Derivative expansions of fermion determinants: Anomaly-induced vertices, Goldstone-Wilczek currents, and Skyrme terms

I. J. R. Aitchison and C. M. Fraser
Department of Theoretical Physics, University of Oxford, 1 Keble Road, Oxford, England OX1 3NP
(Received 9 October 1984)

An algebraic procedure for evaluating functional determinants in powers of derivatives of external fields is used to calculate one-fermion-loop contributions to low-energy effective Lagrangians involving anomaly-induced vertices, Goldstone-Wilczek currents, and Skyrme terms.

I. INTRODUCTION

Radiatively induced contributions to the effective action, which arise specifically from single-fermion loops, are of fundamental importance in many physical processes. Generally, phenomenological applications involve the expansion of the corresponding effective Lagrangians in powers of derivatives of the relevant fields—or equivalently, of the effective amplitudes in powers of momenta. In this paper we employ a general functional method, recently developed by one of us[1] (see also Ref. 2), to calculate such derivative expansions of fermion determinants.

We shall consider three characteristic types of application. The first involves chiral anomalies (which are, of course, associated with fermion loops[3]). We show in Sec. II how to rederive, in a very simple way, those effective interactions between bosons and vector or axial-vector fields [e.g., $\pi^0 \rightarrow \gamma\gamma$ (Ref. 4)], or among bosons alone [e.g., $K\bar{K} \rightarrow 3\pi$ (Ref. 5)], which summarize the low-energy consequences of anomalous Ward identities. Such "anomaly-induced low-energy theorems" form an important class of predictions based on the chiral properties of QCD.

As a second application, we consider the problem of calculating the vacuum fermion-number current in the presence of a background (soliton) field. In their remarkable paper, Goldstone and Wilczek[6] calculated this current by considering certain fermion-loop graphs, expanded in derivatives of the background field. It is, of course, simple to reformulate the problem in terms of an effective-action approach: one introduces a source $s_\mu(x)$ coupled to the fermion-current operator $\bar{\psi}(x)\gamma^\mu\psi(x)$, and retains the term in the (one-loop) connected vacuum functional which is linear in s_μ. This can then be expanded in derivatives of the background field and the Goldstone-Wilczek result obtained quite straightforwardly, as we show in Sec. III. The relation between the functional and graphical methods will also be touched on. The calculation can be easily generalized to include external gauge fields. We show, in some illustrative examples, how the expected[6–8] anomaly-induced contributions to the vacuum current can be efficiently computed by the same method.

The Goldstone-Wilczek[6] calculation provided, among other things, one justification for identifying the Skyrme[9] topological current with the baryon number current, when the soliton in question is a Skyrme soliton.[10] Our last fermion-loop application is to Skyrme-soliton physics. In Skyrme's model,[9] the stability of the soliton against collapse was ensured by the *ad hoc* addition of terms of order $(\partial\phi)^4$ to the (classical) nonlinear σ-model Lagrangian. Since the resulting Lagrangian is treated classically, it can be interpreted as an effective Lagrangian in the field-theory sense. Having, then, invoked a fermion-loop effect for the baryon-number-current result, it seems only consistent to determine its contribution to the $O((\partial\phi)^4)$ terms in the effective Lagrangian. In other words, the possibility exists that such radiative corrections might, of themselves, stabilize the soliton. We present this calculation, a brief version of which has already appeared,[11] in Sec. IV.

We end the present section by outlining the general procedure which we shall apply in all the above cases. We consider a fermion multiplet ψ interacting bilinearly with scalar fields ϕ, gauge fields A, and sources s (any internal-symmetry labels are, for the moment, suppressed). All fields except ψ will be regarded as external and treated classically, since only fermion loops are being considered. The Lagrangian then has the general form

$$\mathscr{L} = \bar{\psi}[i\partial\!\!\!/ - M(\phi, A, s)]\psi \ , \tag{1.1}$$

where the mass functional M depends on the external fields (and could include, of course, a conventional fermion mass). The connected vacuum functional $F(\phi, A, s)$ is given by $F = -i \ln Z(\phi, A, s)$, where

$$Z(\phi, A, s) = \int \mathscr{D}\bar{\psi}\, \mathscr{D}\psi \exp\left[i \int d^4x\, \bar{\psi}[i\partial\!\!\!/ - M(\phi, A, s)]\psi\right] . \tag{1.2}$$

Integrating over the fermion fields yields

$$Z = \det[p\!\!\!/ - M(\phi, A, s)] \ . \tag{1.3}$$

Since the fields ϕ, A, s are external, no Legendre transform is needed to pass from F to the effective action S_{eff}, and so

$$F = S_{\text{eff}} = -i\, \mathrm{Tr} \ln[p\!\!\!/ - M(\phi, A, s)] \ . \tag{1.4}$$

Formula (1.4) is entirely familiar, and undoubtedly represents the effects of all fermion loops in such theories,

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but as it stands it is purely formal and not useful, since it cannot, in general, be evaluated. The difficulty is that the quantities ϕ, A, and s (denoted generically by ϕ for the rest of this section) are all, of course, x dependent for the applications of interest. Thus the functional operations formally indicated in (1.4) can be performed neither in momentum nor in coordinate space. In particular, it is not clear how to manipulate (1.4) into the form

$$S_{\text{eff}} = \int d^4x \, \mathscr{L}_{\text{eff}} \tag{1.5}$$

so as to extract the quantity we are after, the effective Lagrangian.

Recently, a method has been developed[1] which solves this problem. For convenience, we include a brief description of it here. One sets $\phi(x) = \phi_0 + \tilde{\phi}(x)$, where ϕ_0 is a constant field, and expands (1.4) in powers of $\tilde{\phi}(x)$:

$$\begin{aligned} -i \operatorname{Tr} \ln[\not{p} - M(\phi)] &= -i \operatorname{Tr} \ln \left[(\not{p} - M_0) \left[1 - \frac{1}{\not{p} - M_0} \tilde{M} \right] \right] \\ &= -i \operatorname{Tr} \ln(\not{p} - M_0) + i \operatorname{Tr} \frac{1}{\not{p} - M_0} \tilde{M} + \frac{i}{2} \operatorname{Tr} \frac{1}{\not{p} - M_0} \tilde{M} \frac{1}{\not{p} - M_0} \tilde{M} + \cdots , \end{aligned} \tag{1.6}$$

where M_0 is the constant $M(\phi_0)$, and $\tilde{M}$ is $M(\phi) - M(\phi_0)$. There is still a problem with (1.6), apart from the first term, which is that the $\not{p}$ operators do not commute with the $\tilde{\phi}$'s, so that the x and p traces still cannot be disentangled. However, this can easily be overcome by repeated use of the identity

$$\tilde{\phi} \frac{1}{p^2 - M_0{}^2} = \frac{1}{p^2 - M_0{}^2} \tilde{\phi} + \frac{1}{(p^2 - M_0{}^2)^2} [p^2, \tilde{\phi}] + \frac{1}{(p^2 - M_0{}^2)^3} [p^2, [p^2, \tilde{\phi}]] + \cdots , \tag{1.7}$$

together with

$$[p^2, \tilde{\phi}] = \Box \tilde{\phi} + 2ip^\mu \partial_\mu \tilde{\phi} \tag{1.8}$$

and similar expressions for higher commutators. In this way each term in (1.6) can be written as a product of momentum operators on the left and functions of $\tilde{\phi}$, and the derivatives of $\tilde{\phi}$, on the right. A general term then has the form

$$\int d^4p \, F(p^2) \int d^4x \, g(\phi_0, \tilde{\phi}, \partial_\mu \tilde{\phi}) . \tag{1.9}$$

Thus when S_{eff} is expanded in powers of $\partial_\mu \phi$, it can be written as the integral of a local density L_{eff}, as in (1.5), even though the closed expression (1.4) cannot. We write this expansion, symbolically, as

$$\begin{aligned} S_{\text{eff}} &= \int d^4x [-V(\phi) + \tfrac{1}{2} Z(\phi) \partial_\mu \phi \partial^\mu \phi + \cdots] \\ &= \int d^4x \, \mathscr{L}_{\text{eff}} . \end{aligned} \tag{1.10}$$

By further expanding the coefficient functions $V, Z, \ldots$ around $\phi = \phi_0$ one obtains

$$S_{\text{eff}} = \int d^4x [-V(\phi_0) + \text{terms in } \phi_0, \tilde{\phi}, \partial_\mu \tilde{\phi}] . \tag{1.11}$$

By comparing coefficients in (1.11) and (1.6), the coefficient functions $V, Z, \ldots$ in L_{eff} can be determined by a straightforward—indeed mechanical—algebraic procedure. The first term $V(\phi_0)$ is clearly given by $i \operatorname{tr} \ln(\not{p} - M_0)$, which is just the well-known single-fermion-loop expression for the effective potential.[12] The remaining terms provide precisely the effective interactions which are required in low-energy applications. We remark, incidentally, that since all such vertices are *finite*, traditional subtleties associated with the regularization of anomalous graphs are bypassed in our approach. Finally, we note that the method can, of course, be applied to boson loops also; we consider fermion loops here mainly because of their special role in anomalies, and anomaly-related effects.

