Anisotropic Kaluza-Klein cosmologies

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We investigate the class of vacuum five-dimensional spatially homogeneous and anisotropic cosmological models satisfying Einstein's field equations and derive, for some of them, general or particular exact solutions. The analysis of the asymptotic behavior of these solutions shows that Chodos and Detweiler's cosmological dimensional-reduction process is possible for these models, the fifth, unphysical, dimension being allowed to contract to a very small scale.

The idea that space-time has more than four dimensions, the extra ones being compactified so as to be unobservable at the energies currently available, has received much attention recently.[1–8] These multidimensional theories constitute most interesting candidates for a unification of all fundamental interactions, including gravitation, in the framework of a theory of general relativity in $4+d$ dimensions.

The earliest theory of this type due to Kaluza and Klein[9] is five-dimensional and, accordingly, possesses only one extra dimension; this Kaluza-Klein theory represents the first attempt at unifying gravitation and electromagnetism.

In the framework of multidimensional theories, an explanation of the four-dimensionality of the actual universe as well as of the Friedmann-Robertson-Walker form of its metric, has to be found.

An interesting possibility, known as the "cosmological dimensional-reduction process," is based on the idea that, at very early times, all dimensions in the universe were of comparable size but that, later, the scale of the extra dimensions became so small as to be unobservable, by experiencing either a contraction or an expansion at a much lower rate than in the case of the physically observable dimensions. This process was first proposed by Chodos and Detweiler[1] who showed that there exists, in the framework of a pure gravitational theory of Kaluza-Klein type, a vacuum Kasner (Bianchi type-I) solution of Einstein's field equations for which the extra dimension contracts to a very small scale, while the three other spatial dimensions expand isotropically.

The explanation of the smallness of the extra dimensions of the universe by the dynamical evolution of the latter has also been proposed in the case of a more realistic model, i.e., 11-dimensional supergravity.[2,10]

However, Chodos and Detweiler's analysis is based on the study of the very simple anisotropic Bianchi type-I model, which is the less general among spatially homogeneous models. It is then essential to examine if the cosmological dimensional-reduction process is still possible for more general multidimensional spatially homogeneous models.

Leaving aside the problem of stability of the particular Kasner type of solution considered by Chodos and Detweiler, we address here the problem of the evolution of general five-dimensional anisotropic spatially homogeneous models and we consider only vacuum solutions of the field equations. This work is in fact a first step toward a complete understanding of the multidimensional anisotropic cosmologies.

Extending the usual definition of four-dimensional spatially homogeneous models,[11,12] we define a five (N)-dimensional spatially homogeneous model as a five (N)-dimensional space-time possessing a four ($N-1$)-dimensional group of isometry acting simply transitively on four ($N-1$)-dimensional spacelike hypersurfaces. This definition does not probably recover all possible cases of spatially homogeneous models (cf. for instance, for $3+1$ space-times, the supplementary case of the Kantowski-Sachs models[13]) and a detailed analysis of the higher-dimensional isometry groups would be necessary in order to obtain the totality of possible different types of such models. However, the definition adopted here is already sufficiently rich as to recover a large diversity of spatially homogeneous models.

The real four-dimensional Lie algebras have been classified by different authors;[14–17] we have used explicitly the classification given by Fee,[17] which comprises 15 distinct real four-dimensional Lie algebras, some of which (eventually for particular values of the parameters involved in the commutators of the elements of the algebra) generalize the usual three-dimensional Bianchi-type Lie algebras.

The corresponding five-dimensional metrics have been written in the Cartan basis of left-invariant forms (the Killing vectors being then identified with the right-invariant vectors); in this basis, the metric tensor depends on time only. The right- and left-invariant vector fields and forms for each of the real four-dimensional Lie algebras are given explicitly in Fee's work.[17]

For each of the 15 distinct real four-dimensional Lie algebras, we have considered only five-dimensional diagonal metrics in the Cartan basis of left-invariant forms and have written the corresponding Einstein's vacuum field equations, using the algebraic system SHEEP 2, which enables one to change the dimension of the Riemannian manifolds studied.

Some of the five-dimensional space-times considered do not admit any diagonal vacuum solution. For other ones, we have been able to derive general or particular exact solutions, some of which represent direct generalizations to five-dimensional space-times of known vacuum solutions of the Bianchi-type cosmological models;[18] however, certain solutions are typically new and have no spatially homogeneous lower-dimensional counterparts.

Kasner's generalized solution (corresponding to the Abelian algebra denoted in Fee's work as $G0$), which is valid for any dimension of space-time, has been studied in detail by Chodos and Detweiler.[1]

We shall describe here some particularly interesting results obtained for Fee's $G7$, $G8$, and $G11$ four-dimensional Lie algebras [$L(4,4)$, $L(4,2)$, and $L(4,7)$, respectively, in the notation of Patera *et al.*[15]].

