result, which follows from power counting, that gives some useful information about these collinear regions.

Theorem. If a graph contains a line whose momentum is collinear to $+(-)$, then the graph's contribution is leading only if the line attaches on at least one end to another line with collinear to $+(-)$ momentum.

This result is easily proven by working out the power counting for the various types of momentum flows (collinear to $\pm$, soft, hard) along the lines to which the collinear gluon attaches. We do not bother to enumerate the cases here. This result implies that each collinear (to $\pm$) region is connected in the sense described in Sec. III. Namely, there must be a path from each hadron through pure collinear lines to each line collinear to the hadron. As explained in Sec. III, a line collinear to one hadron can "interrupt" the collinear lines that extend from the other hadron [see Fig. 23(c)].

Once we have identified the potentially collinear subgraphs of a given graph, we construct the corresponding main subtraction by making the replacement (2.12) in each gluon that connects a potentially collinear subgraph to its exterior. Since we have eliminated the Glauber region from the currents J_U, J_L, the condition (2.13) is satisfied and the subtraction is a good approximation to the original graph when all the gluons in the potentially collinear to $\pm$ subgraphs are collinear to $\pm$. Also, as described in Sec. III C, we use the subtraction gluon arrowhead and the Ward identity of Fig. 24 to expand all trigluon and q-g vertices; and we drop from the subtraction those terms in which an effectively on-shell propagator is canceled, or a Λ-line arrowhead attaches to an effectively on-shell line. These modifications in the definition of the subtraction, of course, do not change the leading contribution of the subtraction when all gluons in the potentially collinear to $\pm$ subgraphs are collinear to $\pm$.

Now we can give an iterative procedure for constructing a set of subtractions that removes all the collinear contributions. First, in $G^{(n)}$ we construct all main subtractions $M_i^{(n)}$ with $i=n$. Here i is the number of independent momenta that are potentially collinear either to $+$ or to $-$:

$$i = i_1 + i_2 + \tfrac{1}{2} i_3 \ , \tag{4.2}$$

where i_i is the power of g^2 on the interior of the collinear to $+$ region, i_2 is the power of g^2 on the interior of the collinear to $-$ region, and i_3 is the number of subtraction gluons. $M_n^{(n)}$ consists of the subtractions for the cases in which every independent momentum is collinear either to $+$ or to $-$. In $G^{(n)} - M_n^{(n)}$ we construct all main subtractions $M_{n-1}^{(n)}$. That is, the subtractions for the case in which one independent momentum is central and the others are potentially collinear to $+$ or to $-$. Note that in $M_{n-1}^{(n)}$ the potentially collinear subgraphs are subgraphs (possibly improper) of the potentially collinear subgraphs in $M_n^{(n)}$. We continue in this fashion through the subtraction $M_1^{(n)}$. Note that at any stage in the procedure a subtraction may remove a contribution of a type that is, nominally, to be contained in a later subtraction. For example, $M_i^{(n)}$ can contain contributions from the region of momentum space in which $n-i+1$ independent momenta are central, which is the type of contribution to be subtracted by $M_{i-1}^{(n)}$. However, because of the iterative procedure for constructing the subtractions, $M_{i-1}^{(n)}$ removes only the remainder of such contributions in $G^{(n)} - M_n^{(n)} - \cdots - M_i^{(n)}$. Thus, there is no double counting, and the quantity

$$G^{(n)} - M_n^{(n)} - \cdots - M_1^{(n)} \tag{4.3}$$

is free of collinear contributions. This iterative procedure for removing the collinear singularities in momentum space is very similar to the BPH program for subtracting ultraviolet divergences.

The auxiliary subtractions. For each set of subtraction in $M_i^{(n)}$, we construct a set of auxiliary subtractions. The sum of all the auxiliary subtractions for $M_i^{(n)}$ is denoted by $A_i^{(n)}$

$$S_i^{(n)} = M_i^{(n)} + A_i^{(n)} \ . \tag{4.4}$$

The purpose of these subtractions is to allow us to implement the Ward identities so as to obtain the factored form. As pointed out in Sec. III, they vanish in the region of momentum space in which all the subtraction gluons are collinear, so they do not upset the program for removing the collinear contributions that we have already outlined. The auxiliary subtractions are defined iteratively from the main subtractions. There are many possible iteration procedures, and the choice of the most useful one depends on the details of the implementation of the Ward identities. The motivation for the procedure we use will become apparent only when we discuss the use of Ward identities in the next subsection. For this reason, we postpone the discussion of the details of the construction of the auxiliary subtractions until that point.

Before we leave the discussion of the collinear subtractions, a word of caution is in order. Because of the technical difficulty discussed in Sec. III C, a given subtraction can give a leading contribution when some of its collinear gluons have momenta in the collinearity region opposite to the one which the subtraction is designed to remove.

As we have seen, these "wrong collinearity" contributions cancel when one uses Ward identities to sum over graphs to obtain the factored form. However, on a graph-by-graph basis $G-S$ can contain collinear singularities. Thus, in an actual calculation, it is probably simplest to extract collinear singularities by writing the cross section in the factored form (2.6).

(3) Implementation of Ward identities

In applying the Ward identities to the subtractions, we wish to make use of a slight generalization of a theorem on Yang-Mills fields due to 't Hooft.[21] First let us state the original result:

Theorem. In the pure Yang-Mills theory, if we insert an external Λ-line into a Green's function with any number of on-shell external gluons with physical polarization, then the sum over all such insertions is zero (Fig. 40).