II. ANOMALY-INDUCED VERTICES

As our first illustration of the use of fermion determinants, we calculate the $\pi^0 \to 2\gamma$ amplitude, starting from the gauged linear σ model. The Lagrangian is

$$\begin{aligned} \mathscr{L} &= \bar{\psi}(i\not{\partial} - eQ\not{A})\psi - g\bar{\psi}(\sigma + i\boldsymbol{\tau}\cdot\boldsymbol{\pi}\gamma_5)\psi \\ &\quad + \tfrac{1}{2}(\partial_\mu \phi_a)^2 + \tfrac{1}{2}\mu^2 \phi^2 - \tfrac{1}{4}\lambda \phi^4 , \end{aligned} \tag{2.1}$$

where ψ is a massless two-component fermion field (which we take to be the nucleon doublet; one could also use three colored quark doublets); $\phi_a = (\sigma, \boldsymbol{\pi})$ is a four-component meson field; $\phi^2 = \sigma^2 + \boldsymbol{\pi}^2$; and

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

is the charge matrix of the nucleon doublet. With the sign of μ^2 as shown, the SU(2)×SU(2) symmetry is spontaneously broken; we therefore shift the σ field by its vacuum expectation value f, whereupon, writing $\tilde{\sigma} = \sigma - f$, the fermion sector of the Lagrangian then has the form (1.1) with

$$M = m + g\tilde{\sigma} + ig\boldsymbol{\tau}\cdot\boldsymbol{\pi}\gamma_5 + eQ\not{A} , \tag{2.2}$$

where $m = gf$. Hence our expression (1.4) for the one-fermion-loop effective action becomes

$$S_{\text{eff}} = -i \operatorname{Tr} \ln(\not{p} - eQ\not{A} - m - g\tilde{\sigma} - ig\boldsymbol{\tau}\cdot\boldsymbol{\pi}\gamma_5) . \tag{2.3}$$

Equation (2.3) incorporates *all* one-fermion-loop effects; it must therefore include all effects due to anomalies, as these are known to be unmodified by higher-loop corrections.[3,4] Thus to calculate the $\pi^0 \to 2\gamma$ amplitude, all we have to do is expand (2.3) in powers of the fields $\tilde{\sigma}$, $\boldsymbol{\pi}$, and A_μ and their derivatives, and pick out the appropriate terms. It turns out that this is a rather easy calculation. It may seem surprising that this anomaly-induced term can be deduced in a straightforward way from (2.3); but recall that anomalies have traditionally been interpreted as arising from the fact that any gauge-invariant evaluation of quantum corrections will violate the chiral symmetry present in the classical Lagrangian. $S_{\rm eff}$ in (2.3) is explicitly gauge invariant, and this must guarantee the appearance of the anomalous terms in the one-loop action.

To calculate the $\pi^0 \rightarrow 2\gamma$ amplitude, we need the term in the expansion of (2.3) about $\tilde{\sigma}=\pi=A_\mu=0$ of second order in the photon field A_μ and first order in the fields $\tilde{\sigma},\pi$. This term is

$$i\,{\rm Tr}\frac{1}{\not p - m}eQ\not A\frac{1}{\not p - m}eQ\not A\frac{1}{\not p - m}g(\tilde{\sigma}+i\tau\cdot\pi\gamma_5)\ . \qquad (2.4)$$

When we perform the SU(2) trace, this picks out the terms in π_3, the π^0 field, and $\tilde{\sigma}$; we drop the term in $\tilde{\sigma}$, as it is not of interest to us, and we are left with

$$-e^2 g\,{\rm Tr}\frac{1}{p^2-m^2}(\not p+m)\not A\frac{1}{p^2-m^2}(\not p+m)\not A\frac{1}{p^2-m^2}(\not p+m)\gamma_5\pi_3\ . \qquad (2.5)$$

Performing the Dirac trace then gives

$$-4ie^2 gm\,\epsilon_{\mu\nu\lambda\rho}{\rm Tr}\frac{1}{p^2-m^2}\left[p^\mu A^\nu\frac{1}{p^2-m^2}p^\lambda A^\rho+p^\mu A^\nu\frac{1}{p^2-m^2}A^\lambda p^\rho+A^\mu\frac{1}{p^2-m^2}p^\nu A^\lambda p^\rho\right]\frac{1}{p^2-m^2}\pi_3\ , \qquad (2.6)$$

where our convention for $\epsilon_{\mu\nu\lambda\rho}$ is that $\epsilon_{0123}=+1$. We move the p's in the numerators of (2.6) to the left, using the relation

$$f(x)p^\lambda = p^\lambda f(x) - i\partial^\lambda f(x) \qquad (2.7)$$

and find that (2.6) reduces to

$$4ie^2 gm\,\epsilon_{\mu\nu\lambda\rho}{\rm Tr}\frac{1}{p^2-m^2}\partial^\mu A^\nu\frac{1}{p^2-m^2}\partial^\lambda A^\rho\frac{1}{p^2-m^2}\pi_3\ . \qquad (2.8)$$

To find the low-energy limit of the $\pi^0 \rightarrow 2\gamma$ amplitude, which is of second order in momenta, we do not even need to use (1.7) as (2.8) is already of the required order in derivatives of the field. The term we want in $S_{\rm eff}$ is simply

$$4ie^2 gm\,\epsilon_{\mu\nu\lambda\rho}\int d^4x\,\partial^\mu A^\nu\partial^\lambda A^\rho\pi_3\int\frac{d^4p}{(2\pi)^4}\frac{1}{(p^2-m^2)^3}=\frac{e^2}{8\pi^2 f}\int d^4x\,\epsilon_{\mu\nu\lambda\rho}\partial^\mu A^\nu\partial^\lambda A^\rho\pi_3\ , \qquad (2.9)$$

from which we deduce that the amplitude is

$$\frac{e^2}{4\pi^2 f}\epsilon_{\mu\nu\lambda\rho}k_1^\mu k_2^\nu\epsilon_1^\lambda\epsilon_2^\rho\ , \qquad (2.10)$$

where $\epsilon_{i\mu}$, $k_{i\mu}$ are the polarization vector and momentum of photon i. Equation (2.10) agrees with the result given in Ref. 4.

Our second example is a calculation of the five-pseudoscalar vertex in the nonlinear SU(3)×SU(3) chiral model with the pseudoscalar octet coupled to fermions. We will not introduce gauge fields yet (their effects will be considered below), and so at first sight the connection of this vertex with anomalies is rather mysterious. Nevertheless, it has been shown by Wess and Zumino[5] that such a purely pseudoscalar term must appear in the effective action of any model containing pseudoscalars coupled to vector and axial-vector fields in which anomalies appear. Starting from the Ward identities for the divergence of the vector and axial-vector currents, Wess and Zumino derive a functional differential equation for the part of the effective action which summarizes all effects of anomalies—the so-called Wess-Zumino action—and give an explicit solution which [like $S_{\rm eff}$, Eq. (1.4)] cannot itself be written as the integral of some local density, though its expansion in powers of derivatives of the fields can. The Wess-Zumino action is model independent in the sense that, up to an overall multiplicative constant, it does not depend on the details of the fermion sector of the model, but only the structure of the gauge group. Recently Witten[7] has given a geometrical derivation of the Wess-Zumino action for the case of the purely pseudoscalar SU(3)×SU(3) nonlinear chiral model, and has shown that its coefficient is uniquely determined up to an integer of topological origin, which he identifies with the number of colors. In this respect the purely hadronic term in the effective action, which is sensitive to color alone, is in principle independent of the $\pi^0\rightarrow\gamma\gamma$ term, which involves the assumed fermion charges.

Since the Wess-Zumino action is, after all, simply part of the one-fermion-loop contribution to the effective action, it must be possible to calculate its expansion in powers of field derivatives, for any given model, using the methods described in Sec. I. Here, we shall calculate the low-energy contribution to the five-pseudoscalar term in the effective Lagrangian, taking the tree-level fermion-pseudoscalar interaction Lagrangian to be[13]

$$\mathcal{L}=\bar{\psi}(i\not\partial-\mu P)\psi\ , \qquad (2.11)$$

where

$$P=\exp\left[\frac{2i}{f_\pi}\Pi\gamma_5\right],\quad \Pi=\sum_{a=1}^{8}\lambda^a\pi^a\ ,$$

the λ^a are the generators of SU(3), and π^a are the fields of the pseudoscalar octet, transforming nonlinearly under SU(3)× SU(3); ψ is a fermion triplet; and μ is a fermion mass, which will not appear in our final answer. McKay and Munczek[13] have recently calculated this vertex, using graphical methods; they have also considered the effects of adding extra interaction terms to (2.11) and have found that, as expected, these leave the vertex unaffected.

