Classical relativistic constituent particles and composite states. II

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A classical theory of interacting relativistic constituent and composite particles is developed further. The Lorentz-invariant Lagrangian, a function of the single unmeasurable evolution parameter s, is considered for attractive and repulsive harmonic-oscillator forces acting pairwise between constituent particles. Nonrelativistic Newtonian equations of motion can be derived by letting $c \to \infty$ in "equal-time" solutions, but, in general, there is a "surplus" of solutions which have no nonrelativistic counterpart. These solutions are used to construct classical models of strongly interacting composite particles. Asymptotic selection rules and constituent confinement are postulated and lead to space-time conservation laws for systems of scattering composite particles. Constituent four-vectors are linear combinations of "kinematic" terms and "intrinsic" normal modes. The latter are identified with internal symmetries of the composite particles, which are labeled by sets of "intrinsic numbers" analogous to additive quantum numbers. Formation of two- and three-body composite particles follows an exact analogy to the color quark model, in which the meson is composed of a quark and an antiquark of the same color, and the baryon is formed from three quarks of three different colors. Scattering examples are given analogous to $MM \to MM$, $MB \to MB$, and $BB \to BB$. The reactions take place through constituent exchange, and total intrinsic numbers are conserved. There are other similarities to quantum field theory, such as particle-antiparticle pair creation and annihilation, fixed relative values of internal angular momenta, fixed orbital angular momentum, and many-particle systems characterized by a vacuum state (lowest energy state) and the existence of virtual composite particles as well as physically observable composite particles.

I. INTRODUCTION

This paper will develop further the classical theory of relativistic composite particles introduced in an earlier work[1] (hereafter referred to as I). The aim has been multifold: (1) to investigate a particular relativistic classical action-at-a-distance particle theory based on an "evolutionary" parameter s; (2) to develop an alternative to perturbative-type calculations (e.g., by avoiding specifying a Hamiltonian as $H = H_0 + H_1$); (3) to construct models of particle interactions; (4) to examine the extent to which a relativistic classical theory can describe phenomena usually associated with quantum field theory or quantum mechanics.

In I, a nonlocal-Lagrangian formalism, based on an evolution parameter s, was developed to describe classical relativistic many-particle systems. Although the evolution parameter s is not measurable, we adopted the point of view that s describes the evolution of the system in space-time in a manner analogous to time t describing the evolution of a system in space.[2] Solutions to the equations of motion represent particles which are, in general, off the mass shell. It was proposed that these solutions represent *confined* constituent particles which combine into observable composite particles. Sample models were given based upon a harmonic-oscillator potential containing attractive and repulsive couplings. The imposition of constituent confinement led to the conservation laws of energy, momentum, and angular momentum.

The models in I, which describe the scattering of composite particles, exhibit several features not usually associated with classical particle theories: (1) the "creation" and "annihilation" of particles, (2) the description of zero-mass particles, (3) particle interaction by means of constituent exchange, (4) restricted values of angular momenta. That these features appear in a "classical" formalism requires some comment. The Lagrangian adopted in I is Lorentz invariant, that is, it describes the interactions of particles within the context of special relativity. This relativistic Lagrangian yields the Newtonian equations of motion in the limit $c \to \infty$ only in the special case of "equal-time" solutions.[1] In general, the solutions do not have nonrelativistic counterparts, i.e., there is a "surplus" of solutions over and above the ones which reduce to the well-known nonrelativistic solutions.[3] It is precisely such solutions that enabled the formulations of the composite-scattering models. *Thus, one should not expect to find nonrelativistic analogs to such models.*

The models of I were artificial, however. For example, nonforward/backward scattering could occur only in the presence of a zero-mass "catalytic" composite particle. Furthermore, elastic scattering was restricted to forward/backward angles. In this article, these difficulties are removed by extending the harmonic-oscillator Lagrangian to include more than two couplings, and descriptions of two- and three-body composite particle scattering is obtained.

II. HARMONIC-OSCILLATOR EQUATIONS OF MOTION

The Lagrangian for N interacting constituents is given by

$$L(s) = -\left[Mc^2 V[x_1(s), \ldots, x_N(s)] \sum_{I=1}^{N} m_I [\dot{x}_I(s)]^2\right]^{1/2} . \tag{1}$$

(Four-dimensional indices are suppressed throughout.) The metric is $g_{11} = g_{22} = g_{33} = -g_{00} = 1$, and squares of four-vectors are given by $x^2 = -g_{\alpha\beta} x^\alpha x^\beta$. The harmonic-oscillator potential[1] is

$$V = 1 - \frac{1}{2} \sum_{I=1}^{N} \sum_{J=1}^{N} [w_{IJ}(0)/c]^2 (x_I - x_J)^2 , \tag{2}$$

and $M = \sum_{I=1}^{N} m_I$, $w_{IJ}(0) = w_{JI}(0)$. The Lagrangian is singular,[1] and as a result we are allowed to choose one component of one four-vector as an arbitrary function of s. We set

$$X_0 = A_0 s , \tag{3}$$

where X_0 is the fourth component of the center-of-mass vector

$$X(s) = (1/M) \sum_{I=1}^{N} m_I x_I , \tag{4}$$

and A_0 is any constant.

This choice of "gauge" function (3) considerably simplifies the equations of motion, which, for the harmonic-oscillator potential, become

$$\ddot{x}_I(s) = (M/m_I N^2) \sum_{J=1}^{N} (w_{IJ})^2 (x_I - x_J) , \tag{5}$$

where we have defined

$$w_{IJ}{}^2 = (N^2/Mc^2) \left[\sum_{k=1}^{N} m_k \dot{x}_k{}^2 / V\right] [w_{IJ}(0)]^2 . \tag{6}$$

The factor $w_{IJ}{}^2$ is constant in s. However, its presence leads to solutions which behave very differently from the analogous solutions to classical nonrelativistic harmonic-oscillator equations. *Equation (5) is not a simple generalization of the nonrelativistic harmonic-oscillator equations of motion.* [For example, linear combinations of solutions of (5) are not themselves solutions.]

The evolution parameter s is discussed in detail in I. We note here that it is *not* in general associated with the physical clock.

The momenta conjugate to x_I are defined by

$$p_I = \partial L / \partial \dot{x}_I = m_I \dot{x}_I [N w_{IJ}(0)/w_{IJ}] . \tag{7}$$

The Lorentz invariance of L yields ten constants in s:

$$P^\mu = \sum_{I=1}^{N} p_I^\mu(s) = M[N w_{IJ}(0)/w_{IJ}] \dot{X}^\mu \tag{8}$$

and

$$J^{\mu\nu} = \sum_{I=1}^{N} [x_I^\mu(s) p_I^\nu(s) - x_I^\nu(s) p_I^\mu(s)] . \tag{9}$$

Thus we obtain conservation laws in s, and not conservation laws in the observer's time t_{OB}. The P^μ and $J^{\mu\nu}$ obey the Poisson brackets (PB) of the Poincaré group.

III. TWO- AND THREE-BODY HARMONIC-OSCILLATOR SOLUTIONS

A. Two-body harmonic-oscillator solution

This solution was also discussed in I. We summarize the results below.

The constituent four-vectors are given by

$$x_1(s) = \tfrac{1}{2}(As + B) + (m_2/M)[a \exp(iws) + b \exp(-iws)] , \tag{10}$$

$$x_2(s) = \tfrac{1}{2}(As + B) - (m_1/M)[a \exp(iws) + b \exp(-iws)] ,$$

where

$$w \equiv (M^2/4m_1 m_2)^{1/2} w_{12} , \tag{11}$$

and the complex constants are taken such that the $x_I(s)$ are real. Defining

$$w(0) \equiv (M^2/4m_1 m_2)^{1/2} w_{12}(0) , \tag{12}$$

we can express the frequency as

$$[w/w(0)]^2 = (A/c)^2 / \{1 - [w_{12}(0)/c]^2 a \cdot b\} . \tag{13}$$

The total momentum for the system is

$$\mathbf{P} = M_{12} \mathbf{v}_{12} / [1 - (\mathbf{v}_{12}/c)^2]^{1/2} ,$$
$$P_0 = M_{12} c / [1 - (\mathbf{v}_{12}/c)^2]^{1/2} , \tag{14}$$
$$p_2 = M_{12}{}^2 c^2 ,$$

where $\mathbf{v}_{12} = \mathbf{A}/A_0$, and

$$M_{12} = M\{1 - [w_{12}(0)/c]^2 a \cdot b\}^{1/2} . \tag{15}$$

Expressing the angular momentum $\mathbf{J}$ as $\mathbf{J}$(c.m.) + $\mathbf{j}$, we have $\mathbf{J}(\text{c.m.}) = \mathbf{X} \times \mathbf{P}$,

$$\mathbf{j} = (1/M)(\mathbf{x}_1 - \mathbf{x}_2)(m_2 \mathbf{p}_1 - m_1 \mathbf{p}_2)$$
$$= iw(0) M \mathbf{a} \times \mathbf{b} . \tag{16}$$

[For a zero-mass composite, replace $w(0)$ by $c/(a \cdot b)^{1/2}$.] Note that since

$$\mathbf{X}(s) = \mathbf{v}_{12} X_0(s) + \mathbf{B} ,$$

P^μ is independent of the choice of gauge (see, for example, p. 526 of Sudarshan and Mukunda, cited in Ref. 4).