We can immediately arrive at the required generalization by examining the 't Hooft's proof. The first step is to use the trigluon Ward identity (Fig. 26) to decompose all the trigluon vertices into which an arrowhead is inserted. Let us concentrate initially on the convection parts. In these, we either cancel an external line—which gives zero since the external gluons are on-mass-shell—or cancel an internal line. The sum over insertions in which we cancel an internal line adjacent to a given trigluon vertex is canceled by the insertion of the arrowhead into the corresponding quartic vertex, as shown in Fig. 41(a). This is simply the statement that the quartic vertex is the "seagull" for the trigluon vertex. Similarly, the sum of insertions around a quartic vertex is zero, as shown in Fig. 41(b). (We set aside for the moment the contributions in which we cancel a propagator coming out of a ghost-gluon vertex.) Next let us consider the Λ-line parts. For each Λ-line that enters a trigluon vertex, we again decompose the vertex using the Ward identity. Except for the possibility of canceling a propagator coming out of a ghost-gluon or Λ-line—gluon vertex, the convection and quartic vertices give zero as before. Continuing to decompose trigluon vertices in this fashion, we find that there are three possibilities: (1) the Λ-line eventually reaches the exterior to give zero because of the physical polarization [Fig. 42(a)]; (2) the Λ-line doubles back on itself [Fig. 42(b)]; or (3) the Λ-line enters a ghost loop Fig. 42(d). These last two contributions combine with the contributions in which we cancel a propagator coming out of a Λ-line—gluon vertex [Fig. 42(c)] or a propagator coming out of a ghost-gluon vertex [Fig. 42(e)] to give zero.

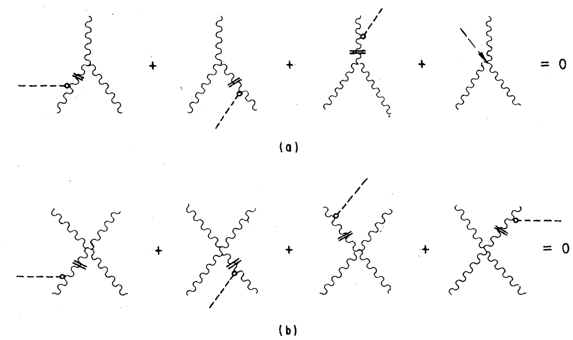


FIG. 41. Identities involving attachment of a Λ-line (a) to the legs of a trigluon vertex and (b) to the legs of a quartic vertex.

The first obvious generalization to the theorem is that we can include quark lines in the Green's function (external quarks on-shell). The new ingredients required are q-g Ward identity (Fig. 6), which we use to decompose all the Λ-line—quark vertices, and the fact that the set of diagrams in which we cancel a propagator coming out of a q-g vertex sums to zero (Fig. 43).

Now let us see how the theorem can be applied to the case of a collinear subtraction. The subtraction graph contains a collinear to $+$ region, from which an arbitrary number of subtraction gluons emerge to enter the collinear to $-$ and central regions (Fig. 44). Only that part of the collinear to $-$ and central regions through which a subtraction gluon momentum flows are effectively off-shell. Because of the connectivity requirement for collinear regions, this means that all the collinear to $-$ and inactive subtraction gluons are in the effectively on-shell region. Furthermore, since our definition of the subtraction excludes from the trigluon decomposition the terms in which Λ-lines pass from the effectively off-shell to the effectively on-shell region, the effectively on-shell gluons that attach to the effectively off-shell region are effectively physically polarized. Thus, we have satisfied most of the conditions necessary to apply the theorem.

Let us number the collinear to $+$ subtraction gluons in an arbitrary fashion, calling the number assigned to each gluon its index. We attempt to generalize the theorem so

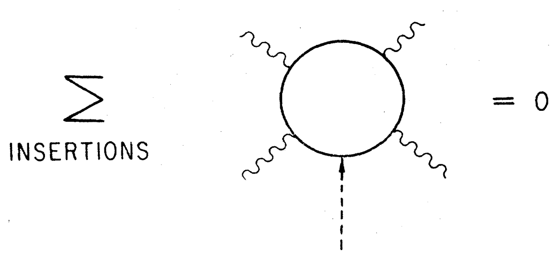


FIG. 40. 't Hooft's theorem for the insertion of a Λ-line into an arbitrary gauge-field diagram. All external lines are on-shell and physically polarized.

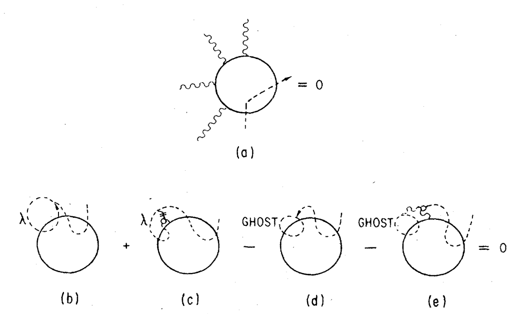


FIG. 42. Remainders after application of the trigluon Ward identity and the identities of Fig. 41. A Λ-line (a) emerges along a physically polarized external line, (b) doubles back on itself, (c) connects to a gluon line that connects to the Λ-line, (d) connects to a ghost loop, and (e) connects to a gluon line that connects to a ghost loop.

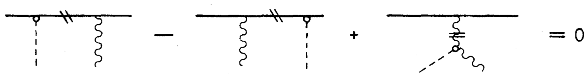


FIG. 43. Identity involving the connection of a Λ-line to the legs of a q-g vertex.

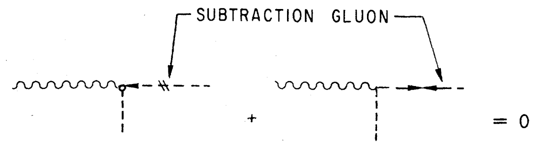


FIG. 45. Cancellation of contributions that arise from a Λ-line—subtraction gluon vertex.

that we can apply it successively to the subtraction gluons, in order of increasing index. This new situation requires several changes to the original argument.

The first change is that new types of terms arise when a Λ-line enters a trigluon vertex involving a subtraction gluon. We decompose the vertex by applying the Ward identity with respect to the Λ-line. The convection term in which the subtraction gluon propagator is canceled and the term in which the Λ-line flows out along the subtraction gluon line combine to give zero (see Fig. 45). The other two terms are treated as in the original argument.

A second change compared to the original argument is that we are missing the convection terms that arise from a trigluon vertex involving two subtraction gluons. This type of contribution is illustrated in Fig. 46. It is at this stage that the auxiliary subtractions enter the picture. First, we pair gluon 1 with each of the other subtraction gluons to form an auxiliary subtraction according to the definition (3.4). The arrowhead of the auxiliary gluon thus formed resides at the location of the arrowhead of the subtraction gluon with the larger index, and the gauge group matrices in (3.4) are in the same representation (fundamental or adjoint) as the gauge group matrix associated with the subtraction gluon of larger index. For purposes of defining the terms a' and a'' in (3.4a), we observe the convention that the first set of arguments is associated with the gluon of lower index (gluon 1). The usual rules for dropping trigluon and q-g vertex terms that are nonleading when the subtraction gluons are collinear to $+$ also apply to the auxiliary subtraction gluons. Now we notice that the A' terms in the auxiliary subtraction supply precisely the missing terms of the type in Fig. 46.