We expand

$$-i\,\mathrm{Tr}\ln[\not p-\mu-\mu(P-1)] \tag{2.12}$$

about $\pi^a=0$. The term of interest to us is

$$\frac{i}{5}\mathrm{Tr}\left[\frac{1}{\not p-m}\mu(P-1)\right]^5 . \tag{2.13}$$

Writing

$$P=1+\frac{2i}{f_\pi}\Pi\gamma_5+\cdots , \tag{2.14}$$

Eq. (2.13) becomes

$$-\frac{32\mu^5}{5f_\pi^{\,5}}\mathrm{Tr}\left[\frac{1}{\not p-m}\Pi\gamma_5\right]^5 . \tag{2.15}$$

Performing the Dirac trace, and using the cyclic property of the trace, we find that (2.15) becomes

$$-\frac{128i\mu^6}{f_\pi^{\,5}}\epsilon_{\mu\nu\lambda\rho}\mathrm{Tr}\frac{p^\mu}{p^2-\mu^2}\Pi\frac{p^\nu}{p^2-\mu^2}\Pi\frac{p^\lambda}{p^2-\mu^2}\Pi\frac{p^\rho}{p^2-\mu^2}\Pi\frac{1}{p^2-\mu^2}\Pi . \tag{2.16}$$

We pull the p's in the numerator through to the left, using (2.7); note that, because of the antisymmetry of $\epsilon_{\mu\nu\lambda\rho}$, all terms of the form $p^\mu p^\nu\Pi$ vanish immediately. Equation (2.16) then becomes

$$\frac{128\mu^6}{f_\pi^{\,5}}\epsilon_{\mu\nu\lambda\rho}\mathrm{Tr}\frac{p^\mu}{p^2-\mu^2}\partial^\nu\Pi\frac{1}{p^2-\mu^2}\partial^\lambda\Pi\frac{1}{p^2-\mu^2}\partial^\rho\Pi\frac{1}{p^2-\mu^2}\Pi\frac{1}{p^2-\mu^2}\Pi . \tag{2.17}$$

We now move all functions of x to the right, using (1.7), to obtain the term of order $(\partial\Pi)^4$:

$$\frac{256i\mu^6}{f_\pi^{\,5}}\int d^4x\,\epsilon_{\mu\nu\lambda\rho}\mathrm{Tr}(\partial^\nu\Pi\partial^\lambda\Pi\partial^\rho\Pi\partial_\alpha\Pi\Pi)\int\frac{d^4p}{(2\pi)^4}\frac{p^\mu p^\alpha}{(p^2-\mu^2)^6}=-\frac{2}{15\pi^2 f_\pi^{\,5}}\epsilon_{\mu\nu\lambda\rho}\int d^4x\,\mathrm{Tr}(\partial^\mu\Pi\partial^\nu\Pi\partial^\lambda\Pi\partial^\rho\Pi\Pi) . \tag{2.18}$$

[The trace in (2.18) is over SU(3) indices.] Equation (2.18) agrees with the result given in Ref. 13, and with the model-independent calculation of Witten;[7] note that if there are N_c fermion colors, Eq. (2.18) will simply be multiplied by N_c.

We now consider the extra low-energy anomalous terms which appear in the effective action when electromagnetic interactions are included in the SU(3)×SU(3) model of (2.11). Witten[7] has derived "by trial and error" an expression for the Wess-Zumino action of a pseudoscalar octet coupled to the electromagnetic field: it is simply a gauge-invariant extension of the purely pseudoscalar Wess-Zumino action. Our result is in agreement with his, when expanded in powers of Π. The mass functional is now

$$M=\mu P+eQ\not A , \tag{2.19}$$

where—taking the fermions to be quarks—the charge matrix Q is

$$\begin{bmatrix}\frac{2}{3} & & \\ & -\frac{1}{3} & \\ & & -\frac{1}{3}\end{bmatrix} .$$

Witten finds two extra terms in the Wess-Zumino action, which, to lowest order in the pseudoscalar fields π, are of the form $\epsilon_{\mu\nu\lambda\rho}A^\mu\partial^\nu\Pi\partial^\lambda\Pi\partial^\rho\Pi$ and $\epsilon_{\mu\nu\lambda\rho}(\partial^\nu A^\lambda)A^\rho\partial^\mu\Pi$; the corresponding terms in the expansion of

$$S_{\mathrm{eff}}(\Pi,A_\mu)=-i\,\mathrm{Tr}\ln(\not p-\mu P-eQ\not A) \tag{2.20}$$

about $\Pi=A_\mu=0$ are

$$\frac{8\mu^3 e}{f_\pi^{\,3}}\mathrm{Tr}\frac{1}{\not p-\mu}\Pi\gamma_5\frac{1}{\not p-\mu}\Pi\gamma_5\frac{1}{\not p-\mu}\Pi\gamma_5\frac{1}{\not p-\mu}Q\not A \tag{2.21}$$

and

$$\frac{2\mu e^2}{f_\pi}\mathrm{Tr}\frac{1}{\not p-\mu}\Pi\gamma_5\frac{1}{\not p-\mu}Q\not A\frac{1}{\not p-\mu}Q\not A . \tag{2.22}$$

When the Dirac traces are performed, and all p^μ's in the numerators moved to the left, Eqs. (2.21) and (2.22) become

$$\frac{32e\mu^4}{f_\pi^{\,3}}\epsilon_{\mu\nu\lambda\rho}\mathrm{Tr}\frac{1}{p^2-\mu^2}\partial^\mu\Pi\frac{1}{p^2-\mu^2}\partial^\nu\Pi\frac{1}{p^2-\mu^2}\partial^\lambda\Pi\frac{1}{p^2-\mu^2}QA^\rho \tag{2.23}$$

and

$$-\frac{8i\mu^2e^2}{f_\pi}\epsilon_{\mu\nu\lambda\rho}\mathrm{Tr}\frac{1}{p^2-\mu^2}\partial^\mu\Pi\frac{1}{p^2-\mu^2}Q\partial^\lambda A^\nu\frac{1}{p^2-\mu^2}QA^\rho . \tag{2.24}$$

We already have the required number of derivatives in the numerators of (2.23) and (2.24), so (1.7) need not be used. Performing the momentum-space integrals, we find that the new terms in the Wess-Zumino action are

$$\int d^4x \left[-\frac{ie}{3f_\pi{}^3}\epsilon_{\mu\nu\lambda\rho}\,\mathrm{Tr}[Q\partial^\nu\Pi\partial^\lambda\Pi\partial^\rho\Pi]A^\mu - \frac{e^2}{4f_\pi}\epsilon_{\mu\nu\lambda\rho}(\partial^\lambda A^\nu)A^\rho\,\mathrm{Tr}(Q^2\partial^\mu\Pi) \right] \tag{2.25}$$

in agreement with Eq. (19) of Ref. 7, when only terms of lowest order in π are retained in the latter. As Witten points out, the $\pi^0\rightarrow 2\gamma$ amplitude can be extracted from the second term in (2.25).

Finally, we remark that the method we have used here to calculate low-energy terms in the Wess-Zumino action can also be used to derive several of the results appearing in recent papers[14] on (2+1)-dimensional axial anomalies, in a very straightforward way.

III. VACUUM FERMION CURRENTS IN THE PRESENCE OF EXTERNAL FIELDS

We consider the problem of calculating the vacuum expectation value of a fermion current, of generic type $j_\mu(x)=\bar\psi(x)\gamma_\mu\psi(x)$, in the presence of background fields ϕ (which may include gauge fields):

$$\langle j_\mu(x)\rangle_\phi = \frac{1}{Z(\phi)}\int \mathscr{D}\bar\psi\,\mathscr{D}\psi\,\bar\psi\gamma_\mu\psi \exp\left[i\int d^4x\,\mathscr{L}(\psi,\bar\psi,\phi)\right], \tag{3.1}$$

where again

$$\mathscr{L}(\psi,\bar\psi,\phi)=\bar\psi[i\partial\!\!\!/ - M(\phi)]\psi \tag{3.2}$$

and

$$Z(\phi)=\int \mathscr{D}\bar\psi\,\mathscr{D}\psi \exp\left[i\int d^4x\,\mathscr{L}(\psi,\bar\psi,\phi)\right]. \tag{3.3}$$

A convenient expression for $\langle j_\mu(x)\rangle_\phi$ can, as usual, be obtained by introducing a source $s^\mu(x)$ coupled to $j_\mu(x)$:

$$\langle j_\mu(x)\rangle_\phi = \frac{\delta}{\delta s^\mu(x)} S_{\mathrm{eff}}(\phi,s)\bigg|_{s=0}, \tag{3.4}$$

where $S_{\mathrm{eff}}(\phi,s)=-i\ln Z(\phi,s)$, and

$$Z(\phi,s)=\int \mathscr{D}\bar\psi\,\mathscr{D}\psi \exp\left[i\int d^4x\,\bar\psi[i\partial\!\!\!/ + s\!\!\!/ - M(\phi)]\psi\right]. \tag{3.5}$$