1. *Interpretation of the solution as a physical state*

The physical free-particle states, that is, the observable composite particles, must form the bases of irreducible representations of the Poincaré group.[4] Such a representation is partially specified by the values of the Casimir operator P^2 and $\mathbf{P}$. It can be easily verified that the internal angular momentum j_k given in (16) satisfies the PB relations of the rotation group and have vanishing $\mathbf{P}$ with P^μ; i.e., they are the generators of the "little" group which leaves P^μ invariant and the representation can be completely specified by, say, P^2 and $\mathbf{P}$, $|\mathbf{j}|$, and j_z. Thus, we may interpret the solution above as follows: The constituents $x_1(1)$ and $x_2(s)$ form an observable composite particle of mass M_{12}, "spin" $\mathbf{j}$, and momentum $\mathbf{P}$. The composite four-vector is the center-of-mass vector $X(s)$.

B. Three-body solution

Equations of motion for the constituents are

$$\ddot{x}_I = -(M/9m_I)\sum_{J=1}^{N} w_{IJ}^2(x_I - x_J) . \tag{17}$$

We consider the case $w_{12}=w_{13}$, $m_2=m_3=m$, and write the solutions in matrix form:

$$\mathbf{x}=\lambda\mathbf{y} . \tag{18}$$

Here

$$\mathbf{x}=\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad \mathbf{y}=\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}, \tag{19}$$

and

$$\lambda=\begin{bmatrix} m/M & 2m/M & 0 \\ m/M & -m_1/M & \frac{1}{2} \\ m/M & -m_1/M & -\frac{1}{2} \end{bmatrix} . \tag{20}$$

The components of $\mathbf{y}$ are given by

$$\begin{aligned} y_1 &= As+B , \\ y_2 &= a_2\exp(iw_2 s)+b_2\exp(-iw_2 s) , \\ y_3 &= a_3\exp(iw_3 s)+b_3\exp(-iw_3 s) \end{aligned} \tag{21}$$

with

$$\begin{aligned} w_2^2 &= (M^2/9mm_1)w_{12}^2 , \\ w_3^2 &= (M/9m)(w_{12}^2+2w_{23}^2) . \end{aligned}$$

The frequencies are given by

$$[w_I/w_I(0)]^2=[w_{IJ}/w_{IJ}(0)]^2=(3m/M)^2(A/c)^2\{1-8[w_{12}(0)/c]^2a_2\cdot b_2-(2/c^2)[w_{12}^2(0)+2w_{23}^2(0)]a_3\cdot b_3\} . \tag{22}$$

The observable composite state has a world line described by the four-vector $X=(m/M)y_1$, and as a basis of an irreducible representation of the Poincaré group it is completely specified by the mass

$$\begin{aligned} M_{123}=M\{&1-(8/c^2)w_{12}^2(0)a_2\cdot b_2 \\ &-(2/c^2)[w_{12}^2(0)+2w_{23}^2(0)]a_3\cdot b_3\}^{1/2} , \end{aligned} \tag{23}$$

three-momentum

$$\mathbf{P}=M_{123}\dot{\mathbf{X}}/[1-(\mathbf{v}_{123}/c)^2]^{1/2} , \tag{24}$$

where $\mathbf{v}_{123}/c=d\mathbf{X}/dX_0$, and internal angular momentum $\mathbf{j}=\mathbf{J}-\mathbf{J}(\text{c.m.})$ with $\mathbf{J}(\text{c.m.})=\mathbf{X}\times\mathbf{P}$. We find

$$\begin{aligned} \mathbf{j}=&-4i\sqrt{mm_1}\,w_{12}(0)\mathbf{a}_2\times\mathbf{b}_2 \\ &-i\sqrt{mM}\,[w_{12}^2(0)+2w_{23}^2(0)]^{1/2}\mathbf{a}_3\times\mathbf{b}_3 . \end{aligned} \tag{25}$$

We were able to diagonalize the three-constituent equations of motion because of the choice $w_{12}=w_{13}$. The resulting solutions allow an interpretation which has an analog in the quark model based on the unitary group SU(3). Consider the matrix λ in (20) which yields the coefficients of the normal coordinates y_1, y_2, and y_3 in each of the three solutions in (18). For the case $m=m_1=M/3$, the coefficient of y_1 (column 1 of λ) is the analog to baryon number, while the coefficients of y_2 (column 2) and y_3 (column 3) play the roles of hypercharge and the third component of isospin, respectively. By allowing $m_1\to\infty$, the general matrix (20) yields a similar interpretation based on SU(2), and is equivalent to the two-body solution in (10). The SU(3) analog will be introduced again in Sec. VII.

The inverse of λ

$$\lambda^{-1}=\begin{bmatrix} m_1/m & 1 & 1 \\ 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & -1 \end{bmatrix} . \tag{26}$$

IV. 4N-CONSTITUENT MODEL

A. Solutions to the equations of motion

In I, models of composite-particle scattering were constructed from constituents of two kinds: "Like" constituents repelled, while "opposite" constituents attracted each other. Here we expand the formulation to include four types of constituents. It is assumed that the system contains N of each kind of constituent, i.e.,

$$q_{IA}, \quad I=1,2,\ldots,N; \quad A=1,2,3,4 .$$

The potential takes the form

$$V=1-\tfrac{1}{2}\sum_{I,J}^{N}\sum_{A,B}^{4}[f_{AB}(0)/c]^2(x_{IA}-x_{JB})^2 . \tag{27}$$

It is convenient to go to matrix notation with general vectors denoted by

$$\mathbf{x}=\begin{bmatrix} x_{I1} \\ x_{I2} \\ x_{I3} \\ x_{I4} \end{bmatrix},$$

and the coupling matrix defined by

$$\begin{aligned} \mathbf{f}^2 &\equiv \{f_{AB}^2\} \\ &\equiv (4N/c^2)\left[\sum_{I=1}^{N}(\dot{\mathbf{x}}_I)^2/V\right]\{f_{AB}^2(0)\} . \end{aligned} \tag{28}$$

We consider the case when $\mathbf{f}^2$ takes the form

$$
\begin{aligned}
&f_{AA}^2 = f_0^2 , \\
&f_{13}^2 = f_{24}^2 = \tfrac{1}{2}(f_x^2 + f_y^2) , \\
&f_{14}^2 = f_{23}^2 = \tfrac{1}{2}(f_y^2 + f_z^2) , \\
&f_{12}^2 = f_{34}^2 = \tfrac{1}{2}(f_x^2 + f_z^2)
\end{aligned}
\tag{29}
$$

with analogous definitions of $f_0(0)$, $f_x(0)$, $f_y(0)$, $f_z(0)$, in terms of $f_{AB}(0)$. The masses are taken to be equal.

With these relations, the coupling matrix $\mathbf{f}^2$ can be diagonalized, and the solutions expressed in terms of normal coordinates. After some algebra, we obtain the solutions

$$\mathbf{x}_I = \mathbf{y}_I + \gamma \mathbf{W} \tag{30}$$

with

$$\gamma = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} = 4\gamma^{-1} , \tag{31}$$

$$
\begin{aligned}
&\mathbf{y}_I = \mathbf{a}_I \exp(\alpha s) + \mathbf{b}_I \exp(-\alpha s) , \\
&\alpha^2 = -\tfrac{1}{4}(f_0^2 + f^2) , \\
&\sum_{I=1}^{N} \mathbf{y}_I = 0 ,
\end{aligned}
\tag{32}
$$

and

$$
\begin{aligned}
&W_1 = As + B , \\
&W_2 = a_2 \exp(\beta s) + b_2 \exp(-\beta s) , \\
&\beta^2 = -\tfrac{1}{4}(f_x^2 + f_y^2 + 2f_z^2) , \\
&W_3 = a_3 \exp(i w_3 s) + b_3 \exp(-i w_3 s) , \\
&w_3^2 = \tfrac{1}{4}(f_x^2 + 2f_y^2 + f_z^2) , \\
&W_4 = a_4 \exp(i w_4 s) + b_4 \exp(-i w_4 s) , \\
&w_4^2 = \tfrac{1}{4}(2f_x^2 + f_y^2 + f_z^2) .
\end{aligned}
\tag{33}
$$

We further take α^2, β^2, w_3^2, and w_3^2 to be positive and greater than zero.