Thus, we first apply the theorem to gluon 1, using the A' terms from the set of auxiliary subtractions that it generates. (We set aside the A'' terms for now.) We would get zero except that one of the external lines to which gluon 1 attaches is always effectively off-shell. This situation is illustrated in Fig. 47. Thus, we obtain a contribution in which gluon 1 attaches to the active-quark line just outside the upper blob in Fig. 44, and there is a factor

$$\frac{\lambda_1}{l_1\cdot n+i\epsilon} \tag{4.5}$$

arising from the replacement (2.12) [see Fig. 48(a)]. Here λ_1 is the group matrix associated with gluon 1. Next, we apply the theorem to gluon 2, constructing the required auxiliary subtractions by pairing it with the remaining subtraction gluons. Again, we use the A' term and set aside the A'' term. Proceeding in this fashion, we eventually remove all the subtraction gluons from the upper blob [Fig. 48(b)] and obtain a result proportional to the factor

$$\frac{\cdots\lambda_3\lambda_2\lambda_1}{\cdots(l_3\cdot n+i\epsilon)(l_2\cdot n+i\epsilon)(l_1\cdot n+i\epsilon)}\ . \tag{4.6}$$

Next we apply the theorem to the contributions involving the A'' terms that we have set aside. The A'' term acts like a subtraction gluon whose momentum is the sum of the momenta of the gluons from which it was derived. We again apply the theorem to the subtraction gluons (or A'' gluon) in order of increasing index. It is convenient to adopt the convention that the index of an A'' gluon is the larger of the two indices of the gluons from which it is constructed. For each gluon to which the theorem is applied, we generate an additional set of auxiliary subtractions—using the new A' terms to remove the gluons from the upper blob and setting aside the new A'' terms. These new A'' terms are treated at the next level of iteration. Proceeding in this fashion, we eventually succeed in removing all subtraction gluons and A'' gluons from the upper blob. (The last to be removed is the A'' term composed iteratively of all the subtraction gluons.) As an illustration, we write out the resulting factors for the case of three subtraction gluons:

$$\frac{\lambda_3\lambda_2\lambda_1}{(l_3)(l_2)(l_1)}+\frac{\lambda_3\tilde{a}''(1;2)}{(l_3)}+\frac{\tilde{a}''(1;3)\lambda_2}{(l_2)}+\frac{\tilde{a}''(2;3)\lambda_1}{(l_1)}$$
$$+\tilde{a}''(\tilde{a}''(1;2);3)+\tilde{a}''(2;\tilde{a}''(1;3))\ , \tag{4.7a}$$

where

$$(l_i)=(l_i\cdot n+i\epsilon)$$

and

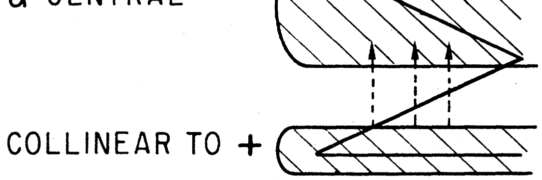


FIG. 44. Schematic representation of a subtraction graph containing collinear to $+$ subtraction gluons.

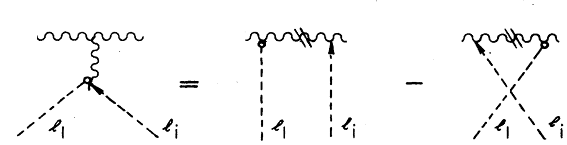


FIG. 46. Decomposition of the "missing term" in the gauge-invariance sum into the A' terms of an auxiliary subtraction.

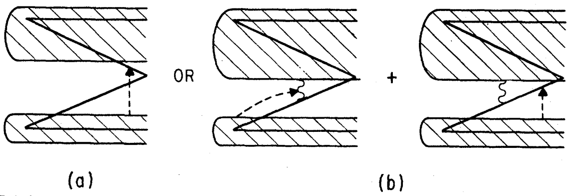


FIG. 47. Graphs in which a collinear to + subtraction gluon attaches to an external line of a potentially collinear to − subgraph. The external line can be effectively off-shell. Thus these graphs lead to a nonzero contribution upon application of the generalized 't Hooft's theorem.

$$\tilde{a}''(i,j)=\frac{[\lambda_i,\lambda_j]}{(l_i)(l_i+l_j)} .$$

It is easy to see that the expression (4.7a) is equal to

$$\sum_{\substack{\text{permutations}\\ \text{of } 1,2,3}} \frac{\lambda_1\lambda_2\lambda_3}{(l_1)(l_1+l_2)(l_1+l_2+l_3)} , \tag{4.7b}$$

which is the desired eikonal form. The expression analogous to (4.7a) for the case of n subtraction gluons is rather complicated. However, we can arrive at a simplified expression by noting that, due to our convention for the index of an A'' gluon, the term

$$\frac{\lambda_1\lambda_2\cdots\lambda_n}{(l_1)(l_1+l_2)\cdots(l_1+\cdots+l_n)} \tag{4.8}$$

appears exactly once, namely, in the expression

$$\tilde{a}''(\tilde{a}''(\cdots\tilde{a}''(1;2);3\cdots;n) . \tag{4.9}$$

Furthermore, the main subtractions are symmetric with respect to interchange of gluon indices and, because of the symmetry of $A(i;j)$ with respect to interchange of arguments, the auxiliary subtractions that we add at each iteration are also symmetric. Thus the final expression for the main plus auxiliary subtractions has the eikonal factor

$$\sum_{\substack{\text{permutations}\\ \text{of } 1,\ldots,n}} \frac{\lambda_1\lambda_2\cdots\lambda_n}{(l_1)(l_1+l_2)\cdots(l_1+\cdots+l_n)} . \tag{4.10}$$

At this stage we have shown, by application of the Ward identities, that we can put the main subtractions plus the auxiliary subtractions generated by the collinear to + subtraction gluons into a form in which all the collinear to + subtraction gluons attach to an eikonal line that connects to the active antiquark line just outside the central and collinear to − subgraphs. A similar application of Ward identities allows us to move the connections of the collinear to − subtraction gluons out of the central subgraphs to obtain the factored form. (For each main subtraction and auxiliary subtraction generated by the collinear to + subtraction gluons, the collinear to − subtraction gluons generate additional auxiliary subtractions.)