We therefore need to calculate

$$S_{\mathrm{eff}}(\phi,s\!\!\!/)=-i\,\mathrm{Tr}\ln[p\!\!\!/ + s\!\!\!/ - M(\phi)] \tag{3.6}$$

expanded to first order in s:

$$S_{\mathrm{eff}}^{(1)}(\phi,s) = -i\,\mathrm{Tr}\frac{1}{p\!\!\!/ - M(\phi)}s\!\!\!/ \tag{3.7}$$

$$= \int d^4x\,\langle j_\mu(x)\rangle_\phi s^\mu(x). \tag{3.8}$$

Our notation emphasizes that the vacuum current $\langle j_\mu(x)\rangle_\phi$ depends on the external fields $\phi(x)$. Following the approach outlined in Sec. I [cf. Eq. (1.10)], we expand the current in powers of $\partial^\mu\phi(x)$:

$$\langle j_\mu(x)\rangle_\phi = \cdots + S(\phi^2)\epsilon_{\mu\nu\lambda\rho}\epsilon_{abcd}\phi_a(x)\partial^\nu\phi_b\partial^\lambda\phi_c\partial^\rho\phi_d + \cdots, \tag{3.9}$$

where we have singled out a term of Goldstone-Wilczek form (see below) relevant to the case of a pseudoscalar field ϕ. By comparing the expansions of both (3.7) and (3.8) about $\phi=\phi_0$, as before, we can obtain the coefficient functions $S(\phi),\ldots$.

One small technicality deserves comment. The expansion of (3.7) about $\phi=\phi_0$ proceeds via

$$S_{\mathrm{eff}}^{(1)}(\phi,s) = -i\,\mathrm{Tr}(\{(p\!\!\!/ - M_0)[1-(p\!\!\!/ - M_0)^{-1}\widetilde{M}]\}^{-1}s\!\!\!/)$$

$$= -i\left[\mathrm{Tr}\frac{1}{p\!\!\!/ - M_0}s\!\!\!/ + \mathrm{Tr}\frac{1}{p\!\!\!/ - M_0}\widetilde{M}\frac{1}{p\!\!\!/ - M_0}s\!\!\!/ + \mathrm{Tr}\left[\frac{1}{p\!\!\!/ - M_0}\widetilde{M}\right]^2\frac{1}{p\!\!\!/ - M_0}s\!\!\!/ + \cdots\right], \tag{3.10}$$

where $M_0=M(\phi_0)$. As explained in Sec. I, such an expression is not really meaningful until the x and p traces have been separated: only then can it be compared with the corresponding expansion of (3.8). Nevertheless, it is permissible to perform on (3.7) the functional differentiation indicated in (3.4), so as to obtain the formal expression

$$\langle j_\mu(y)\rangle_\phi = -i\,\mathrm{Tr}\frac{1}{p\!\!\!/ - M(\phi)}\gamma_\mu\delta(x-y). \tag{3.11}$$

Expanding (3.11) about ϕ_0 yields

$$\langle j_\mu(y)\rangle_\phi = -i\left[\mathrm{Tr}\frac{1}{\not p - M_0}\gamma_\mu\delta(x-y) + \mathrm{Tr}\frac{1}{\not p - M_0}\widetilde{M}\frac{1}{\not p - M_0}\gamma_\mu\delta(x-y) + \mathrm{Tr}\frac{1}{\not p - M_0}\widetilde{M}\frac{1}{\not p - M_0}\widetilde{M}\frac{1}{\not p - M_0}\gamma_\mu\delta(x-y) + \cdots\right], \tag{3.12}$$

which is, indeed, the appropriate derivative of (3.10), and which may be directly compared with (3.9). The notation in (3.11) and (3.12) implies that Tr includes the integral $\int d^4x$. In using (3.12), one must obviously be careful about cyclically reordering the factors in the traces, and then commuting the p's to the left [so as to end up with the desired form (1.9)]. In this process, derivatives of the δ function must not be omitted: they correspond, of course, to commutation of p operators with $\not x$ in (3.10). We shall find it most convenient simply to retain the δ functions at the right of all expressions, and not permute the factors in the traces. Alternatively one can, of course, retreat to (3.10).

We proceed with the calculation of the Goldstone-Wilczek[6] fermion current, for the case of a background field $\phi_a \equiv (\sigma,\boldsymbol{\pi})$ ($a=0$ to 3) transforming as the four-dimensional representation of the chiral SU(2)$\times$SU(2) symmetry of the SU(2) σ model. The mass functional is taken to be

$$M(\phi) = g\sigma + ig\boldsymbol{\tau}\cdot\boldsymbol{\pi}\gamma_5 \tag{3.13}$$

and ψ is an isospinor. The leading term of $\langle j_\mu(y)\rangle_\phi$ expanded in powers of $\partial_\mu\phi$ is the SU(2)$\times$SU(2)-invariant Lorentz four-vector

$$S(\phi^2)\epsilon_{\mu\nu\lambda\rho}\epsilon_{abcd}\phi_a(y)\partial^\nu\phi_b(y)\partial^\lambda\phi_c(y)\partial^\rho\phi_d(y)\ . \tag{3.14}$$

The form (3.14) is dictated by symmetry principles. The fermion current is an isoscalar, and has the quantum numbers of the ω. Isospin and/or G parity forbid $\omega\to(\sigma,\pi)$ and $\omega\to\sigma\sigma$, $\pi\pi$, or $\sigma\pi$; the first coupling they allow is $\omega\to 3\pi$. Since the π's are pseudoscalars, momenta (or derivatives) must be introduced and coupled to form a (Lorentz) axial vector, so that the complete current is an ordinary vector. This leads to the $\epsilon_{\mu\nu\lambda\rho}$ product in (3.14). O(4) invariance then implies the ϵ_{abcd} product in (3.14), which is correctly Bose symmetric.

We now expand (3.14) about $\phi=\phi_0$. The leading term in powers of $\partial\widetilde{\phi}$ is

$$S(\phi_0^2)\epsilon_{\mu\nu\lambda\rho}\epsilon_{abcd}\phi_{0a}\partial^\nu\widetilde{\phi}_b\partial^\lambda\widetilde{\phi}_c\partial^\rho\widetilde{\phi}_d\ . \tag{3.15}$$

This arises uniquely from the term

$$-i\,\mathrm{Tr}\frac{1}{\not p - M_0}\widetilde{M}\frac{1}{\not p - M_0}\widetilde{M}\frac{1}{\not p - M_0}\widetilde{M}\frac{1}{\not p - M_0}\gamma_\mu\delta(x-y) \tag{3.16}$$

in (3.12). Using SU(2)$\times$SU(2) invariance to choose $\phi_{0a}=(\sigma_0,\mathbf{0})$, and isolating the terms in (3.16) which produce $\epsilon_{\mu\nu\lambda\rho}$ from the Dirac trace, Eq. (3.16) reduces to

$$8g^4\sigma_0\epsilon_{ijk}\epsilon_{\mu\nu\lambda\rho}\left[-\mathrm{Tr}\frac{1}{X}p^\nu\widetilde{\pi}_i\frac{1}{X}p^\lambda\widetilde{\pi}_j\frac{1}{X}p^\rho\widetilde{\pi}_k\frac{1}{X}\delta(x-y) + \mathrm{Tr}\frac{1}{X}p^\nu\widetilde{\pi}_i\frac{1}{X}p^\lambda\widetilde{\pi}_j\frac{1}{X}\widetilde{\pi}_k\frac{1}{X}p^\rho\delta(x-y) - \mathrm{Tr}\frac{1}{X}p^\nu\widetilde{\pi}_i\frac{1}{X}\widetilde{\pi}_j\frac{1}{X}p^\lambda\widetilde{\pi}_k\frac{1}{X}p^\rho\delta(x-y) + \mathrm{Tr}\frac{1}{X}\widetilde{\pi}_i\frac{1}{X}p^\nu\widetilde{\pi}_j\frac{1}{X}p^\lambda\widetilde{\pi}_k\frac{1}{X}p^\rho\delta(x-y)\right], \tag{3.17}$$

where $X=p^2-m^2$, with (as before) $m^2=g^2\phi_0^2$. Remarkably enough, one finds that (3.17) reduces [restoring manifest SU(2)$\times$SU(2) symmetry] to

$$8ig^4\epsilon_{abcd}\epsilon_{\mu\nu\lambda\rho}\phi_{0a}\,\mathrm{Tr}\frac{1}{X}\partial^\nu\widetilde{\phi}_b\frac{1}{X}\partial^\lambda\widetilde{\phi}_c\frac{1}{X}\partial^\rho\widetilde{\phi}_d\frac{1}{X}\delta(x-y)\ , \tag{3.18}$$