The frequencies are calculated using (6) or (28):

$$
\begin{aligned}
[\alpha/\alpha(0)]^2 &= [\beta/\beta(0)]^2 \\
&= [w_3/w_3(0)]^2 = [w_4/w_4(0)]^2 \\
&= (4N)^2 (A/c)^2 \Big/ \Bigg[1 + N[4\alpha(0)/c]^2 \sum_{I=1}^{N} \mathbf{a}_I \cdot \mathbf{b}_I + 4[4N\beta(0)/c]^2 a_2 \cdot b_2 \\
&\qquad - 4[4Nw_3(0)/c]^2 a_3 \cdot b_3 - 4[4Nw_4(0)/c]^2 a_4 \cdot b_4 \Bigg] ,
\end{aligned}
\tag{34}
$$

where $\alpha(0)$, $\beta(0)$, $w_3(0)$ and $w_4(0)$ are defined in terms of $f_0(0)$, $f_x(0)$, $f_y(0)$, $f_z(0)$, in analogy to α, β, w_3, and w_4 in (33).

The solutions x_{IA} are characterized by a term dependent on I and linear combinations of the normal coordinates W_A. We shall denote y_{IA} as the "kinematic" term and the normal coordinates as the "intrinsic" terms.

In solution (30) for the $4N$ constituents, there are $8\times(4N)-2$ arbitrary constants, which could be *mathematically* determined by specifying initial conditions in the evolution parameter s. Since s does not correspond to the physical clock, we approach the problem differently. The number of arbitrary constants is reduced by imposing boundary conditions on the constituents, namely that they cannot exist as free particles, but that asymptotically in s they bind together into observable composite states (see also the discussion in I). For the $4N$ system at hand, more specifically, we impose the following.

B. Boundary conditions and selection rules

Assume there is an "interaction interval" in s, defined by

$$-s_0 < s < s_0 \tag{35}$$

for which the couplings $f_0(0)$, $f_x(0)$, $f_y(0)$, $f_z(0)$, etc., take the form analogous to (29). Assuming also that s_0 is large, i.e., tends to infinity. For physical solutions to the system, impose the condition that as $s \to \pm s_0$, and for s outside the interaction interval, the constituents must pair up into two-body harmonic-oscillator states taking the form described in Sec. III. That is, they must form the bases for irreducible representations of the Poincaré group. This implies that for

$$s < -s_0 \text{ and } s > s_0 ,$$

the couplings appearing in the exponential terms must vanish. Further, the frequencies associated with the oscillations must be equal. These conditions are met provided that the couplings in the intervals $|s| > s_0$ satisfy

$$f_0^2(0) = f_z^2(0) = -f_x^2(0) = -f_y^2(0) . \tag{36}$$

Solutions outside the interaction interval are considered in detail in Sec. V.

Selection rules will also be imposed upon the solutions by specifying the initial and final configurations of constituents. For example, Fig. 1 contains a schematic representation of two-composite scattering (discussed in detail in Sec. VI).

In the Appendix, the solutions and their derivatives are matched at $s = \pm s_0$ to the solutions outside the interac-

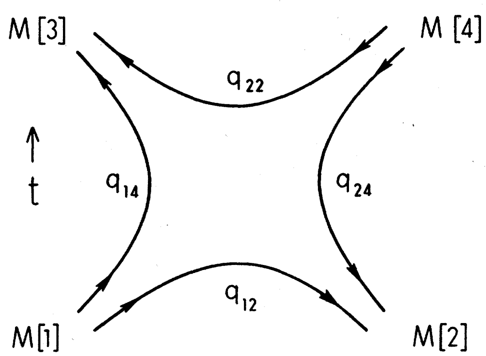

FIG. 1. The scattering of two composite particles via exchange of constituents. The arrows denote the sense of increasing s, while the observer's time axis is in the vertical direction. The constituents pair up as $s \rightarrow \pm s_0$ to form composites $M[J]$, $J=1,2,3,4$.

tion interval, giving relations for the arbitrary constants. Rather than introduce unnecessary additional complications in the notation, we shall retain the exponential form of the solutions with the understanding that as $s \rightarrow \pm s_0$, the exponential terms should be replaced by terms linear in s.

In order to meet the boundary conditions imposed above, the frequencies w_3 and w_4 within the interaction interval must be independent of the kinematics of the composite particles. That is, the internal state of a composite particle should not depend on the kinematic behavior of other composites in the system. *We shall make the following assumption: The individual terms* $\mathbf{y}_I$ *determine the kinematic behavior of the composite particles; the solutions* W_A *are considered to be an intrinsic part of the total system.* This leads to the important frequency condition given below.

1. The frequency condition

Examination of the expression for the frequencies (34) shows that for it to be independent of external kinematics, i.e., independent of the constants $\mathbf{a}_I$, $\mathbf{b}_I$, the following condition must hold:

$$\sum_{I=1}^{N} \mathbf{a}_I \cdot \mathbf{b}_I = 0 \; . \tag{37}$$

Equation (34) then reduces to

$$\begin{aligned}
[\alpha/\alpha(0)]^2 &= [\beta/\beta(0)]^2 \\
&= [w_3/w_3(0)]^2 = [w_4/w_4(0)]^2 \\
&= (A)^2/\{(c/4N)^2 + 4[\beta(0)]^2 a_2 \cdot b_2 - 4[w_3(0)]^2 a_3 \cdot b_3 - 4[w_4(0)]^2 a_4 \cdot b_4\} \; .
\end{aligned} \tag{38}$$

C. Energy, momentum, and angular momentum for the $4N$-constituent system

The conservation laws for the system of $4N$ constituents appear in terms of the parameter s. The conserved quantities P^μ and $J^{\mu\nu}$ cannot be identified with the physical and angular momentum of the system unless the evolution parameter can be identified with the physical clock, as was done for the isolated two- and three-body systems of Sec. III. However, as is demonstrated in the examples below, s is not in general the parameter that describes the evolution of the system as the observer measures it.

Nevertheless, at the observer's times $t_{\mathrm{OB}} = \pm\infty$, boundary conditions imply that if any constituent particles are present, they must pair up into observable composites. We can then, at these asymptotic times, define the energy, momentum, and angular momentum as the sum of the corresponding quantities for the composite states. We shall find that the imposition of constituent confinement implies the space-time conservation laws.

D. Formation of composite states

To simplify the scattering examples and to illustrate the main features of the constituent model, we shall restrict the examples considered in this article to the case defined by

$$f_0{}^2 = f_z{}^2, \;\; f_x{}^2 = f_y{}^2 \; . \tag{39}$$

Equation (39) implies that $\alpha=\beta$ and $w_3=w_4=w$. The $4N$-constituent solutions are given by

$$\begin{aligned}
x_{I1} &= [A_{I1}\exp(\alpha s) + B_{I1}\exp(-\alpha s)] \\
&\quad + (W_3 + W_4) + (As + B) \; , \\
x_{I2} &= [A_{12}\exp(\alpha s) + B_{I2}\exp(-\alpha s)] \\
&\quad + (W_3 - W_4) + (As + B) \; , \\
x_{I3} &= [A_{I3}\exp(\alpha s) + B_{I3}\exp(-\alpha s)] \\
&\quad - (W_3 + W_4) + (As + B) \; , \\
x_{I4} &= [A_{I4}\exp(\alpha s) + B_{I4}\exp(-\alpha s)] \\
&\quad - (W_3 - W_4) + (As + B) \; ,
\end{aligned} \tag{40}$$

where

$$\begin{aligned}
A_{I1,3} &= a_{I1,3} + a_2 \; , \\
B_{I1,3} &= b_{I1,3+b_2} \; , \\
A_{I2,4} &= a_{I2,4} - a_2 \; , \\
B_{I2,4} &= b_{I2,4} - b_2 \; ,
\end{aligned} \tag{41}$$

and

$$\sum_{I=1}^{N} a_{IA} = \sum_{I=1}^{N} b_{IA} = 0 \ . \tag{42}$$

In order for a composite to be formed from q_{IA} and q_{JB}, the relative vector must satisfy

$$x_{IA} - x_{JB} = \text{oscillatory terms only as } s \to \pm s_0 \ .$$

Furthermore, if the composite is to be a free particle, its four-vector must satisfy

$$\tfrac{1}{2}(x_{IA} + x_{JB}) = \text{linear terms only as } s \to \pm s_0 \ .$$

Thus, only q_{I1} and q_{J3}, or q_{I2} and q_{J4}, can form composite states. (In terms of the coupling constants, the attractive couplings form composite states.)

The constituent solutions given by (40) could be used to construct models similar to the examples in I. However, to avoid the presence of catalytic zero-mass composites, we shall instead adopt a different approach. Solutions of the $4N$ model outside the interaction region are examined in the next section. These lead to an alternative definition of the "physical" constituents as linear combinations of the solutions x_{IA}. We shall also assign various intrinsic properties, or intrinsic numbers, to the physical constituents based upon the columns of the matrix γ.