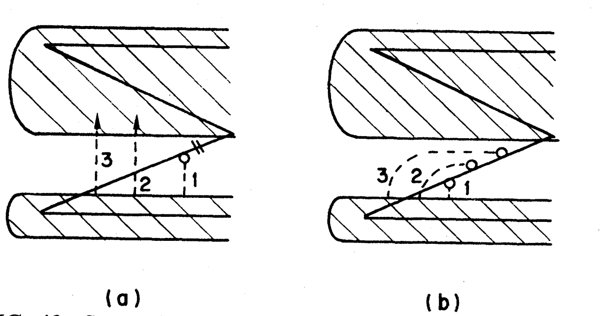


FIG. 48. Successive application of the generalized 't Hooft's theorem to collinear to + subtraction gluons.

(4) The factored form

The factored form of the Drell-Yan cross section, the convolution of a structure function for meson 1 and a structure function for meson 2 with a central cross section, is shown schematically in Fig. 3. Note that σ_{central} has connections only to the active-quark and antiquark legs. Furthermore (since we are excluding, for the moment, the very soft region), the active lines entering σ_{central} are effectively on-shell. Recall that, in the iterative subtraction procedure, the potentially collinear subgraphs for each graph in $S_{i-1}^{(n)}$ are subsets of the potentially collinear subgraphs for some graph in $S_i^{(n)}$. (For example, each graph that contributes to $\mathscr{P}_{q/1}^{(j)}\mathscr{P}_{\bar{q}/2}^{(n-i-j)}$ has potentially collinear subgraphs that are subgraphs of the potentially collinear subgraphs in $\mathscr{P}_{q/1}^{(j+1)}\mathscr{P}_{\bar{q}/2}^{(n-i-j)}$.) Thus, the remnants of the subtractions that remain in σ_{central} have just the form of the subtractions for those interactions that involve only the active lines. We calculate σ_{central}, then, by computing the difference, with active q and $\bar{q}$ lines on-shell, between the $q\bar{q}$ annihilation graphs and their corresponding collinear subtractions [see (2.6e)]. In practice, the collinear subtractions $S_{q\bar{q}}^{(n)}$ are probably computed most easily from the eikonal form obtained after application of the Ward identities. The $S_{q\bar{q},i}^{(n)}$ can be determined from the q and $\bar{q}$ structure functions by iterating the equations

$$S_{q\bar{q},i}^{(n)}=\sigma_{\text{central}}^{(n-i)}\sum_{k=0}^{i}\mathscr{P}_{q/1}^{(k)}\mathscr{P}_{\bar{q}/2}^{(i-k)} , \tag{4.11a}$$

$$\sigma_{\text{central}}^{(j)}=\sigma_{q\bar{q}}^{(j)}-\sum_{k=1}^{j}S_{q\bar{q},k}^{(j)} \tag{4.11b}$$

[which follow from (2.6)], starting with $S_{q\bar{q},1}^{(1)}$.

(5) The very soft region

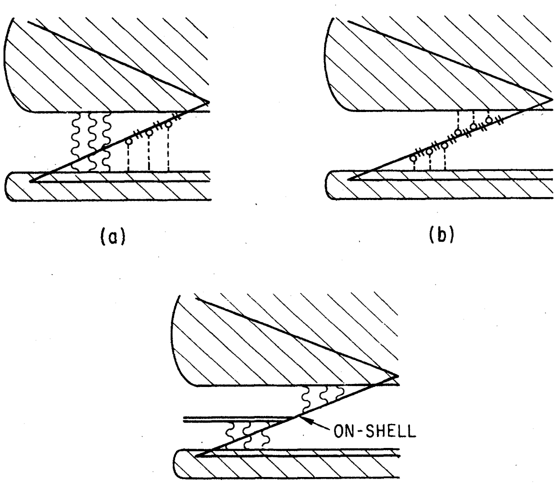
Finally, let us discuss the effects of the very soft momentum region, which so far we have omitted from our analysis. The key to analyzing this momentum region is the power-counting result that gluons with very soft momentum must attach either to each other or to collinear lines, if one is to obtain a leading contribution. Furthermore, if an end of a very soft gluon attaches to a collinear to + or − line, then that end must contract into a current that is collinear to + or −. Using these facts we can make a simple modification of the subtraction procedure that we have already outlined in order to incorporate the effects of the very soft gluons.

First we partition momentum space, calling all gluons with $|l_\mu| \leq \lambda$ "very soft gluons." Here $m^2/P \ll \lambda \ll m$; for example, we could take $\lambda = m^{3/2}/P^{1/2}$. There is no need to identify the "very soft collinear gluons"—those with l^+l^-, $l_\perp{}^2 \ll m^2$, and $l^+ \gg l^-$ or $l^- \gg l^+$—since these can be treated in essentially the same way as the ordinary collinear gluons. (Contributions in which the "effectively off-shell" arguments fail for the very soft collinear gluons are power suppressed.) We construct collinear subtractions exactly as before, except that now the very soft gluons can be attached to lines in the collinear region, since the very soft momentum does not interrupt the collinear momentum flow. In constructing the subtractions, we make the replacement (2.12) only for gluons that connect one collinear region of a graph to the other or to the soft and hard central regions. The replacement (2.12) is not a good approximation for collinear gluons attached to very soft gluons. (It is also a bad approximation for very soft collinear gluons attached to central gluons, but such contributions are power suppressed.) Then, application of Ward identities allows us to remove the subtraction gluon attachments for the soft and hard central regions, but not the very soft region. Thus, we have the situation depicted in Fig. 49(a). The subtraction gluon scalar remnants attach to the active-quark (or antiquark) line at a point just outside the last soft or hard central gluon attachment. The very soft gluons attach (in all possible ways) to lines inside a collinear region. We can move these soft gluon attachments out of these collinear regions in the following way: we make the replacements

$$g_{\mu\nu} \rightarrow \frac{n^+_\mu l_\nu}{(l\cdot n^+ - i\epsilon)} \tag{4.12a}$$

for a gluon whose ν end attaches a collinear to $+$ line,

$$g_{\mu\nu} \rightarrow \frac{l_\mu n^-_\mu}{(l\cdot n^- + i\epsilon)} \tag{4.12b}$$