The diagonal five-dimensional metric corresponding to Fee's $G7$ algebra can be written as

$$ds^2=e^{2A}(-dt^2+dw^2)+\sum_{i=1}^{3} g_{ii}(\omega^i)^2 , \tag{1}$$

where A and all g_{ii}'s are functions of time t only and the spatial Cartan basis of left-invariant forms is given by

$$\begin{aligned}&\omega^1=e^{-Pw}dx^1 ,\\ &\omega^2=e^{-w}dx^2 ,\\ &\omega^3=e^{-w}(dx^3-w\,dx^2) ,\\ &\omega^4=dw ,\end{aligned} \tag{2}$$

where P is an arbitrary constant. The commutators characteristic of the Lie algebra generated by the corresponding Killing (right-invariant) vectors, i.e.,

$$\begin{aligned}&X_i=\frac{\partial}{\partial x^i} \quad (i=1,2,3) ,\\ &X_4=(x^3+x^1)\frac{\partial}{\partial x^3}+x^2\frac{\partial}{\partial x^2}+Px^1\frac{\partial}{\partial x^1}+\frac{\partial}{\partial w}\end{aligned} \tag{3}$$

have the following form:

$$\begin{aligned}&[X_3,X_4]=X_3 ,\\ &[X_2,X_4]=X_2+X_4 ,\\ &[X_1,X_4]=PX_1 .\end{aligned} \tag{4}$$

Einstein's vacuum field equation $R_{23}=0$ imposes $P=-2$. An exact solution of the field equations can, in this case, be derived; it is given by

$$\begin{aligned}&g_{11}=t^{2/3} ,\\ &g_{22}=t^{5/3}(t^{\gamma}+t^{-\gamma}) ,\\ &g_{33}=t^{-1/3}/(t^{\gamma}+t^{-\gamma}) ,\\ &g_{44}=e^{2A}=e^{3t^2}t^{(\gamma^2-1/3)/2}(t^{\gamma}+t^{-\gamma}) ,\end{aligned} \tag{5}$$

where γ is a positive constant.

The study of the asymptotic time behavior of this exact solution shows that, for any (positive) value of γ, g_{11}, g_{22}, and g_{44} tend toward infinity while g_{33} vanishes when $t\to\infty$: this is typically the behavior found by Chodos and Detweiler[1] in their study of the five-dimensional Kasner solution, leading thus to a contraction of the physically unobservable dimension and an expansion of the three physical dimensions.

The spatially homogeneous five-dimensional model corresponding to the $G8$ four-dimensional algebra can easily be generalized to any dimension; its metric can be written, in the case of an n-dimensional algebra (latin indices running from 1 to $n-1$) as

$$ds^2=e^{2A(t)}(-dt^2+dw^2)+\sum_i (e^{B_i(t)+P_i w}dx^i)^2 , \tag{6}$$

where the P_i's are constant. The commutators of the corresponding Killing vector fields generating the Lie algebra, i.e.,

$$\begin{aligned}&X_j=\frac{\partial}{\partial x^j} \quad (j=1,\dots,n-1) ,\\ &X_n=\sum_i P_i x^i\frac{\partial}{\partial x^i}+\frac{\partial}{\partial w}\end{aligned} \tag{7}$$

are given by

$$[X_j,X_n]=P_jX_j . \tag{8}$$

Einstein's vacuum field equations, computed by means of SHEEP 2, have the following form (the subscript zero refers to the time variable $x^0\equiv t$):

$$\begin{aligned}&e^{2A}R_{00}=\dot\Omega\dot A-\ddot\Omega-\ddot A-\sum_i \dot B_i{}^2=0 ,\\ &e^{2A}R_{jj}=\ddot B_j+\dot\Omega\dot B_j-\sigma P_j=0 ,\\ &e^{2a}R_{0n}=\sigma\dot A-\sum_i P_i\dot B_i=0 ,\\ &e^{2A}R_{nn}=\ddot A+\dot\Omega\dot A-\sum_i P_i{}^2=0 ,\end{aligned} \tag{9}$$

where

$$\Omega(t)=\sum_i B_i(t)=\ln R(t) \tag{10a}$$

and

$$\sigma=\sum_i P_i . \tag{10b}$$

Summing the $(n-1)(jj)$-field equations, it becomes

$$\ddot\Omega+\dot\Omega^2=\sigma^2 \tag{11a}$$

or

$$\ddot R=\sigma^2 R \tag{11b}$$

which can easily be integrated:

$$\dot R^2=\sigma^2R^2+\mu , \tag{12}$$

where μ is an integration constant.