where our convention for ϵ_{abcd} is that $\epsilon_{0ijk}=\epsilon_{ijk}$. Since we only need the term $O((\partial\widetilde{\phi})^3)$ in order to collect $S(\phi_0^2)$, we may at once replace (3.18) by

$$8ig^4\epsilon_{abcd}\epsilon_{\mu\nu\lambda\rho}\phi_{0a}\int d^4x\,\partial^\nu\widetilde{\phi}_b\partial^\lambda\widetilde{\phi}_c\partial^\rho\widetilde{\phi}_d\delta(x-y)\int\frac{d^4p}{(2\pi)^4}\frac{1}{(p^2-m^2)^4} = -\frac{1}{12\pi^2\phi_0^4}\epsilon_{abcd}\epsilon_{\mu\nu\lambda\rho}\phi_{0a}\partial^\nu\widetilde{\phi}_b\partial^\lambda\widetilde{\phi}_c\partial^\rho\widetilde{\phi}_d\ , \tag{3.19}$$

whence, comparing with (3.15), we obtain

$$S(\phi_0^2) = -\frac{1}{12\pi^2\phi_0^4}\ . \tag{3.20}$$

It should be clear from the fact that (3.15) is an *expansion* of (3.14) that we can replace ϕ_0 in (3.20) by $\phi(x)$ to obtain

$$\langle j_\mu \rangle_\phi = -\frac{1}{12\pi^2\phi^4}\epsilon_{abcd}\epsilon_{\mu\nu\lambda\rho}\phi_a\partial^\nu\phi_b\partial^\lambda\phi_c\partial^\rho\phi_d \ , \tag{3.21}$$

which is the Goldstone-Wilczek result for the linear SU(2)×SU(2) σ model.

We have presented this calculation in terms of a direct evaluation of (3.11). As remarked above, however, we could also have used (3.4), retaining the part of S_{eff} which is linear in s^μ and has the tensorial structure indicated in (3.14). This is precisely an anomaly term of the type considered in the previous section. By way of varying the example, we note that Witten's expression[7] for the Goldstone-Wilczek current in nonlinear SU(3)×SU(3) [e.g., Eq. (29) of Ref. 7] drops out very easily in this way: comparing (3.6) with (2.20), we see that $\langle j_\mu \rangle_\Pi$ can immediately be obtained from (2.25) by replacing $-eQA\!\!\!/$ by $s\!\!\!/$. We find

$$\langle j_\mu \rangle_\Pi = \frac{i}{3f_\pi^{\,3}}\epsilon_{\mu\nu\lambda\rho}\mathrm{Tr}(\partial^\nu\Pi\partial^\lambda\Pi\partial^\rho\Pi) \ , \tag{3.22}$$

as expected.[7]

Though this paper advocates a functional approach to the evaluation of fermion-loop effects, we should like at this point to indicate the connection with the graphical method used by Goldstone and Wilczek[6] (see Ref. 1 for further comments on the relation between the two approaches). Clearly, this entails a momentum-space formulation. We expand the one-loop effective action $S_{\text{eff}}^{(1)}(\phi,s)$ in powers of ϕ about ϕ_0, and in s_μ about $s_\mu=0$— retaining only the term linear in s_μ:

$$S_{\text{eff}}^{(1)}(\phi,s) = -i\sum_n \frac{1}{n!}\int \frac{d^4p_1}{(2\pi)^4}\cdots\frac{d^4p_n}{(2\pi)^4}\frac{d^4k}{(2\pi)^4}(2\pi)^4\delta\left[k+\sum_{i=1}^n p_i\right]\Gamma_\mu^{(n)}(-p_1,\ldots,-p_n,-k;\phi_0)\tilde{\phi}(p_1)\cdots\tilde{\phi}(p_n)s^\mu(k) \ , \tag{3.23}$$

where $\Gamma_\mu^{(n)}(p_1,\ldots,p_n,k;\phi_0)$ (see Fig. 1) is the fermion loop, in momentum space, with one external s line and n external ϕ lines (internal indices being understood), evaluated using "shifted"[1,12] Feynman rules; p_i, k are ingoing momenta. Actually, since the interactions we are considering are linear in ϕ, the effect of the shift is trivial, and amounts simply to using the propagator $i(p\!\!\!/-M_0)^{-1}$ and interaction vertex $\tilde{M}$. Comparing (2.32) with (3.8) we deduce

$$\langle j_\mu(x)\rangle_\phi = \int\frac{d^4k}{(2\pi)^4}e^{ik\cdot x}\Bigg[-i\sum_n\frac{1}{n!}\int\frac{d^4p_1}{(2\pi)^4}\cdots\frac{d^4p_n}{(2\pi)^4}(2\pi)^4\delta\left[k+\sum_{i=1}^n p_i\right] \times\Gamma_\mu^{(n)}(-p_1,\ldots,-p_n,-k;\phi_0)\tilde{\phi}(p_1)\cdots\tilde{\phi}(p_n)\Bigg] \ . \tag{3.24}$$

Expansion in powers of $\partial_\mu\tilde{\phi}$ corresponds, of course, to a momentum expansion of $\Gamma_\mu^{(n)}$. The first nonvanishing contribution comes from $\Gamma_\mu^{(3)}$ and, as we shall see, has the expected Lorentz structure $\epsilon_{\mu\nu\lambda\rho}p_1^\nu p_2^\lambda p_3^\rho$ [cf. Eq. (3.15)]. Reinstating now the SU(2)×SU(2) index, and taking $\phi_{0a}=(\sigma_0,\mathbf{0})$ as before, $\Gamma_\mu^{(3)}$ is (for a given ordering of the external lines)

$$-\mathrm{Tr}\left[\int\frac{d^4p}{(2\pi)^4}i\gamma_\mu\frac{(p\!\!\!/+g\sigma_0)}{p^2-m^2}(g\tau_i\gamma_5,-ig)\frac{(p\!\!\!/+p\!\!\!/_1+g\sigma_0)}{(p+p_1)^2-m^2}(g\tau_j\gamma_5,-ig)\frac{(p\!\!\!/+q\!\!\!/+g\sigma_0)}{(p+q)^2-m^2}(g\tau_k\gamma_5,-ig)\frac{(p\!\!\!/+r\!\!\!/+g\sigma_0)}{(p+r)^2-m^2}\right] , \tag{3.25}$$

where $q=p_1+p_2$, $r=p_1+p_2+p_3$, and the trace is over internal and Dirac indices; the notation $(g\tau_i\gamma_5, -ig)$, for example, means that the vertex $g\tau_i\gamma_5$ is taken for an external π_i line, and $-ig$ for an external σ line. Performing the internal and Dirac traces, Eq. (3.25) reduces to

$$8i\epsilon_{ijk}g^4\sigma_0\epsilon_{\mu\nu\lambda\rho}\int\frac{d^4p}{(2\pi)^4}\frac{1}{p^2-m^2}\frac{1}{(p+p_1)^2-m^2}\frac{1}{(p+q)^2-m^2}\frac{1}{(p+r)^2-m^2}p_1^\nu p_2^\lambda p_3^\rho \ . \tag{3.26}$$

Dropping the p_i from the denominators, we obtain for the leading term

$$\Gamma_\mu^{(3)} = -\frac{1}{12\pi^2}\frac{\sigma_0}{\phi_0^{\,4}}\epsilon_{ijk}\epsilon_{\mu\nu\lambda\rho}p_1^\nu p_2^\lambda p_3^\rho \ . \tag{3.27}$$

There are six such terms with different orderings of the ϕ lines; inserting this into (3.24) and performing the integrations yields the leading contribution to $\langle j_\mu(x)\rangle_\phi$:

$$\langle j_\mu(x)\rangle_\phi = -\frac{1}{12\pi^2}\frac{\sigma_0}{\phi_0^{\,4}}\epsilon_{ijk}\epsilon_{\mu\nu\lambda\rho}\partial^\nu\tilde{\pi}_i\partial^\lambda\tilde{\pi}_j\partial^\rho\tilde{\pi}_k \ , \tag{3.28}$$

whence [cf. Eq. (3.15)] we recover

$$S(\phi_0^{\,2}) = -\frac{1}{12\pi^2\phi_0^{\,4}} \tag{3.29}$$

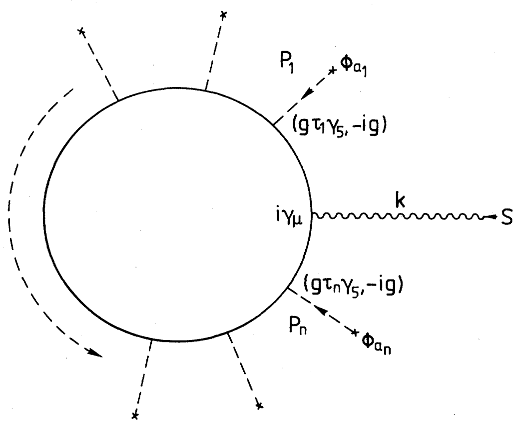

FIG. 1. The vertex $\Gamma_\mu^{(n)}(p_1,\ldots,p_n,k;\,\phi_0)$.

as before, Eq. (3.20). In general, it is clear that if the coefficient of $\epsilon_{\mu\nu\lambda\rho}p_1^\nu p_2^\lambda p_3^\rho$ in $\Gamma_\mu^{(3)}$ is written as S_{bcd} (where the internal index b goes with momentum p_1, etc.), then

$$S(\phi_0{}^2)\phi_{0a}\epsilon_{abcd}=S_{bcd}\ . \tag{3.30}$$

In this approach, it is the use of shifted Feynman rules which corresponds to the expansion of (3.14) about $\phi=\phi_0$ and allows identification of (3.28) with (3.15)—and thence, via the connection between (3.14) and (3.15), the reconstruction of the complete current (3.21).