(a) (b)

(c)

FIG. 49. Results of application of the generalized 't Hooft theorem (a) to the collinear subtraction gluons and (b) to the collinear subtraction gluons and very soft gluons. (c) an equivalent form for (b) in terms of a meson structure function times a graph in which the active antiquark is on-shell.

for a gluon whose μ end attaches to a collinear to $-$ line, and

$$g_{\mu\nu} \rightarrow \frac{(\frac{1}{2})l_\mu l_\nu}{(l\cdot n^+ - i\epsilon)(l\cdot n^- + i\epsilon)} , \tag{4.12c}$$

for a gluon whose μ end attaches to a collinear to $+$ line and ν end attaches to a collinear to $-$ line. Here

$$n^+ = (1,0,0) , \quad n^- = (0,1,0) . \tag{4.12d}$$

Because of the rules given above for the currents into which a very soft gluon can contract, (4.12) is always a good approximation for very soft gluons. Now we can apply Ward identities, using the factor l on the end of the very soft gluon propagator that attaches to the collinear region. The procedure is very similar to that employed in disentangling the collinear subtraction gluons from the opposite collinearity region and the central region. (The contributions in which the l^ν of a Λ-line contracts into the n_ν of a subtraction gluon are power suppressed.) Note that, in the present case, in implementing the Ward identities, we are free to add contributions analogous to the auxiliary subtractions, since these vanish when both gluons in an auxiliary gluon pair are in the very soft region. The result of application of the Ward identities is that all the very soft gluon attachments are moved to a point on the active-antiquark line between the collinear subtraction gluon attachment and the outermost soft or hard central attachment point, as shown in Fig. 49(b). But this is precisely the result that one would obtain by making the substitution (4.12) and applying Ward identities to the very soft gluon attachments on the active-antiquark line in Fig. 49(c)—provided that one treats the point on the active-antiquark line just outside the last very soft gluon attachment as if it were on-shell.

Thus, we arrive again at the factored form of Fig. 3. The net result of the analysis of the very soft region is the instruction that σ_{central} must be computed with external active fermion lines on-shell. This is the resolution of the off-shell—on-shell ambiguity.

B. Cancellation of soft contributions

If one integrates over transverse momentum, Q_T, of the Drell-Yan pair up to order $Q_T = Q_T^0$, then the contributions containing central gluons with $|l_\perp| \ll Q_T^0$ cancel. In order to show this we make use of an argument similar to that invented by CSS[23] to show the connection between Drell-Yan and deep-inelastic structure functions. First we note that, because the gluons are in the central region, the propagators along the active-quark line take on the eikonal form

$$\frac{1}{P(\sum_i l_i^+) + i\epsilon} \tag{4.13a}$$

on the left half of the graph, and

$$\frac{1}{-P(\sum_i l_i^+) + i\epsilon} \tag{4.13b}$$

on the right half of the graph. Here we have routed the real gluon momenta through the Drell-Yan photon. There is an important cancellation between real and virtual gluons. In order to make it manifest, we consider the uncut graph of Fig. 50. (In this graph all propagator denominators have $+\ i\epsilon$.) Now suppose we move some central gluon attachments from the left-quark eikonal line to the right-quark eikonal line—preserving the order of attachments so that the color factor is unchanged. Provided that we integrate over Q_T and $l_{i\perp} \ll Q_T{}^0$, only the quark eikonal propagators are affected by the change of gluon connections. From power counting, we know that if $l_{i\perp} \ll l_{j\perp}$, then the attachment of the ith gluon to the eikonal line must be outside that of the jth gluon. Thus, we can move the attachments of the gluons with $l_\perp \ll Q_T{}^0$ without disturbing the harder gluons. If there are m soft gluon connections to the right-quark eikonal line and N-m connections to the left-quark eikonal line, the corresponding propagators and vertices contribute a factor

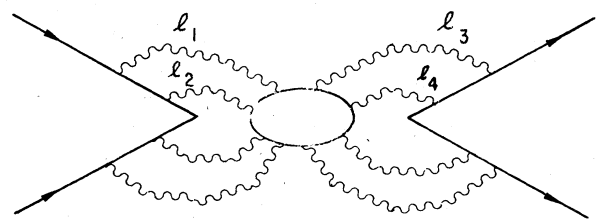


FIG. 50. Example of momentum labeling for the discussion of cancellation of soft contributions in σ_{central}.

$$E_m^N(l_1^+,\ldots,l_N^+)=\left[\frac{1}{l_{m+1}^+ +i\epsilon}\,\frac{1}{l_{m+1}^+ +l_{m+2}^+ +i\epsilon}\cdots\frac{1}{l_{m+1}^+ +\cdots +l_N^+ +i\epsilon}\right]\times\left[\frac{1}{-l_m^+ +i\epsilon}\,\frac{1}{-l_{m-1}^+ -l_m^+ +i\epsilon}\cdots\frac{1}{-l_1^+ -\cdots -l_m^+ +i\epsilon}\right]. \tag{4.13c}$$

Note that

$$E_0^N(l_1^+,\ldots,l_N^+)=\frac{1}{l_1^+ +i\epsilon}\,\frac{1}{l_1^+ +l_2^+ +i\epsilon}\cdots\frac{1}{l_1^+ +l_2^+ +\cdots +l_N^+ +i\epsilon}\,,$$
$$E_N^N(l_1^+,\ldots,l_N^+)=\frac{1}{-l_N^+ +i\epsilon}\,\frac{1}{-l_{N-1}^+ -l_N^+ +i\epsilon}\cdots\frac{1}{-l_1^+ -l_2^+ -\cdots -l_N^+ +i\epsilon}\,. \tag{4.13d}$$