Two cases have to be considered, following the value of the parameter σ.

(a) $\sigma\neq 0$ (σ can then, without restriction, be normalized to $+1$). The field equation $R_{00}=0$ imposes the constraint $\mu\geq 0$. If $\mu=0$, we obtain a special solution of Eqs. (9) generalizing to any dimension the particular four-dimensional solution obtained by Collins[19] and Evans[20] for the vacuum Bianchi type-VI_h model:

$$ds^2 = \exp\left[\sum P_i{}^2 t\right](-dt^2+dw^2) + \sum_i (e^{P_i(t+w)}dx^i)^2 \quad (13a)$$

with

$$\sum_i P_i = 1 . \quad (13b)$$

On the other hand, if $\mu=1$, we obtain the following general solution:

$$ds^2=(\sinh t)^{2\sum_i P_i{}^2}(\tanh t/2)^{2\sum_i P_i\alpha_i}(-dt^2+dw^2)+\sum_i (\sinh t)^{2P_i}(\tanh t/2)^{2\alpha_i}(e^{P_i w}dx^i)^2 , \quad (14a)$$

where

$$\sum_i P_i=1,\ \sum_i \alpha_i=0,\ \sum_i \alpha_i{}^2=1+\sum_i P_i{}^2 . \quad (14b)$$

This solution appears as an $(n+1)$-dimensional generalization of Ellis and MacCallum's[21] solution for the vacuum Bianchi type-VI_h model (containing also as particular cases Bianchi type-III and type-V models).

(b) $\sigma=0$. The solution obtained in this case is the following one:

$$ds^2=\exp\left[\sum_i P_i{}^2t^2/2\right]t^{\left[\sum_i \alpha_i{}^2-1\right]}(-dt^2+dw^2)+\sum_i t^{2\alpha_i}(e^{P_i w}dx^i)^2 , \quad (15a)$$

where

$$\sum_i \alpha_i=1,\ \sum_i P_i=0,\ \sum_i \alpha_i P_i=0 . \quad (15b)$$

The asymptotic time behavior of the spatial part of the exact solutions (13a), (14a), and (15a) [with the constraints (13b), (14b), and (15b), respectively] can be, in the five-dimensional case, one among the three following possible different types: four, three, or two spatial dimensions can be expanding monotonically while the other ones would be contracting; this second type of behavior is in agreement with Chodos and Detweiler's cosmological dimensional-reduction process.

In the $G11$ case, which has no spatially homogeneous three-dimensional counterpart, it is possible to derive an exact particular solution of Einstein's vacuum field equations, given by

$$\begin{aligned} g_{44}&=-g_{00} \\ &=(\sinh\sigma t)^{4(P^2+P+1)/\sigma^2}\left[\tanh\frac{\sigma t}{2}\right]^{\mp 8(P-1)C} , \\ g_{11}&=(\sinh\sigma t)^{4(2P^2+5P+2)/3\sigma^2}\left[\tanh\frac{\sigma t}{2}\right]^{\pm 4(P-1)C} , \\ g_{22}&=(\sinh\sigma t)^{4(P+2)^2/3\sigma^2}\left[\tanh\frac{\sigma t}{2}\right]^{\mp 4(4P+5)C} , \\ g_{33}&=(\sinh\sigma t)^{4(2P=1)^2/3\sigma^2}\left[\tanh\frac{\sigma t}{2}\right]^{\mp 4(5P+4)C} , \end{aligned} \quad (16)$$

where σ and C are given, respectively, by

$$\sigma^2=\tfrac{1}{6}[27(P+1)^2+(P-1)^2] \quad (17a)$$

and

$$C=\frac{[(P-1)^2+9(P+1)^2]^{1/2}}{3\sqrt{6}\,\sigma^2} , \quad (17b)$$

P being an arbitrary constant.

The corresponding five-dimensional metric has the form (1), with the spatial Cartan basis of left-invariant forms given by

$$\begin{aligned} \omega^1&=e^{-(P+1)w}(dx^1-x^3dx^2) , \\ \omega^2&=e^{-w}dx^2 , \\ \omega^3&=e^{-Pw}dx^3 , \\ \omega^4&=dw . \end{aligned} \quad (18)$$

All g_{ii}'s ($i=1$ to 4) tend to infinity for $t\to\infty$, with the exception of g_{11} which, for $P\in\,]-2,-\tfrac{1}{2}[$, tends toward zero.

We have thus shown that Chodos and Detweiler's cosmological dimensional-reduction process is present in each model considered here and we conjecture that this would be true for each five-dimensional spatially homogeneous cosmological model. A detailed analysis of the whole class of such models is in progress.

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