We may also obtain the Goldstone-Wilczek result[6] for the vacuum current for the case in which the chiral symmetry is gauged. We introduce two isotriplets of vector and axial-vector gauge potentials, V_i^μ and A_i^μ [i is the SU(2), and μ the Lorentz, index]; ordinary derivatives then become covariant

$$\partial^\mu\to D^\mu\equiv\partial^\mu+\tfrac{1}{2}ig\boldsymbol{\tau}\cdot\mathbf{V}^\mu+\tfrac{1}{2}ig\boldsymbol{\tau}\cdot\mathbf{A}^\mu\gamma_5$$

when acting on isospinors, and the mass functional is

$$M(\phi,V,A)=\tfrac{1}{2}g\boldsymbol{\tau}\cdot\not{\mathbf{V}}+\tfrac{1}{2}g\boldsymbol{\tau}\cdot\not{\mathbf{A}}\gamma_5+ig\boldsymbol{\tau}\cdot\boldsymbol{\pi}\gamma_5+g\sigma\ . \tag{3.31}$$

The current $\langle j_\mu\rangle_{\phi,V,A}$ will now have a term of the form (3.14) with ∂ replaced by D. In addition, however, there will be a new piece depending on the O(4) field strength tensor. V_i^μ and A_i^μ make up the components of the six-dimensional regular representation A_{ab}^μ of O(4), according to $A_{0i}^\mu\equiv A_i^\mu$ and $A_{ij}^\mu\equiv\epsilon_{ijk}V_k^\mu$. The field-strength tensor is then

$$F_{ab}^{\mu\nu}=\partial^\mu A_{ab}^\nu-\partial^\nu A_{ab}^\mu+\cdots\ , \tag{3.32}$$

where we shall not be interested in the explicit form of the terms in (3.32) which are bilinear in the A's. The extra contribution to (3.14) has the O(4)- and Lorentz-invariant structure

$$G(\phi^2)\epsilon_{\mu\nu\lambda\rho}\epsilon_{abcd}\phi_a F_{bc}^{\nu\lambda}D^\rho\phi_d\ . \tag{3.33}$$

The leading term in the expansion of (3.33) about $\phi=\phi_0$ and $A_{ab}=0$ is

$$G(\phi_0{}^2)\epsilon_{\mu\nu\lambda\rho}\epsilon_{abcd}\phi_{0a}(\partial^\nu A_{bc}^\lambda-\partial^\lambda A_{bc}^\nu)\partial^\rho\phi_d \tag{3.34}$$

which arises from the terms

$$-i\left[\mathrm{Tr}\frac{1}{\not{p}-M_0}\left[g\frac{\boldsymbol{\tau}\cdot\not{\mathbf{V}}}{2}+g\frac{\boldsymbol{\tau}\cdot\not{\mathbf{A}}}{2}\gamma_5\right]\frac{1}{\not{p}-M_0}\widetilde{M}\frac{1}{\not{p}-M_0}\gamma_\mu\delta(x-y)\right.$$
$$\left.+\mathrm{Tr}\frac{1}{\not{p}-M_0}\widetilde{M}\frac{1}{\not{p}-M_0}\left[g\frac{\boldsymbol{\tau}\cdot\not{\mathbf{V}}}{2}+g\frac{\boldsymbol{\tau}\cdot\not{\mathbf{A}}}{2}\gamma_5\right]\frac{1}{\not{p}-M_0}\gamma_\mu\delta(x-y)\right] \tag{3.35}$$

in the expansion of (3.11) in powers of the fields, where $\widetilde{M}=M(\phi)-M(\phi_0)$, and $M_0=ig\boldsymbol{\tau}\cdot\boldsymbol{\pi}_0\gamma_5+g\sigma_0$ as usual. Performing similar manipulations to those which led to (3.18) from (3.16), we find that the relevant terms (3.35) reduce to

$$-2ig^3\epsilon_{\mu\nu\lambda\rho}\epsilon_{abcd}\phi_{0a}\left[\mathrm{Tr}\frac{1}{X}\partial^\nu A_{bc}^\lambda\frac{1}{X}\partial^\rho\phi_d\frac{1}{X}\delta(x-y)-\mathrm{Tr}\frac{1}{X}\partial^\lambda A_{bc}^\nu\frac{1}{X}\partial^\rho\phi_d\frac{1}{X}\delta(x-y)\right]\ . \tag{3.36}$$

Once again, since we need retain only $O(\partial A\,\partial\phi)$ terms to compare with (3.34), we obtain directly

$$G(\phi_0{}^2)=-\frac{g}{16\pi^2\phi_0{}^2} \tag{3.37}$$

from which [cf. Eq. (3.33)] the new contribution to the current is

$$-\frac{g}{16\pi^2\phi^2}\epsilon_{abcd}\epsilon_{\mu\nu\lambda\rho}\phi_a F_{bc}^{\nu\lambda}D^\rho\phi_d\ , \tag{3.38}$$

in agreement with Ref. 6, and containing the expected anomalous divergence.

IV. SKYRME-SOLITON PHYSICS

As our last, and technically most intricate, application of the derivative-expansion technique, we shall calculate the fermion-loop contribution to the $O((\partial\phi)^4)$ terms in the effective Lagrangian of the linear and nonlinear σ models; the motivation for this was explained in the Introduction—see also Ref. 11. The nonlinear model has the Lagrangian

$$\mathscr{L}_2=\tfrac{1}{2}(\partial_\mu\phi_a)^2\ , \tag{4.1}$$

where the index a runs from 0 to 3 with $\phi=(\sigma,\boldsymbol{\pi})$ and ϕ_a is constrained by $\sigma^2+\boldsymbol{\pi}^2=f^2$, where f is a symmetry-breaking parameter. Treating (4.1) as a classical field theory, Skyrme[9] showed that field configurations characterized by nontrivial topology are possible; however, simple scaling arguments[9] imply that they would be unstable against collapse if the dynamics is given solely by (4.1). Skyrme proposed the addition of the term (in our notation)

$$\mathscr{L}_4^{(s)}=\frac{s^2}{f^4}[(\partial_\mu\phi_a\partial_\nu\phi_a)^2-(\partial_\mu\phi_a)^4]\ , \tag{4.2}$$

where s is a dimensionless parameter. With the addition of this term, static finite-energy field configurations are possible, carrying a conserved topological quantum number. Since ϕ_a is treated as a classical field, we may interpret $\mathscr{L}_2$ as an effective Lagrangian: indeed, the associated action is well known to generate π-irreducible vertices which are exact to second order in the momenta for a wide class of theories exhibiting spontaneously broken chiral symmetry.[15] Thus $\mathscr{L}_2$ is, from today's perspective, rather securely based as part of the low-energy phenomenology of QCD, in which pions are the Goldstone modes of the associated broken chiral symmetry. It is then natural to try and *calculate,* from some chosen quantum field theory, the $O((\partial\phi)^4)$ terms in the effective Lagrangian, $\mathscr{L}_4$. As remarked in the Introduction, we take the view that at least those $O((\partial\phi)^4)$ terms generated by fermion loops should be included, since this particular type of radiative effect has already been employed in the calculation of the Goldstone-Wilczek current, which in turn allows identification of the Skyrme topological current with physical currents.[16] We shall be interested to see whether such $O((\partial\phi)^4)$ terms have anything like the form (4.2) proposed phenomenologically by Skyrme.