V. SOLUTIONS OUTSIDE THE INTERACTION INTERVAL: PHYSICAL CONSTITUENTS

The boundary conditions force the constituents to pair up into composite states outside the interaction interval, i.e., when $s < -s_0$ or $s > s_0$. Thus, the conditions (36) follow. The equations of motion outside the interaction interval can be expressed as

$$\ddot{\mathbf{z}}_I = (1/4N)\mathbf{f}^2 \mathbf{z}_I \ . \tag{43}$$

Using the matrix

$$\delta = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix} \tag{44}$$

to diagonalize $\mathbf{f}^2$ instead of γ in (31), we obtain solutions of the form

$$\begin{aligned} z_{I1} &= A_{I1}s + B_{I1} + [a_{I1}\exp(iws) + b_{I1}\exp(-iws)] \ , \\ z_{I3} &= A_{I3}s + B_{I3} - [a_{I1}\exp(iws) + b_{I1}\exp(-iws)] \ , \\ z_{I2} &= A_{I2}s + B_{I2} + [a_{I2}\exp(iws) + b_{I2}\exp(-iws)] \ , \\ z_{I4} &= A_{I4}s + B_{I4} - [a_{I2}\exp(iws) + b_{I2}\exp(-iws)] \ , \end{aligned} \tag{45}$$

where from (3)

$$\sum_{I=1}^{N} \sum_{A=1}^{4} z_{IA0} = A_0 s \ . \tag{46}$$

In terms of the couplings $(f_{AB}{}^2)$, Eq. (36) implies

$$f_{12} = f_{14} = f_{23} = f_{34} = 0 \ , \tag{47}$$

or the system breaks up into two identical but noninteracting systems, one consisting of the constituents q_{I1} and q_{I3}, and the other q_{I2} and q_{I4}.

We shall now make the following supposition: The constituent solutions within the interaction interval must "match" the solutions (45) at the end points defined by $\pm s_0$. This leads us to *define* the physical constituent solutions within the interaction interval by the linear combinations of the original solutions given below:

$$\begin{aligned} z_{I1} &= \tfrac{1}{2}(x_{I1} + x_{I2}) \ , \\ z_{I2} &= \tfrac{1}{2}(x_{I1} - x_{I2}) \ , \\ z_{I3} &= \tfrac{1}{2}(x_{I3} + x_{I4}) \ , \\ z_{I4} &= \tfrac{1}{2}(x_{I3} - x_{I4}) \ . \end{aligned} \tag{48}$$

Note that the solutions (48) are symmetric or antisymmetric under the exchange of the "original" constituents which become identical outside the interaction interval.

In the remainder of the paper *the "constituent solutions" will be taken to mean the physical constituent solutions unless otherwise indicated.* The notation q_{IA} will hereafter refer to the physical constituents.

It is important to recognize that although the solutions z_{IA} *satisfy harmonic-oscillator equations of motion outside the interaction interval, the equations of motion are not harmonic oscillator in form within the interaction interval.* There are two ways to interpret this. (1) The physical constituents satisfy more complicated equations of motion than the harmonic-oscillator equations within the interaction interval, or (2) since the constituents are not considered to be observed particles, we may treat them as mathematical entities, taking linear combinations of them to construct the four-vectors of physically measurable states. As we shall later see, the latter interpretation is strongly implied. We still retain the nomenclature classical to describe the formalism, however, since the theory is mathematically "deterministic" if only one knew all the "initial" conditions at some value of s.

In the next section, we examine the implications of these new definitions on the description of composite scattering.

VI. EXCITATION OF PHYSICAL COMPOSITE STATES: THE p AND THE x COMPOSITE PARTICLES

Inside the interaction interval, the constituent solutions fall into two groups. The first can be written as

$$\begin{aligned} z_{I1} &= a_{I1}\exp(\alpha s) + b_{I1}\exp(-\alpha s) + W_1 + W_3 \ , \\ z_{I3} &= a_{I3}\exp(\alpha s) + b_{I3}\exp(-\alpha s) + W_1 - W_3 \ , \end{aligned} \tag{49}$$

where W_1 and W_3 are given in (33), and

$$\sum_{I=1}^{N} a_{I1} = \sum_{I=1}^{N} b_{I1} = \sum_{I=1}^{N} a_{I3} = \sum_{I=1}^{N} b_{I3} = 0 \ . \tag{50}$$

The second group is

$$z_{I2}=a_{I2}\exp(\alpha s)+b_{I2}\exp(-\alpha s)+W_2+W_4 \; ,$$
$$z_{I4}=a_{I4}\exp(\alpha s)+b_{I4}\exp(-\alpha s)+W_2-W_4 \; , \tag{51}$$

with W_2 and W_4 given by (33), and

$$\sum_{I=1}^{N} a_{I2}=\sum_{I=1}^{N} a_{I4}=\sum_{I=1}^{N} b_{I2}=\sum_{I=1}^{N} b_{I4}=0 \; . \tag{52}$$

The two systems of solutions are related through the equation for the frequencies, which retains the form of (50) for the newly defined constants in (49) and (51):

$$[\alpha/\alpha(0)]^2=[w/w(0)]^2=(A)^2/\{(c/4N)^2+4[\alpha(0)]^2a_2\cdot b_2-4[w(0)]^2(a_3\cdot b_3+a_4\cdot b_4)\} \; , \tag{53}$$

and through the frequency condition, which also retains the form of (37):

$$\sum_{I=1}^{N}\sum_{A=1}^{4} a_{IA}\cdot b_{IA}=0 \; . \tag{54}$$

Just as for the system of equations (30), the arbitrary constants in the solutions (49) and (51) can be determined mathematically by specifying initial conditions in the parameter s. However, as we shall see, these do not correspond to physical initial conditions as measured in the laboratory. The initial-value problem will be discussed in more detail in the scattering example to follow. Here, we just mention that a complete set of initial conditions in s precludes any difficulties associated with causality.

It has already been assumed that the kinematic behavior of the composites is determined by the terms $a_{IA}\exp(\alpha s)$ and $b_{IA}\exp(-\alpha s)$, and that the solutions W_A are to be associated with intrinsic properties of the composites. Denoting the solutions (49) as the p constituents, and the solutions (51) as the x constituents, we shall define the coefficients of W_1 and W_2 as the P number and the X number, respectively (see, for example, the discussion in Sec. II regarding interpretation of the three-body composite state). Let us further assign the constituents the intrinsic properties denoted as the L number and B number, defined as the coefficients of W_3 and W_4, respectively. Thus two-body composite states must be constructed from constituents of opposite L number or of opposite B number: The two-body composites are characterized by zero L number and zero B number.

At this point, let us consider particle and antiparticle interpretations. As already noted, the constituents which turn around in time (see Fig. 1) can be reinterpreted as constituent-anticonstituent annihilation or pair production. Similarly, composites with the opposite sense of s can be interpreted as composities and anticomposites.

The model of this article differs from the usual quark model: The two-constituent states are not composed of a particle and antiparticle, but of two particles of opposite L or B number.

We shall consider again the particle-antiparticle interpretation in the scattering examples below.

A. Excitation of the x composite states

Assume that all the p-constituent solutions are identically equal to zero. For a system of $2N$ x composites, the lowest energy state (vacuum) corresponds to

$$a_{I2}=a_{I4}=b_{I2}=b_{I4}=a_2=b_2=0 \; . \tag{55}$$

In this case, the constituents (and composites) are all virtual particles, oscillating about the origin of space-time. In the examples to follow, the x constituents are excited into configurations representing composite scattering at arbitrary angles. Thus we have referred to them as the x (for "extended") constituents. All two-constituent x composite particles will be designated as M particles.

Setting the p solutions to zero places a constraint on the internal states of the x composite particles. Since $A=0$, the right-hand side of the expression for the frequency (53) must become indeterminate, or the internal amplitudes of the oscillating constituents satisfy

$$4[\alpha^2(0)a_2\cdot b_2-w_4{}^2(0)a_4\cdot b_4]=-(c/4N)^2 \; . \tag{56}$$

Consider now the excitation of the four constituents

$$q_{12},q_{14},q_{22},q_{24} \; , \tag{57}$$

as shown in Fig. 1. The constituent four-vectors are

$$z_{12}=A_{12}\exp(\alpha s)+B_{12}\exp(-\alpha s)+W_4 \; ,$$
$$z_{22}=A_{22}\exp(\alpha s)+B_{22}\exp(-\alpha s)+W_4 \; ,$$
$$z_{14}=A_{14}\exp(\alpha s)+B_{14}\exp(-\alpha s)-W_4 \; , \tag{58}$$
$$z_{24}=A_{24}\exp(\alpha s)+B_{24}\exp(-\alpha s)-W_4 \; ,$$

where

$$A_{I2}=a_{I2}+a_2,\;\; A_{I4}=a_{I4}+a_2 \; ,$$
$$B_{I2}=b_{I2}+b_2,\;\; B_{I4}=b_{I4}+b_2 \; , \tag{59}$$

and

$$A_{12}+A_{22}=A_{14}+A_{24}=Na_2 \; ,$$
$$B_{12}+B_{22}=B_{14}+B_{24}=Nb_2 \; . \tag{60}$$

The four constituents are assumed to form the configuration shown schematically in Fig. 1. The arrows denote the sense of increasing s, and the observer's time axis is in the vertical direction. The constituents q_{12} and q_{14}, and q_{22} and q_{24}, pair up as $s\to -s_0$ to form composites $M[1]$ and $M[4]$, respectively. As $s\to +s_0$, constituents q_{12} and q_{24}, and q_{22} and q_{14} pair up to form composites $M[2]$ and $M[3]$, respectively. Note that the *constituents* q_{12} and q_{22} turn around in the observer's time. We can interpret the trajectory of q_{12} as representing a constituent and an anticonstituent annihilating in time, and the trajectory q_{22} as representing the creation of a constituent and anticonstituent pair. Thus we can interpret the diagram as follows. The initial state consists of two composites, $M[1]$ and $M[2]$ which subsequently come together and through the exchange of constituents lose their identity and yield the final state consisting of composites $M[3]$ and $M[4]$.