Let us consider first the quantity $\widetilde{E}{}_m^N$ obtained by changing the sign of the $i\epsilon$ in each of the eikonal denominators in the second factor of (4.13c):

$$\widetilde{E}{}_m^N(l_1^+,\ldots,l_N^+)=\left[\frac{1}{l_{m+1}^+ +i\epsilon}\,\frac{1}{l_{m+1}^+ +l_{m+2}^+ +i\epsilon}\cdots\frac{1}{l_{m+1}^+ +\cdots +l_N^+ +i\epsilon}\right]\times(-1)^m\left[\frac{1}{l_m^+ +i\epsilon}\,\frac{1}{l_{m-1}^+ +l_m^+ +i\epsilon}\cdots\frac{1}{l_1^+ +\cdots +l_m^+ +i\epsilon}\right]. \tag{4.14}$$

Now, $\widetilde{E}{}_m^N$ has the property that

$$\sum_{m=0}^{N}\widetilde{E}{}_m^N(l_1^+,\ldots,l_N^+)=0\,, \tag{4.15}$$

which we prove in the Appendix.[24] Thus, the eikonal factor associated with the sum over gluon attachments (with a given color factor) is

$$\sum_{m=0}^{N}E_m^N(l_1^+,\ldots,l_N^+)=\sum_{m=0}^{N}[E_m^N(l_1^+,\ldots,l_N^+)-\widetilde{E}{}_m^N(l_1^+,\ldots,l_N^+)]\,. \tag{4.16}$$

We can use the identity

$$\frac{1}{-x+i\epsilon}=-2\pi i\delta(x)-\frac{1}{x+i\epsilon} \tag{4.17}$$

to rewrite each eikonal denominator in the second factor of E_m^N [Eq. (4.13c)]. Then, the term involving no δ functions is precisely $\widetilde{E}{}_m^N$, and it cancels in the difference between E_N^m and $\widetilde{E}{}_N^m$ in (4.16). Each of the remaining terms contains at least one δ function, whose argument is a sum of l_i^+'s. We can use this constraint on the l_i^+'s to show that, for the constrained momenta, the l_i^- contours can be deformed into the lhp so that l_i is collinear to $-$. [The arguments are essentially the same as those given in Sec. IV A (1) for the case of gluons attached to spectators.] But, owing to the collinear subtractions, contributions to σ_{central} are leading only for all gluons in the central region. That is, the contour deformation argument shows that the contribution is power suppressed.

Thus, all contributions in which gluons with $l_\perp \ll Q_T{}^0$ attach to active lines cancel. There is, then, an effective lower cutoff on the transverse momentum of the external lines of σ_{central}. This transverse momentum cutoff on external lines ensures that no internal loop integration can become divergent in the momentum range less than the cutoff. As a consequence, the inclusive cross section is infrared finite, and contributions with loop momenta for which $l_\perp \ll Q_T{}^0$ are nonleading.

C. Strong factorization

For completeness, we outline here a proof, suggested by CSS,[23] of the connection (1.1) between the Drell-Yan and deep-inelastic structure functions. Let us consider the structure function for the meson that is moving in the + direction $\mathscr{P}_{\bar{q}/2}(x,\mathbf{k}_\perp)$, integrated over $\mathbf{k}_\perp$, and the corresponding deep-inelastic structure function $f_{\bar{q}/2}(x)$ shown in Fig. 51. Since, in deep-inelastic scattering, one of the active lines goes into the final state, the eikonal line in $f_{\bar{q}/2}(x)$ is timelike; that is, the eikonal vector $\tilde{n}$ is given by

$$\tilde{n}=(1,1,0)\ . \tag{4.18}$$

Thus, the deep-inelastic structure functions differ from the Drell-Yan structure function in that

$$\frac{1}{n\cdot l+i\epsilon}\rightarrow\frac{1}{-\tilde{n}\cdot l+i\epsilon} \tag{4.19a}$$

in the eikonal propagators, and

$$n_\mu\rightarrow-\tilde{n}_\mu \tag{4.19b}$$

in the eikonal vertices. The minus sign in (4.19b) represents the relative sign in the coupling of the Yang-Mills field to a quark, as opposed to an antiquark.

Suppose that each of the gluons that attach to the eikonal lines is collinear to +. Then

$$\frac{n_\mu J^\mu}{n\cdot l+i\epsilon}\simeq\frac{J^+}{l^+}\simeq\frac{-\tilde{n}_\mu J^\mu}{-\tilde{n}\cdot l+i\epsilon}\ , \tag{4.20}$$

since J^μ, the current to which the gluon couples, is necessarily collinear to +. Thus, the two structure functions receive equal contributions from this region of momentum space.

However, we can show that the contributions to the structure functions in which gluons with central momentum attach to the eikonal lines cancel between real and

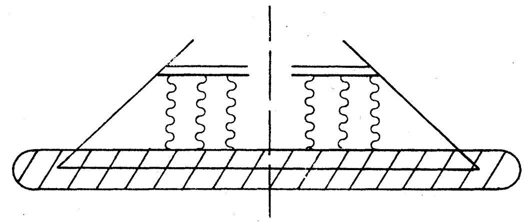


FIG. 51. Schematic representation of the deep-inelastic meson structure function.

virtual emission. As in the discussion of the soft cancellations, we consider the uncut graphs for the structure functions. For the Drell-Yan structure function in time-ordered perturbation theory, we never obtain an intermediate state that cuts through the eikonal lines, since these are orthogonal to the time direction. That is, the eikonal propagator has no l^0 dependence. Thus, we are free to choose the sign of the $i\epsilon$ in the eikonal propagators in the uncut graph—which we take to be positive on the left-hand side of the graph and negative on the right-hand side. The sign of the $i\epsilon$ in the eikonal propagators is also irrelevant in the uncut deep-inelastic structure function, but for a different reason. The poles in the lightlike eikonal propagators are at $l^0=0$; that is, they correspond to zero-energy states. As a consequence, it is immaterial whether we pick up the positive- or negative-energy pole in forming an intermediate state. Again we choose plus signs for the $i\epsilon$'s on the left-hand eikonal and minus signs for the $i\epsilon$'s on the right-hand eikonal.