Though we shall eventually specialize to the nonlinear σ model, we shall first present the results for the more general linear model. We consider meson fields coupled to fermions via the Lagrangian

$$\mathscr{L}=\tfrac{1}{2}(\partial_\mu\phi_a)^2+\tfrac{1}{2}\mu^2\phi^2-\tfrac{1}{4}\lambda\phi^4+\bar\psi(i\partial\!\!\!/-ig\boldsymbol{\tau}\cdot\boldsymbol{\pi}\gamma_5-g\sigma)\psi\ , \tag{4.3}$$

where ψ is a massless isospinor fermion field, $\phi_a=(\sigma,\boldsymbol{\pi})$, and $\phi^2=\sigma^2+\boldsymbol{\pi}^2$ is not yet constrained to equal a constant. Thus, we are consistently taking the same $M(\phi)$ as in the Goldstone-Wilczek calculation of Sec. III [see Eq. (3.13)]. We seek the $O((\partial\phi)^4)$ terms in the expansion of

$$S_{\rm eff}=-i\,{\rm Tr}\ln(p\!\!\!/-ig\boldsymbol{\tau}\cdot\boldsymbol{\pi}\gamma_5-g\sigma)\ ; \tag{4.4}$$

we do this by expanding about a constant field ϕ_{0a}, as in all our previous examples.

First consider the expansion of $S_{\rm eff}$ in powers of $\partial_\mu\phi_a$. There are ten independent (i.e., not related by integrations by parts) terms with four derivatives of ϕ_a, which we choose as follows:

$$\begin{aligned}S_{\rm eff}=\cdots+\int d^4x[&Y_1(\phi^2)(\partial_\mu\phi_a)^4+Y_2(\phi^2)(\partial_\mu\phi_a\partial_\nu\phi_a)^2+Y_3(\phi^2)(\phi_a\Box\phi_a)^2\\&+Y_4(\phi^2)\phi_a\partial_\mu\partial_\nu\phi_a\partial^\mu\phi_b\partial^\nu\phi_b+Y_5(\phi^2)\phi_b\Box\phi_b(\partial_\mu\phi_a)^2+Y_6(\phi^2)\phi^2(\Box\phi_a)^2\\&+Y_7(\phi^2)(\phi_a\partial_\mu\phi_a)^2(\partial_\mu\phi_b)^2+Y_8(\phi^2)\phi_a\partial_\mu\phi_a\phi_b\partial_\nu\phi_b\partial^\mu\phi_c\partial^\nu\phi_c\\&+Y_9(\phi^2)\phi_a\partial_\mu\partial_\nu\phi_a\phi_b\partial^\mu\phi_b\phi_c\partial^\nu\phi_c+Y_{10}(\phi^2)(\phi_a\partial_\mu\phi_a)^4]+\cdots\ .\end{aligned} \tag{4.5}$$

We now put $\phi_a(x)=\phi_{0a}+\tilde\phi_a(x)$ in (4.5), where ϕ_{0a} is a constant field, and retain terms of $O(\tilde\phi^4)$. There are 19 such terms, of which one is not independent of the rest. After performing some integrations by parts, to eliminate this dependent term and to bring some of the other terms into a more convenient form, we find that (4.5) becomes

$$\begin{aligned}\int d^4x[&S_1(\partial_\mu\tilde\phi_a)^4+S_2(\partial_\mu\tilde\phi_a\partial_\nu\tilde\phi_a)^2+S_3(\tilde\phi_a\Box\tilde\phi_a)^2+S_4\tilde\phi_a\partial_\mu\partial_\nu\tilde\phi_a\partial^\mu\tilde\phi_b\partial^\nu\tilde\phi_b+S_5\tilde\phi_b\Box\tilde\phi_b(\partial_\mu\tilde\phi_a)^2\\&+S_6\tilde\phi^2(\Box\tilde\phi_a)^2+T_1\phi_{0a}\partial_\mu\tilde\phi_a\phi_{ob}\partial_\mu\tilde\phi_b(\partial_\nu\tilde\phi_c)^2+T_2\phi_{0a}\partial_\mu\tilde\phi_a\phi_{0b}\partial_\nu\tilde\phi_b\partial^\mu\tilde\phi_c\partial^\nu\tilde\phi_c\\&+T_3(\phi_0\cdot\tilde\phi)\phi_{0a}\partial_\mu\partial_\nu\tilde\phi_a\partial^\mu\tilde\phi_b\partial^\nu\tilde\phi_b+T_4(\phi_0\cdot\tilde\phi)\partial_\mu\partial_\nu\tilde\phi_a\partial^\mu\tilde\phi_a\phi_{0b}\partial^\nu\tilde\phi_b\\&+T_5\tilde\phi_a\partial_\mu\partial_\nu\tilde\phi_a\phi_{0b}\partial^\mu\tilde\phi_b\phi_{0c}\partial^\nu\tilde\phi_c+T_6\phi_{0a}\partial_\mu\partial_\nu\tilde\phi_a\tilde\phi_b\partial^\mu\tilde\phi_b\phi_{0c}\partial^\nu\tilde\phi_c+T_7(\phi_0\cdot\tilde\phi)^2(\Box\tilde\phi_a)^2\\&+T_8(\phi_0\cdot\tilde\phi)\phi_{0a}\Box\tilde\phi_a\tilde\phi_b\Box\tilde\phi_b+T_9\tilde\phi^2(\phi_{0a}\Box\tilde\phi_a)^2+U_1(\phi_0\cdot\tilde\phi)^2(\phi_{0a}\Box\tilde\phi_a)^2\\&+U_2(\phi_0\cdot\tilde\phi)\phi_{0a}\partial_\mu\partial_\nu\tilde\phi_a\phi_{0b}\partial^\mu\tilde\phi_b\phi_{0c}\partial^\nu\tilde\phi_c+U_3(\phi_0\cdot\tilde\phi)\phi_{0a}\Box\tilde\phi_a(\phi_{0b}\partial_\mu\tilde\phi_b)^2]\ ,\end{aligned} \tag{4.6}$$

where the coefficients S_i,T_i,U_i, which are related to the Y_i, are functions of $\phi_0{}^2$. Only ten of these are independent; they give the coefficients $Y_i(\phi_0{}^2)$, as follows:

$$\begin{aligned}&S_1=Y_1,\ \ S_6=Y_6+\phi_0{}^2\frac{dY_6}{d\phi_0{}^2}\ ,\quad S_2=Y_2,\ \ T_1=Y_7-2\frac{dY_5}{d\phi_0{}^2}\ ,\\&S_3=Y_3,\ \ T_2=Y_8\ ,\quad S_4=Y_4,\ \ T_5=Y_9\ ,\quad S_5=Y_5,\ \ U_3=-Y_{10}\ .\end{aligned} \tag{4.7}$$

The other eight satisfy the following relations:

$$\begin{aligned}&T_3=2\frac{dY_4}{d\phi_0{}^2}=2\frac{dS_4}{d\phi_0{}^2}\ ,\qquad T_4=-4\frac{dY_5}{d\phi_0{}^2}=-4\frac{dS_5}{d\phi_0{}^2}\ ,\\&T_6=2Y_9=2T_5,\ \ T_7=2\left[2\frac{dY_6}{d\phi_0{}^2}+\phi_0{}^2\frac{d^2Y_6}{d(\phi_0{}^2)^2}\right]=2\frac{dS_6}{d\phi_0{}^2}\ ,\end{aligned} \tag{4.8}$$

$$T_8=4\frac{dY_3}{d\phi_0{}^2}=4\frac{dS_3}{d\phi_0{}^2}\,,\quad T_9=\frac{dY_3}{d\phi_0{}^2}=\tfrac{1}{4}T_8\,,$$

$$U_1=2\frac{d^2Y_3}{d(\phi_0{}^2)^2}=2\frac{dT_9}{d\phi_0{}^2}\,,\quad U_2=2\left[\frac{dY_9}{d\phi_0{}^2}-Y_{10}\right]=2\left[\frac{dT_5}{d\phi_0{}^2}+U_3\right].$$

The relations (4.8) provide a very useful check when we come to read off the coefficients S_i, T_i, U_i from the expansion of (4.4).

We now put $\phi_a(x)=\phi_{0a}+\tilde{\phi}_a(x)$ in (4.4), and expand that, again keeping the term of $O(\tilde{\phi}^4)$. This term is

$$\frac{ig^4}{4}\mathrm{Tr}\left[\frac{1}{\not p-ig\boldsymbol{\tau}\cdot\boldsymbol{\pi}_0\gamma_5+g\sigma_0}(i\boldsymbol{\tau}\cdot\tilde{\boldsymbol{\pi}}\gamma_5+\tilde{\sigma})\right]^4=\frac{ig^4}{4}\mathrm{Tr}\left[\frac{1}{p^2-m^2}(\not p-ig\boldsymbol{\tau}\cdot\boldsymbol{\pi}_0\gamma_5+g\sigma_0)(i\boldsymbol{\tau}\cdot\tilde{\boldsymbol{\pi}}\gamma_5+\tilde{\sigma})\right]^4,\qquad(4.9)$$

where $m^2=g^2\phi_0{}^2$. After performing the Dirac and SU(2) traces, Eq. (4.9) becomes