We have assumed that particle and antiparticle four-vectors are identified as having time components which are increasing and decreasing functions of s, respectively. Thus $M[1]$ and $M[3]$ are composites, and $M[2]$ and $M[4]$ are anticomposites. If we further assume that the X number is equal but opposite in sign for the composite (constituent) and the anticomposite (anticonstituent), then it follows that

$$a_2 = -b_2 \; . \tag{61}$$

Applying the boundary conditions at $t_{OB} = \pm\infty$ yields the relations

$$\begin{aligned} &A_{12} = A_{24}, \quad A_{22} = A_{14} \; , \\ &B_{12} = B_{14}, \quad B_{22} = B_{24} \; , \end{aligned} \tag{62}$$

The asymptotic composite four-vectors become:

Incoming states

$$\begin{aligned} Z_1 &= \tfrac{1}{2}(z_{12} + z_{14})\,|_{s\to -s_0} \\ &= B_{12}\exp(-\alpha s) \; , \\ Z_2 &= \tfrac{1}{2}[z_{12}(-s) + z_{24}(-s)]\,|_{s\to -s_0} \\ &= A_{12}\exp(-\alpha s) \; . \end{aligned} \tag{63}$$

Outgoing states

$$\begin{aligned} Z_3 &= \tfrac{1}{2}(z_{22} + z_{14})\,|_{s\to s_0} \\ &= A_{22}\exp(\alpha s) \; , \\ Z_4 &= \tfrac{1}{2}[z_{22}(-s) + z_{24}(-s)]\,|_{s\to s_0} \\ &= B_{22}\exp(\alpha s) \; . \end{aligned} \tag{64}$$

With the boundary conditions, the frequency condition (54) becomes

$$(A_{12} + A_{22})\cdot(B_{12} + B_{22}) = 2Na_2\cdot b_2 \; , \tag{65}$$

or, applying (60),

$$N^2 a_2\cdot b_2 = 2Na_2\cdot b_2 \; . \tag{66}$$

For the case $a_2\cdot b_2 \neq 0$, this implies

$$N = 2 \; , \tag{67}$$

or all the constituents in the system are excited into the configuration of Fig. 1.

The relations (60) and (61) yield

$$(A_{12} + A_{22}) = -(B_{12} + B_{22}) \; , \tag{68}$$

so that from (63) and (64),

$$[(dZ_1/ds) + (dZ_2/ds)]\,|_{s\to -s_0} = [(dZ_3/ds) + (dZ_4/ds)]\,|_{s\to s_0} \; . \tag{69}$$

Consider a two-body harmonic-oscillator system composed of constituents I and J which have a relative vector equal to

$$a\exp(iws) + b\exp(-iws) \; .$$

Assume that each constituent has mass $m(\text{eff})$ and that they interact via coupling $w(0,\text{eff})$. Then from (13), the frequency is equal to

$$w = G[1 - (\mathbf{v_{IJ}}/c)^2]^{1/2}(dX_{IJ0}/ds)/2M_{IJ}(\text{eff}) \; , \tag{70}$$

where

$$M_{IJ}(\text{eff}) = 2m(\text{eff})\{1 - [w(0,\text{eff})/c]^2 a\cdot b\}^{1/2} \; , \tag{71}$$

and we have set

$$G = w(0,\text{eff})m(\text{eff}) \; . \tag{72}$$

X_{IJ0} is the fourth component of the composite's four-vector, and G is assumed to be the same constant for every composite formed in a given system.

From (70), it follows that the relation between the energy of the composite and the derivative of the fourth component of its four-vector is[5]

$$dX_{IJ0}/ds = \tfrac{1}{2}(w/Gc)E_{IJ} \; . \tag{73}$$

Thus, Eq. (69) implies the *conservation of energy and momentum in the observer's time.* Similarly, the orbital angular momentum (which is zero) and the internal angular momenta ("spin") are also conserved.

It follows that the masses of the scattering composites of Fig. 1 are given by

$$M_J = (G/2wc)[1 - (\mathbf{v}_J/c)^2]^{1/2} dZ_{J0}/ds \; , \quad J = 1,2,3,4 \; . \tag{74}$$

The composite velocities, defined by dZ_J/dZ_{J0}, are

$$\begin{aligned} \mathbf{v}_1/C &= \mathbf{B}_{12}/B_{120} \; , \\ \mathbf{v}_2/C &= \mathbf{A}_{12}/A_{120} \; , \\ \mathbf{v}_3/C &= \mathbf{A}_{22}/A_{220} \; , \\ \mathbf{v}_4/C &= \mathbf{B}_{22}/B_{220} \; . \end{aligned} \tag{75}$$

From (60) and (61), it follows that specifying the constant a_2 and the initial energies and velocities of the incoming composites $M[1]$ and $M[2]$ completely determines the energies and momenta of the final state. However, the constant a_2 cannot be determined from the kinematics of the initial state. We have already assumed that a_2 along with the "internal" constants a_4 and b_4 are determined by the intrinsic properties of the system. These, at least in part, may depend on the selection rules, i.e., which composite states exist in the observer's initial and final states. Clearly this is a problem which needs to be further investigated.

The internal states of all four composites are described by the same relative vector which equals

$$a_4\exp(iws) + b_4\exp(-iws) \; , \tag{76}$$

where the frequency $w=w_4$ is given by

$$[w/w(0)]^2=(4N)^2(A/c)^2/\{1+4[4N\alpha(0)/c]^2a_2\cdot b_2 4[4Nw(0)/c]^2a_4\cdot b_4\} . \tag{77}$$

However, since $A=0$, the right-hand side of the equation must become indeterminate for nontrivial solutions, or the internal constants must satisfy (56). Note that the composites all have equal spin:

$$j=i2G\,\mathbf{a}_4\times\mathbf{b}_4 . \tag{78}$$

B. Excitations of the p composites

Consider now the excitations of composites from the system of q_{I1}, q_{I3} constituents. The q_{I2} and q_{I4} constituent solutions are taken to be identically equal to zero. As long as any of the 1—3 constituents are virtual, the constants A and B must equal zero.[6] Because of the conditions (50), the system is equivalent to the models of the type produced in I: Excitation of four constituents q_{11}, q_{21}, q_{13}, q_{23} leads to forward/backward scattering of equal-mass composites, excitation of the appropriate six constituents leads to the second model of I. However, no matter how many constituents are excited, it is not difficult to verify that elastic scattering can take place in the forward/backward direction only. Thus, the composites behave very much like point particles. For this reason we have referred to the 1—3 constituents as the p (for "point-like") constituents and the resulting composites as the p composites.

In summary, redefining the four-vectors for the physical constituents in terms of symmetric and antisymmetric combinations of the original constituent four-vectors leads to the descriptions of composites of the p type and the x type, respectively. The p composite particles can scatter only in the restricted sense of the models of I, e.g., only forward/backward elastic scattering. The x composites can scatter at arbitrary angles. For the x composite scattering, the initial energies and velocities of the composites are not enough to uniquely determine the final state. The intrinsic parameter a_2 must also be known.

In the last section, solutions to the equations of motion were obtained for the constituents outside the interaction interval. In those solutions, the orbital angular momenta can take on arbitrary values. Thus we can interpret the model as follows: Only constituents that are "aimed" for a head-on collision will scatter, and in this sense, the composites act like point particles with a zero interaction range. However, scattering at arbitrary angles can take place for the x composites, so that if the observer is unable to measure the initial zero-orbital angular momentum state, he may interpret the scattering as due to the composite being extended in space and/or as having a finite interaction range.