Now we can easily see that the contributions in which central gluons attach to the eikonal lines cancel. In the Drell-Yan case, for a graph with m central gluon connections on the right eikonal line and N-m central gluon connections on the left eikonal line, the corresponding propagators and vertices contribute a factor

$$E^N_{\mathrm{DY},m}=\left[\frac{1}{l_{m+1}\cdot n+i\epsilon}\,\frac{1}{(l_{m+1}+l_{m+2})\cdot n+i\epsilon}\cdots\frac{1}{(l_{m+1}+\cdots+l_N)\cdot n+i\epsilon}\right]\times\left[\frac{1}{-l_m\cdot n-i\epsilon}\,\frac{1}{(-l_{m-1}-l_m)\cdot n-i\epsilon}\cdots\frac{1}{(-l_1-\cdots-l_m)\cdot n-i\epsilon}\right]. \tag{4.21}$$

Here we have used the fact (from power counting) that all the soft gluon attachments must be to the outside of the hard gluon attachments. Since we have integrated over k_T, we can move central gluon attachments from the left eikonal to the right eikonal without changing the graphical factors, other than in the eikonal lines. (As usual, we choose the order of attachments so as to maintain the original color factor.) Summing over all such attachments, we obtain

$$\sum_{m=0}^{N}E^N_{\mathrm{DY},m}=0\ , \tag{4.22}$$

where we have made use of the combinatorial identity (4.15). In the case of the deep-inelastic structure function, we obtain a factor identical to (4.21) except that $n\rightarrow-\tilde{n}$. After summing over all central gluon connections to the eikonal lines with a given color factor, we again obtain zero.

V. CONCLUDING REMARKS

Having established in detail the mechanisms by which strong and weak factorizations occur, we can now address the question raised in the Introduction concerning the "transparency" of nuclear matter to a fast-moving quark. The statement of strong factorization does not imply that an incoming quark undergoes no interactions with spectators as it traverses a nucleus. Rather, weak factorization tells us that, provided the target-length condition (2.26) is satisfied, the net effect of such interactions is to append an eikonal line to the target wave function. Strong factorization tells us that the outgoing quark in deep-inelastic scattering undergoes a similar set of spectator interactions, producing an equivalent eikonal line, so that if we compare the Drell-Yan process to deep-inelastic scattering, the spectator interactions cancel. One can regard the eikonal line, which makes the structure functions gauge invariant, as a gauge artifact. Indeed, for a suitable choice of axial gauge, the interactions in which a gluon attaches to the eikonal vanish.

Finally, let us summarize the general properties of quantum chromodynamics that are crucial in implementing the factorization program. Contour deformation arguments allow us to eliminate Glauber contributions and to show that the spectator interactions proceed via collinear gluons. The propagator $i\epsilon$'s, which play a central role in these arguments, are, of course, just the momentum-space realization of the *causality* of the theory. In fact, Collins, Soper, and Sterman[8] have suggested an heuristic space-time picture for elucidating the relevant momentum flows and graphical structures that is based on the causality principle. A feature of the Drell-Yan process that is essential to the causality arguments and the contour deformations is the presence of a *hard (short-distance) interaction* in the basic process. Another essential feature is the cancellation of final-state interactions, which is a consequence of the *unitarity* of the theory. The *power-counting* properties of a vector theory with dimensionless coupling constant enable us to identify the graphical structures that could give leading contributions for various combinations of loop momenta. For example, power counting allows us to see that the collinear subgraphs are connected and that very soft gluons must attach to collinear lines or to each other. Power counting is also crucial in showing that the Grammer-Yennie replacements yield a good approximation in certain momentum regions. Of course, it is precisely the nature of power counting in a vector theory that leads to a potential violation of factorization in the first place. In a scalar theory, the factorization-violating contributions are power suppressed—but the extra numerator powers of momentum in a vector theory destroy this simple picture. Much as in the case of the renormalization program, it is the *gauge invariance* of the Yang-Mills theory that allows us to tame the unruly power behavior of the vector interaction, for it is the Ward identity manifestation of gauge invariance that enables us to organize the cross section into a factored form.

ACKNOWLEDGMENTS

This work was begun in collaboration with S. J. Brodsky and G. P. Lepage as part of an effort that was aimed at understanding the role of initial-state interactions in hadronic collisions. Their insights into this problem played an important part in my initial attempts to work out a factorization program. I also wish to acknowledge numerous illuminating discussions with J. Collins, D. Soper, and G. Sterman, who first suggested several of the techniques used in this work. I would like to thank J. C. Taylor for pointing out an error in an early version of the proof of cancellation of soft contributions. R. Tucci offered many useful suggestions for improving the original manuscript, for which I am most grateful. The early stages of this work were carried out at the Stanford Linear Accelerator Center, where support was received from the U. S. Department of Energy. This work was performed under the auspices of the United States Department of Energy.

APPENDIX: PROOF OF A COMBINATORIAL IDENTITY

In this Appendix we prove the identity (4.15) for the sum over connections (with a given color factor) to two eikonal lines. The proof proceeds by induction. First we assume that (4.15) holds for some N. Then for the case of $N+1$ gluons we have

$$\begin{aligned}\sum_{m=0}^{N+1}\widetilde{E}_m^{N+1}(l_1,\ldots,l_{N+1})&=\sum_{m=0}^{N}\widetilde{E}_m^{N+1}(l_1,\ldots,l_{N+1})+\widetilde{E}_{N+1}^{N+1}(l_1,\ldots,l_{N+1})\\&=\sum_{m=0}^{N}\widetilde{E}_m^{N}(l_1,\ldots,l_n)\frac{1}{l_{m+1}+\cdots+l_{N+1}+i\epsilon}\\&\quad+(-1)^{N+1}\left[\frac{1}{l_{N+1}+i\epsilon}\cdots\frac{1}{l_1+\cdots+l_{N+1}+i\epsilon}\right].\end{aligned}\tag{A1}$$