$$\begin{aligned}&2ig^8[\phi_0{}^4T_{abcd}+8(\phi_{0c}\phi_{0d}-\delta_{cd})\phi_{0a}\phi_{0b}]\mathrm{Tr}\frac{1}{p^2-m^2}\tilde{\phi}_a\frac{1}{p^2-m^2}\tilde{\phi}_b\frac{1}{p^2-m^2}\tilde{\phi}_c\frac{1}{p^2-m^2}\tilde{\phi}_d\\&+8ig^6g_{\mu\nu}[2(\delta_{ab}\phi_{0c}\phi_{0d}-\delta_{bd}\phi_{0a}\phi_{0c}+\delta_{ad}\phi_{0b}\phi_{0c})-\phi_0{}^2T_{abcd}]\mathrm{Tr}\frac{p^\mu}{p^2-m^2}\tilde{\phi}_a\frac{p^\nu}{p^2-m^2}\tilde{\phi}_b\frac{1}{p^2-m^2}\tilde{\phi}_c\frac{1}{p^2-m^2}\tilde{\phi}_d\\&+4ig^6g_{\mu\nu}[2(\delta_{bd}\phi_{0a}\phi_{0c}-2\delta_{cd}\phi_{0a}\phi_{0b}+\delta_{ac}\phi_{0b}\phi_{0d})+\phi_0{}^2T_{abcd}]\mathrm{Tr}\frac{p^\mu}{p^2-m^2}\tilde{\phi}_a\frac{1}{p^2-m^2}\tilde{\phi}_b\frac{p^\nu}{p^2-m^2}\tilde{\phi}_c\frac{1}{p^2-m^2}\tilde{\phi}_d\\&+2ig^4[T_{abcd}(g_{\mu\nu}g_{\lambda\rho}-g_{\mu\lambda}g_{\nu\rho}+g_{\mu\rho}g_{\nu\lambda})-i\epsilon_{\mu\nu\lambda\rho}\epsilon_{abcd}]\mathrm{Tr}\frac{p^\mu}{p^2-m^2}\tilde{\phi}_a\frac{p^\nu}{p^2-m^2}\tilde{\phi}_b\frac{p^\lambda}{p^2-m^2}\tilde{\phi}_c\frac{p^\rho}{p^2-m^2}\tilde{\phi}_d\,,\end{aligned}\qquad(4.10)$$

where

$$T_{abcd}=\delta_{ab}\delta_{cd}-\delta_{ac}\delta_{bd}+\delta_{ad}\delta_{bc}\,.$$

Equation (4.10) can now be expanded in powers of derivatives of ϕ_a, by moving momentum operators to the left and functions of x to the right, using the relation (1.7). (The calculation is unfortunately very much more complicated algebraically than those presented in Secs. II and III, where we were usually looking for terms of at most third order in derivatives, and where the trace was always contracted with $\epsilon_{\mu\nu\lambda\rho}$, which eliminated many terms.) After a good deal of algebra and some integrations by parts to bring this expansion into the same form as (4.6), we find that the coefficients S_i, T_i, and U_i are, in units of $1/60\pi^2\phi_0{}^4$,

$$\begin{aligned}&S_1=-\tfrac{19}{8},\ T_1=\frac{1}{\phi_0{}^2}\,,\quad S_2=\tfrac{15}{4},\ T_2=-\frac{38}{3\phi_0{}^2}\,,\\&S_3=-\tfrac{1}{2},\ T_3=-\frac{10}{\phi_0{}^2}\,,\quad S_4=\tfrac{5}{2},\ T_4=-\frac{8}{\phi_0{}^2}\,,\\&S_5=-1,\ T_5=-\frac{8}{3\phi_0{}^2}\,,\quad S_6=-\tfrac{5}{4},\ T_6=-\frac{16}{3\phi_0{}^2}\,,\\&U_1=-\frac{6}{\phi_0{}^4},\ T_7=\frac{5}{\phi_0{}^2}\,,\quad U_2=\frac{2}{\phi_0{}^4},\ T_8=\frac{4}{\phi_0{}^2}\,,\quad U_3=-\frac{7}{\phi_0{}^4},\ T_9=\frac{1}{\phi_0{}^2}\,.\end{aligned}\qquad(4.11)$$

We see that these satisfy the constraints (4.8). From (4.7), we can deduce the coefficients $Y_i(\phi_0{}^2)$; we may now replace ϕ_0 by $\phi(x)$, and (4.5) becomes

$$\begin{aligned}S_{\mathrm{eff}}=\frac{1}{60\pi^2}\int d^4x\,\Big[&-\frac{19}{8\phi^4}(\partial_\mu\phi_a)^4+\frac{15}{4\phi^4}(\partial_\mu\phi_a\partial_\nu\phi_a)^2-\frac{1}{2\phi^4}(\phi_a\Box\phi_a)^2+\frac{5}{2\phi^4}\phi_a\partial_\mu\partial_\nu\phi_a\partial^\mu\phi_b\partial^\nu\phi_b\\&+\frac{5}{4\phi^2}(\Box\phi_a)^2-\frac{1}{\phi^4}\phi_a\Box\phi_a(\partial_\mu\phi_b)^2+\frac{5}{\phi^6}(\phi_a\partial_\mu\phi_a)^2(\partial_\nu\phi_b)^2\\&-\frac{38}{3\phi^6}\phi_a\partial_\mu\phi_a\phi_b\partial_\nu\phi_b\partial^\mu\phi_c\partial^\nu\phi_c-\frac{8}{3\phi^6}\phi_a\partial_\mu\partial_\nu\phi_a\phi_b\partial^\mu\phi_b\phi_c\partial^\nu\phi_c+\frac{7}{\phi^8}(\phi_a\partial_\mu\phi_a)^4\Big]\,.\end{aligned}\qquad(4.12)$$

Expression (4.12) is the complete one-fermion-loop contribution to the $O((\partial\phi)^4)$ terms in the effective action, in the model of (4.3). It may well have independent applications, but for the moment we are interested in the special case in which the nonlinear constraint is imposed. To this order in $\hbar$ and derivatives it is sufficient[17] simply to set $\phi^2=f^2$ in (4.12), where $f^2=\mu^2/\lambda$ is the minimum of the classical potential in (4.3). The last four terms in (4.12) now vanish identically as we have $\phi_a\partial_\mu\phi_a=0$. The remaining six are no longer independent; since now $\phi_a\partial_\mu\partial_\nu\phi_a=-\partial_\mu\phi_a\partial_\nu\phi_a$ and $\phi_a\Box\phi_a=-(\partial_\mu\phi_a)^2$, there are only three independent nonlinear invariants, and the final contribution to the effective Lagrangian is

$$\mathscr{L}_4^{(1f)\mathrm{NL}}=\frac{1}{48\pi^2 f^4}[(\partial_\mu\phi_a\partial_\nu\phi_a)^2-\tfrac{3}{2}(\partial_\mu\phi_a)^4+f^2(\Box\phi_a)^2]\,. \tag{4.13}$$

We may now compare (4.13) with the Skyrme form (4.2). We first note that the first two terms of (4.13) do have the same signs as those in (4.2). Furthermore, the "additional" $-\frac{1}{2}(\partial_\mu\phi_a)^4$ contribution gives a positive contribution to the static energy (though it leads to a Hamilton for time-dependent solutions which is unbounded below). Thus it might seem possible that these radiative corrections could stabilize the soliton against collapse. However, the third term $[(\Box\phi_a)^2]$ remains to be considered. Previous discussions[19,20] of the possible radiative origin of the Skyrme terms have omitted this third term, on the basis that only the first two are required as counterterms in renormalizing the nonlinear σ model to one-loop order. However, there is nothing to rule out a finite term of this form in general. Our calculation shows explicitly that such a term is induced by the one-fermion-loop contribution, and we have also found that in the linear σ model (without fermions) the boson loops generate such a finite term.[21] It appears that the effect of this term is to cause the total static energy to become negative, thus indicating a possible instability of the ground state: this conclusion is arrived at by MacKenzie *et al.*[18] from a consideration of a regime of g and N (the number of fermion species) for which the one-loop calculation should be dominant, and it is also confirmed in numerical calculations by Ripka.[22] Though more detailed numerical studies will be undertaken[23] to study the effect of this term further, it seems clear now that the calculated $\mathscr{L}_4$ terms do not stabilize the soliton, and indeed give a negative contribution to the energy.

In conclusion, we should note that the status of (4.12) or (4.13) is rather different from the anomaly-related effects of the earlier sections. In those cases, general topological arguments were available to guarantee that the form of the amplitude was model independent. By contrast, the $O((\partial\phi)^4)$ terms we have calculated are surely dynamical in origin and model dependent. In particular, there is no *a priori* reason why (in this nonanomalous part of the Lagrangian) boson loops should not also be considered. We have, in fact, also calculated the $O((\partial\phi)^4)$ terms arising from ϕ loops, starting again from (4.3). We hope to report on this elsewhere[21] and also on the gauged extension of this model, which may have interesting applications to a heavy Higgs sector.[17,20,24,25]

ACKNOWLEDGMENT

One of us (C.M.F.) would like to thank Somerville College, Oxford for financial support.

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