VII. ADDITION OF "TINT"

A. Constituent-particle solutions

The coupling matrix is now enlarged in order to describe both two- and three-body composite states and their interactions. In analogy to "color" in the quark model, the new attribute will be denoted as "tint." The potential V is assumed to be

$$V=1-\tfrac{1}{2}\sum_{I,J}^{N}\sum_{A,B}^{4}\sum_{C,D}^{3}[f_{AB}(0)C_{CD}(0)/c]^2(x_{IA}^{C}-x_{JB}^{D})^2 , \tag{79}$$

where the couplings are factorizable into the "f couplings" and the "tint couplings." The constituent masses are taken to be equal.

Following Sec. II, we define new constants f_{AB} and C_{CD} through their product:

$$[f_{AB}C_{CD}/f_{AB}(0)C_{CD}(0)]^2=(12N)\left[\sum_{I=1}^{N}\dot{\mathbf{x}}_I^T\cdot\dot{\mathbf{x}}_I/Vc^2\right] , \tag{80}$$

where the vector $\dot{\mathbf{x}}_I$ has components $(\dot{x}_{IA}^{c})$. Tint-coupling matrices are defined by

$$\mathbf{C}^2=(C_{CD}{}^2) . \tag{81}$$

In analogy to the color quark model, it will be assumed that the constituents of different C behave similarly; i.e., the matrix $\mathbf{C}^2$ is given the form

$$\mathbf{C}^2=\begin{bmatrix} C_1{}^2 & C_{12}{}^2 & C_{12}{}^2 \\ C_{12}{}^2 & C_1{}^2 & C_{12}{}^2 \\ C_{12}{}^2 & C_{12}{}^2 & C_1{}^2 \end{bmatrix} . \tag{82}$$

The matrix $\mathbf{C}^2$ is diagonalized by the matrix λ given in (20) in Sec. III with $m_1=m=\frac{1}{3}M$. The matrix $\mathbf{f}^2$ is diagonalized by γ in (31). We continue to examine the special case where $f_x{}^2=f_y{}^2$, and $f_0{}^2=f_z{}^2$. The solutions may be written as

$$\mathbf{x}_I=\mathbf{y}_I+\lambda\otimes\gamma\mathbf{W} , \tag{83}$$

where

$$\sum_{I=1}^{N}\mathbf{y}_I=0 , \tag{84}$$

$$\ddot{\mathbf{y}}_I=\alpha^2\mathbf{y}_I , \tag{85}$$

$$\alpha^2=\tfrac{1}{6}(f_0{}^2+f_x{}^2)(C_1{}^2+2C_{12}{}^2) , \tag{86}$$

and

$$\ddot{\mathbf{W}}=\beta^2\mathbf{W} \tag{87}$$

with

$$\beta^2=-\tfrac{1}{12}[(f_0{}^2+f^2)(C_1{}^2+2C_{12}{}^2)1\otimes 1-\Omega_A{}^2\otimes\Omega_c{}^2] , \tag{88}$$

and

$$\Omega_A{}^2=\gamma^{-1}f^2\gamma=\begin{vmatrix} 2(f_0{}^2+f_x{}^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & f_0{}^2-f_x{}^2 & 0 \\ 0 & 0 & 0 & f_0{}^2-f_x{}^2 \end{vmatrix}, \tag{89}$$

and

$$\Omega_c{}^2=\lambda^{-1}\mathbf{C}^2\lambda=\begin{bmatrix} C_1{}^2+2C_{12}{}^2 & 0 & 0 \\ 0 & C_1{}^2-C_{12}{}^2 & 0 \\ 0 & 0 & C_1{}^2-C_{12}{}^2 \end{bmatrix}. \tag{90}$$

We have, then,

$$\begin{aligned}
&(\beta^2)_{11}{}^{11}=0\ , \\
&(\beta^2)_{11}{}^{22}=(\beta^2)_{11}{}^{33}<0\ , \\
&(\beta^2)_{22}^{CC}=\alpha^2>0,\ \ C=1,2,3\ , \\
&(\beta^2)_{33}{}^{11}=(\beta^2)_{44}{}^{11}>0\ , \\
&(\beta^2)_{33}^{CC}=(\beta^2)_{44}^{CC}<0,\ \ C=2,3\ ,
\end{aligned} \tag{91}$$

where the positivity assumptions have been added. Thus, the solutions for $W_A{}^C$ exhibit the following behavior:

$W_1{}^1$ linear in s ,

$W_1{}^{2,3}$ oscillatory in s ,

W_2^c exponential in s ,

$W_{3,4}{}^1$ exponential in s ,

$W_{3,4}{}^2, W_{3,4}{}^3$, oscillatory in s .

In Sec. V, solutions outside the interaction interval $-s_0<s<s_0$ were obtained under the assumption that the asymptotic constituents must pair up into free composite particles. As a result, attractive and repulsive forces became exactly equal and opposite in magnitude, i.e.,

$$f_{11}{}^2(0)=f_{33}{}^2(0)=-f_{13}{}^2(0)\ ,$$

and

$$f_{22}{}^2(0)=f_{44}{}^2(0)=-f_{24}{}^2(0)$$

(outside interaction interval). Keeping the same assumption when tint is added, we find that $C_1{}^2(0)=-2C_{12}{}^2(0)$ when s is outside the interval. Thus, if s is outside the interaction interval, all terms in the constituent solutions become linear in s except $W_3{}^2$, $W_4{}^2$, $W_3{}^3$, and $W_4{}^3$, which remain oscillatory with equal frequencies. We will find that these terms are associated with the internal motion of the composite states. The remaining terms describe the space-time evolution of the composite's center-of-energy four-vector.

B. Definition of the physical constituents and intrinsic numbers

Following Secs. V and VI, we define the physical constituents by the following linear combinations of the solutions (83):

$$\begin{aligned}
&\mathbf{z}_{I1}=\tfrac{1}{2}(\mathbf{x}_{I1}+\mathbf{x}_{I2})\ , \\
&\mathbf{z}_{I2}=\tfrac{1}{2}(\mathbf{x}_{I1}-\mathbf{x}_{I2})\ , \\
&\mathbf{z}_{I3}=\tfrac{1}{2}(\mathbf{x}_{I3}+\mathbf{x}_{I4})\ , \\
&\mathbf{z}_{I4}=\tfrac{1}{2}(\mathbf{x}_{I3}-\mathbf{x}_{I4})\ ,
\end{aligned} \tag{92}$$

The solutions once again fall into two groups which can be expressed as follows.

p constituents:

$$\begin{aligned}
&\mathbf{z}_{I1}=\mathbf{a}_{I1}\exp(\alpha s)+\mathbf{b}_{I1}\exp(-\alpha s)+\lambda(\mathbf{W}_1+\mathbf{W}_3)\ , \\
&\mathbf{z}_{I3}=\mathbf{a}_{I3}\exp(\alpha s)+\mathbf{b}_{I3}\exp(-\alpha s)+\lambda(\mathbf{W}_1-\mathbf{W}_3)\ ,
\end{aligned} \tag{93}$$

where

$$\sum_{I=1}^{N}\mathbf{a}_{I1}=\sum_{I=1}^{N}\mathbf{a}_{I3}=\sum_{I=1}^{N}\mathbf{b}_{I1}=\sum_{I=1}^{N}\mathbf{b}_{I3}=0\ . \tag{94}$$

x constituents:

$$\begin{aligned}
&\mathbf{z}_{I2}=\mathbf{a}_{I2}\exp(\alpha s)+\mathbf{b}_{I2}\exp(-\alpha s)+\lambda(\mathbf{W}_2+\mathbf{W}_4)\ , \\
&\mathbf{z}_{I4}=\mathbf{a}_{I4}\exp(\alpha s)+b_{I4}\exp(-\alpha s)+\lambda(\mathbf{W}_2-\mathbf{W}_4)
\end{aligned} \tag{95}$$

with

$$\sum_{I=1}^{N}\mathbf{a}_{I2}=\sum_{I=1}^{N}\mathbf{a}_{I4}=\sum_{I=1}^{N}\mathbf{b}_{I2}=\sum_{I=1}^{N}\mathbf{b}_{I4}=0\ . \tag{96}$$

Using (80), (79), (83), and the solutions above to calculate the frequencies, and imposing the frequency condition

$$\sum_{I=1}^{N}\sum_{A=1}^{4}\sum_{C=1}^{3}a_{IA}^C\cdot b_{IA}^C=0\ , \tag{97}$$

we obtain

$$\begin{aligned}
[w_1^C/w_1^C(0)]^2&=[w^C/w^C(0)]^2 \\
&=[\alpha/\alpha(0)]^2 \\
&=(4N)^2(A/c)^2\bigg/\left[1+12(4N/c)^2\sum_{A=1}^{4}\sum_{C=1}^{3}[(\beta^2(0))_{AA}^{CC}\{\lambda^\dagger\lambda\}_{CC}a_A^C\cdot b_A^C]\right].
\end{aligned} \tag{98}$$

The x and p constituents are related through the frequency (98) and through the frequency condition (97).