(Here we have dropped the superscript $+$ on the momenta to simplify the notation.) Now,

$$\begin{aligned}\sum_{m=0}^{N}\widetilde{E}_m^{N}(l_1,\ldots,l_N)\frac{1}{l_{m+1}+\cdots+l_{N+1}+i\epsilon}&=\sum_{m=0}^{N}\widetilde{E}_m^{N}(l_1,\ldots,l_N)\frac{1}{l_{N+1}+i\epsilon}\\&\quad+\sum_{m=0}^{N}\widetilde{E}_m^{N}(l_1,\ldots,l_N)\left[\frac{1}{l_{N+1}+i\epsilon}-\frac{1}{l_1+\cdots+l_{N+1}+i\epsilon}\right]\\&=-\sum_{m=0}^{N-1}\widetilde{E}_m^{N}(l_1,\ldots,l_N)\frac{l_{m+1}+\cdots+l_N}{l_1+\cdots+l_{N+1}+i\epsilon}\,\frac{1}{l_{N+1}+i\epsilon}\,,\end{aligned}\tag{A2}$$

where we have used (4.15) to obtain the last line of (A2), and have changed the upper limit of the sum from N to $N-1$, since the term with $m=N$ vanishes. Substituting (A2) into the right-hand side of (A1), we have

$$-\sum_{m=0}^{N}\tilde{E}_m^N(l_1,\ldots,l_N)\frac{l_{m+1}+\cdots+l_N}{l_{m+1}+\cdots+l_{N+1}+i\epsilon}\frac{1}{l_{N+1}+i\epsilon}+(-1)^N\left[\frac{1}{l_N+l_{N+1}+i\epsilon}\cdots\frac{1}{l_1+\cdots+l_{N+1}+i\epsilon}\right]\frac{1}{l_{N+1}+i\epsilon}$$
$$=-\frac{1}{l_{N+1}+i\epsilon}\sum_{m=0}^{N}\tilde{E}_m^N(l_1,l_2,\ldots,l_{N-1},l_N+l_{N+1})=0\ . \qquad \text{(A3)}$$

In the last line of (A3), we have again used (4.15). Now, (4.15) holds trivially for the case $N=1$, where it gives

$$\frac{1}{l_1+i\epsilon}-\frac{1}{l_1+i\epsilon}=0\ . \qquad \text{(A4)}$$

This completes the inductive argument.

[1]S. D. Drell and T. M. Yan, Phys. Rev. Lett. **25**, 316 (1970); J. L. Cardy and G. A. Winbow, Phys. Lett. **52B**, 95 (1974); C. E. DeTar, S. D. Ellis, and P. V. Landshoff, Nucl. Phys. **B87**, 176 (1975), have argued, using pre-QCD assumptions about the power behavior of parton amplitudes, that the effects of initial- and final-state spectator interactions cancel in the Drell-Yan cross section.

[2]H. D. Politzer, Nucl. Phys. **B129**, 301 (1977).

[3]Stephen B. Libby and George Sterman, Phys. Rev. D **18**, 3252 (1978); Phys. Lett. **78B**, 618 (1978); R. K. Ellis, H. Georgi, M. Machacek, H. D. Politzer, and G. C. Ross, Nucl. Phys. **B152**, 285 (1979); S. Gupta and A. H. Mueller, Phys. Rev. D **20**, 118 (1979); John Collins and George Sterman, Nucl. Phys. **B185**, 172 (1981).

[4]G. T. Bodwin, S. J. Brodsky, and G. P. Lepage, Phys. Rev. Lett. **47**, 1799 (1981); in *Particles and Fields,* Proceedings of the Banff Summer Institute, Banff, Canada, 1981, edited by A. N. Kamal and A. C. Capri (Plenum, New York, 1983), Vol. 2. A report on this work was also presented in *Perturbative Quantum Chromodynamics,* proceedings of the Tallahassee Conference, 1981, edited by D. W. Duke and J. F. Owens (AIP, New York, 1981); in *Multiparticle Dynamics 1982,* proceedings of the XIIIth International Symposium, edited by E. W. Kittel, W. Metzger, and A. Stergiou (World Scientific, Singapore, 1983).

[5]W. W. Lindsay, D. A. Ross, and C. T. Sachrajda, Nucl. Phys. **B214**, 61 (1983); **B222**, 189 (1983).

[6]W. W. Lindsay, Nucl. Phys. **B231**, 397 (1984).

[7]S. J. Brodsky, in *Stillwater Workshop,* proceedings of the Workshop on Nonperturbative QCD, Oklahoma State University edited by Kimball A. Milton and Mark A. Samuel (Birkhauser, Boston, Mass., 1983).

[8]J. C. Collins, D. E. Soper, and G. Sterman, Phys. Lett. **109B**, 388 (1982).

[9]J. C. Collins, D. E. Soper, and G. Sterman, Phys. Lett. **126B**, 275 (1983).

[10]J. J. Aubert *et al.*, Phys. Lett. **123B**, 275 (1983).

[11]While this manuscript was in preparation, I learned of an outline of an all-orders proof of factorization due to CSS. J. C. Collins, D. E. Soper, and G. Sterman, Phys. Lett. **134B**, 263 (1984); Rahul Basu, Anibal J. Ramalho, and George Sterman, Nucl. Phys. **B244**, 221 (1984), have demonstrated that certain subleading-twist contributions to the Drell-Yan cross section do not factor.

[12]P. V. Landshoff and W. J. Stirling, Z. Phys. C **14**, 251 (1982).

[13]J. Frenkel, J. G. M. Gatheral, and J. C. Taylor, Nucl. Phys. **B223**, 307 (1984), have also given arguments to show that the leading-twist contribution to the Drell-Yan cross section is infrared finite.

[14]J. C. Collins, D. E. Soper, and G. Sterman, Nucl. Phys. **B223**, 381 (1983).

[15]G. T. Bodwin, C. Y. Lo, J. D. Stack, and J. D. Sullivan, Phys. Lett. **92B**, 337 (1980); F. Khalafi, Z. Phys. C **18**, 57 (1983), and references therein.

[16]G. Sterman, Phys. Rev. D **17**, 2773 (1977), has given an analysis of soft and collinear mass singularities that includes a detailed discussion of power counting.

[17]G. 't Hooft and M. T. Veltman, Nucl. Phys. **B44**, 189 (1973).

[18]CSS have given a detailed analysis of one-loop factorization in Ref. 14. Here, we present a simplified discussion that contains only those ingredients that we need for the all-orders proof.

[19]G. Grammer, Jr. and D. R. Yennie, Phys. Rev. D **8**, 4332 (1973).

[20]R. E. Cutkosky, J. Math. Phys. **1**, 429 (1960); M. Veltman, Physica **29**, 186 (1963).

[21]G. 't Hooft, Nucl. Phys. **B33**, 173 (1971).

[22]I thank John Collins for suggesting an approach along these lines.

[23]G. Sterman (private communication).

[24]This result is also derived in S. Libby and G. Sterman, Phys. Rev. D **19**, 2468 (1979).