In Sec. VI, the four columns of the matrix γ were defined as the P, X, L, and B numbers of the constituents, respectively. The three columns of the λ matrix are now added as the T, Y, and I numbers, respectively. Collectively, we shall denote all of the above as intrinsic numbers, in analogy to the additive quantum numbers of quantum particle physics. The corresponding intrinsic numbers of composite states are the sums of the intrinsic numbers of the individual constituents.

The excitation of the p constituents follows in the similar manner to Sec. VI, and are not considered further in this paper. The x constituents are discussed below.

C. Formation of x composite states

The p-constituent solutions are assumed to be identically zero. It follows that $A=0$, and from (98), the intrinsic amplitudes of the x composites are constrained by

$$12(4N/c)^2 \sum_{A=1}^{4} \sum_{C=1}^{3} [(\beta^2(0))^{CC}_{AA}\{\lambda^\dagger\lambda\}_{CC} a^C_A \cdot b^C_A] = -1 \ . \tag{99}$$

It is convenient to reexpress the solutions z^C_{IA} in the form

$$\mathbf{z}_{I2} = \mathbf{A}_{I2}\exp(\alpha s) + \mathbf{B}_{I2}\exp(-\alpha s) + \lambda \mathbf{W}_4 \ ,$$
$$\mathbf{z}_{I4} = \mathbf{A}_{I4}\exp(\alpha s) + \mathbf{B}_{I4}\exp(-\alpha s) - \lambda \mathbf{W}_4 \tag{100}$$

with

$$\sum_{I=1}^{N} \mathbf{A}_{I2,4} = \sum_{I=1}^{N} (\mathbf{a}_{I2,4} + \lambda \mathbf{a}_2) = N\lambda \mathbf{a}_2 \ ,$$
$$\sum_{I=1}^{N} \mathbf{B}_{I2,4} = \sum_{I=1}^{N} (\mathbf{b}_{I2,4} + \lambda \mathbf{b}_2) = N\lambda \mathbf{b}_2 \ . \tag{101}$$

The frequency condition becomes

$$\sum_{I=1}^{N} \sum_{A=1}^{4} \mathbf{A}_{IA} \cdot \mathbf{B}_{IA} = 2N(\lambda \mathbf{a}_2)^{T} \cdot (\lambda \mathbf{b}_2) \ . \tag{102}$$

The internal oscillatory terms $W_4{}^2$ and $W_4{}^3$ must cancel in forming composite four-vectors. This can be accomplished in one of the three ways below.

M composites (B=0):

$$\tfrac{1}{2}(z^C_{I2}+z^C_{J4}) = [\tfrac{1}{2}(a^C_{I2}+a^C_{J4}) + \{\lambda\}_{CD} a^D_2]\exp(\alpha s) + [\tfrac{1}{2}(b^C_{I2}+b^C_{J4}) + \{\lambda\}_{CD} b^D_2]\exp(-\alpha s) \ . \tag{103}$$

B composites (B=1):

$$\tfrac{1}{3}(z_{I2}{}^1+z_{J2}{}^2+z_{K2}{}^3) = \tfrac{1}{3}(a_{I2}{}^1+a_{J2}{}^2+a_{k2}{}^3+a_2{}^1)\exp(\alpha s) + \tfrac{1}{3}(b_{I2}{}^1+b_{J2}{}^2+b_{k2}{}^3+b_2{}^1)\exp(-\alpha s) + W_4{}^1 \ . \tag{104}$$

B composites (B=−1):

$$\tfrac{1}{3}(z_{I4}{}^1+z_{J4}{}^2+z_{k4}{}^3) = \tfrac{1}{3}(a_{I4}{}^1+a_{J4}{}^2+a_{k4}{}^3+a_2{}^1)\exp(\alpha s) + \tfrac{1}{3}(b_{I4}{}^1+b_{J4}{}^2+b_{k4}{}^3+b_2{}^1)\exp(-\alpha s) - W_4{}^1 \ . \tag{105}$$

All x composite states have $X=1$. By requiring all anticomposites (obtained by reversing the sign of s) to have $X=-1$, we obtain the relation

$$a^c_2 = -b^c_2 \ . \tag{106}$$

For simplicity, we shall examine solutions for which $W_4{}^1=0$. By limiting the examples to such systems, we find that there is no way to physically distinguish the $B=1$ and $B=-1$, since the four-vectors take the same form and we have assumed the internal states are not measurable. In addition, the intrinsic property of tint is not measurable either.

In the paper so far, we have left the asymptotic four-vectors in the exponential form, with the understanding that as $s \to \pm s_0$, the exponents are to be replaced by terms linear in s. We shall continue to do so.

D. M-M scattering

The 12 constituents

$$q^C_{I2}, q^C_{I4}, \quad I=1,2, \quad C=1,2,3 \ , \tag{107}$$

are excited into the configuration of Fig. 2. The schematic diagram represents the existence of three different systems corresponding to two-body scattering of M particles. Each of the three is characterized by a different tint.

Applying the boundary conditions at $s \to \pm s_0$ to the constituent solutions (100), we obtain the asymptotic M-composite four-vectors.

Incoming:

$$Z^C_1 = B^C_{12}\exp(-\alpha s), \quad Z^C_2 = A^C_{12}\exp(-\alpha s) \ , \tag{108}$$

Outgoing:

$$Z^C_3 = A^C_{22}\exp(\alpha s), \quad Z^C_4 = B^C_{22}\exp(\alpha s) \tag{109}$$

with

$$\sum_{I=1}^{2} A^C_{I2} = N\{\lambda\}_{CD} a^D_2 = -\sum_{I=1}^{2} B^C_{I2} \ . \tag{110}$$

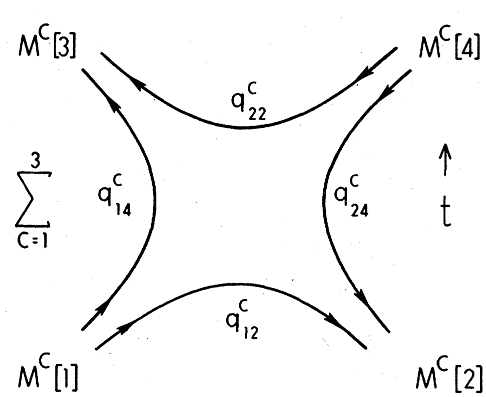

FIG. 2. Twelve constituents are excited to form three separate scattering systems corresponding to $MM \to MM$. Each of the three is characterized by a different value of tint C. The example in the text corresponds to the superposition of the three systems.

It follows from (110) that

$$(dZ_1^C/ds + dZ_2^C/ds)\,|_{s\to -s_0} = (dZ_3^C/ds + dZ_4^C/ds)\,|_{s\to s_0}\ , \tag{111}$$

or space-time conservation laws hold separately for each of the systems.

Substituting (110) into the frequency condition (102) yields

$$N^2(\lambda \mathbf{a}_2)^2 = 2N(\lambda \mathbf{a}_2)^2\ . \tag{112}$$

For $\mathbf{a}_2 \neq 0$, it follows that $N=2$, or all the constituents of the system are excited into the scattering states. For $\mathbf{a}_2 = 0$, arbitrary N is allowed, but the example reduces to the cases described in I, and leads, for example, to elastic scattering only in the forward/backward direction.

In the general case $\mathbf{a}_2 \neq 0$, the frequency condition is satisfied provided there exists three scattering systems of differing tint. The physically interesting case is when the M-composite four-vectors have superimposed trajectories, i.e.,

$$A_{I2}{}^1 = A_{I2}{}^2 = A_{I2}{}^3,\ \ B_{I2}{}^1 = B_{I2}{}^2 = B_{I2}{}^3\ . \tag{113}$$

In this case, the observer sees only two M composites in the initial and final states, respectively. Note that the superposition of the M-composite center-of-mass four-vectors does not imply superpositions of the *constituent* trajectories.

E. M-B scattering

To verify the space-time conservation laws for M-M scattering, we implicitly used (73) to relate energy of the M composite to the fourth component of its four-vector:

$$dZ_{IJ0}/ds = \tfrac{1}{2}(w/Gc)E_{IJ}\ . \tag{73}$$

For a three-body harmonic-oscillator state, or B particle, the analogous relation is

$$dZ_{IJK0}/ds = \tfrac{1}{3}(w/Gc)E_{IJK}\ , \tag{114}$$

where $w = w_4{}^2 = w_4{}^3$. G is given by (72), and it is assumed that for a given system of constituents, G is the same for all composites.

Consider the example where 18 constituents, namely,

$$q_{I2}^C, q_{I4}^C,\ \ C=1,2,3,\ \ I=1,2,\ldots,6\ , \tag{115}$$

are excited into the configuration shown in Fig. 3. The schematic illustration represents six separate reactions $MB \to MB$. From (101), it follows that

$$\sum_{I=1}^{6} A_{I2}^C = \sum_{I=1}^{6} A_{I4}^C = N\{\lambda\}_{CD} a_2^D = -\sum_{I=1}^{6} B_{I2}^C = -\sum_{I=1}^{6} B_{I4}^C\ . \tag{116}$$

In order to describe two-composite scattering, we take the case when the composite four-vectors are superimposed, noting that the six reactions are interpreted as a single one to the observer, who is not able to distinguish the different tints or $A=2$ and $A=4$ B particles. Denoting the physically observable composite four-vectors by

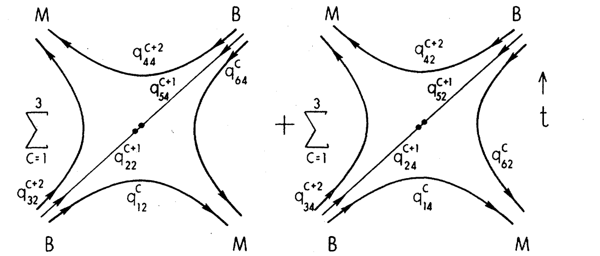

FIG. 3. Eighteen constituents are excited into the six separate reactions $MB \to MB$. (Note that for $C=3$, the notation $C+1$ stands for 1, etc.) The example in the text corresponds to the superposition of the six systems.

$$\begin{aligned} &Z_B^{\rm IN} = B_{\rm IN}\exp(-\alpha s),\ \ Z_A^{\rm IN} = A_{\rm IN}\exp(-\alpha s)\ ,\\ &Z_B{}^{\rm OUT} = B_{\rm OUT}\exp(\alpha s),\ \ Z_A^{\rm OUT} = A_{\rm OUT}\exp(\alpha s)\ , \end{aligned} \tag{117}$$

we apply the boundary conditions

$$\begin{aligned} &B_{I2}^C = B_{I4}^C = B_{\rm IN},\ \ C=1,2,3,\ \ I=1,2,3\ ,\\ &B_{I2}^C = B_{I4}^C = B_{\rm OUT},\ \ C=1,2,3,\ \ I=4,5,6\ ,\\ &A_{22}^C = A_{24}^C = A_{54}^C = A_{52}^C = 0,\ \ C=1,2,3\ ,\\ &A_{32}^C = A_{44}^C = A_{34}^C = A_{42}^C = A_{\rm OUT},\ \ C=1,2,3\ ,\\ &A_{12}^C = A_{14}^C = A_{64}^C = A_{62}^C = A_{\rm OUT},\ \ C=1,2,3\ . \end{aligned} \tag{118}$$

Then the relations (101) yield

$$2(A_{\rm IN} + A_{\rm OUT}) = (N/3)a_2{}^1 = -3(B_{\rm IN} + B_{\rm OUT})\ , \tag{119}$$

and from (73) and (114), the space-time conservation laws follow.

After applying the boundary conditions, the frequency condition becomes

$$Na_2{}^1 = 6a_2{}^1\ , \tag{120}$$

or, for the general case $a_2{}^1 \neq 0$, it follows that $N=6$. All constituents of the $N=6$ system are excited into the scattering M and B composite particles.

The intrinsic numbers for the M-B system remain unchanged in the initial and final state, and are given by

$$X=0,\ \ B=0,\ \ T=\tfrac{2}{3},\ \ Y=0,\ I=0\ . \tag{121}$$

F. B-B scattering

As a final example, take the case of B-B scattering where 48 constituents

$$q_{I2}^C, q_{I4}^C,\ \ C=1,2,3,\ \ I=1,2,\ldots,8\ , \tag{122}$$

are excited into the configurations depicted in Fig. 4. Again take the case when the systems are superimposed. The observer sees two B composites in the initial and final state which have the following four-vectors

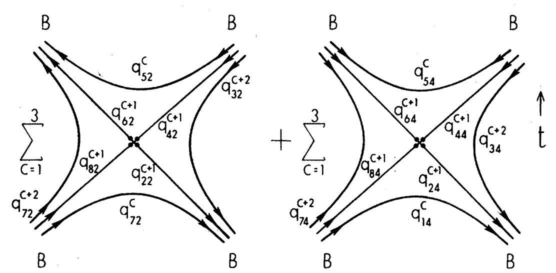

FIG. 4. Forty-eight constituents are excited into the six reactions $BB \to BB$, which are superimposed in the example considered in the text.

Incoming:

$$Z_1 = B_{\rm IN} \exp(-\alpha s) ,$$

$$Z_2 = A_{\rm IN} \exp(-\alpha s) .$$

Outgoing:

$$Z_3 = A_{\rm OUT} \exp(\alpha s) ,$$

$$Z_4 = B_{\rm OUT} \exp(\alpha s) , \tag{123}$$

where the boundary conditions below have been applied.

$$\begin{aligned}
&B_{12}^C = B_{14}^C = B_{82}^C = B_{72}^C = B_{85}^C = B_{74}^C = B_{\rm IN} ,\\
&B_{32}^C = B_{42}^C = B_{52}^C = B_{34}^C = B_{44}^C = B_{54}^C = B_{\rm OUT} ,\\
&A_{82}^C = A_{85}^C = A_{42}^C = A_{44}^C = 0 ,\\
&A_{12}^C = A_{22}^C = A_{32}^C = A_{14}^C = A_{24}^C = A_{34}^C = A_{\rm IN} ,\\
&A_{52}^C = A_{62}^C = A_{72}^C = A_{54}^C = A_{64}^C = A_{74}^C = A_{\rm OUT} ,\\
&B_{22}^C = B_{62}^C = B_{24}^C = B_{64}^C = 0 .
\end{aligned} \tag{124}$$

From (101), it follows that

$$3(A_{\rm IN} + A_{\rm OUT}) = (N/3) a_2{}^1 = -3(B_{\rm IN} + B_{\rm OUT}) , \tag{125}$$

which, in turn, implies the space-time conservation laws.

The frequency condition becomes

$$(A_{\rm IN} + A_{\rm OUT}) \cdot (B_{\rm IN} + B_{\rm OUT}) = -N (a_2{}^1)^2/9 , \tag{126}$$

or, for $a_2{}^1 \neq 0$, $N=9$. The remaining constituents q_{92}^c and q_{94}^c can be taken to be virtual particles.

Once again, all intrinsic numbers are conserved in the reaction.

VIII. SUMMARY

Restricting our attention at first to the simpler case of two-constituent particles, we introduced a model based on a system consisting of $4N$ constituents, q_{IA}, $I = 1, \ldots, N$, $A = 1,2,3,4$. (That is, there are N each of four kinds of constituents.) It was found that constituent solutions are linear combinations of a kinematic term and a set of normal coordinates W_A, the latter being subsequently identified with the intrinsic properties of the composite particles. Their coefficients are denoted as intrinsic numbers.

Boundary conditions which imply the confinement of the constituents and the asymptotic formation of composite particles were formulated. This led to a constraint, denoted as the "frequency condition," which is a statement that the internal behavior of the composite particles cannot depend upon the kinematic behavior of the composites.

At this point, it would have been possible to construct scattering examples similar to those of I. However, all such examples suffer the same drawback, namely, the presence of zero-mass particles. Therefore, in Sec. V, we introduced a mechanism to avoid this difficulty. It was demonstrated that asymptotically in s the system of constituents breaks up into two identical but noninteracting systems. This suggests the definition of a new set of constituent solutions which are symmetric or antisymmetric under the exchange of identical constituents. These linear combinations of the original solutions (i.e., the solutions obtained in Sec. IV), which we labeled the physical constituents, were used to construct composite-particle four-vectors for finite s as well.

With the aid of the physical constituents, realistic examples of composite scattering were introduced in Sec. VI. The system of $4N$ physical constituents breaks up naturally into two systems of $2N$ constituents each: We have called the first system the p (pointlike) constituents; and the second system the x (extended) constituents. Each system is characterized by its own set of intrinsic numbers.

The excited p composite particles can undergo elastic scattering in the forward/backward angles only. Thus they behave much like point particles which do not scatter unless they suffer head-on collisions. Constituents of the second system form the x composites. The scattering between x composites can occur at arbitrary angles *although the relative orbital angular momentum is zero both in the initial and the final states.* The initial-value problem was discussed.

In Sec. VII, the system of constituent particles was enlarged to 3×4 N constituents q_{IA}^C (where $C = 1,2,3$). Constituents characterized by different values of C are assumed to behave similarly. Constituent solutions are linear combinations of a kinematic term and a set of normal coordinates W_A^c. In analogy to color in the quark model, we called the new attribute of the constituents tint. Asymptotic two-body composites are formed by constituent pairs of the same tint. The three-body composite particles are composed of three constituents all of different tint. It was assumed that tint is not an observable property of the composites. A reinterpretation of the scattering system as linear combinations of physically indistinguishable systems was made. Examples of x composite scattering were given, involving two- and three-constituent composite particles. Intrinsic numbers are conserved in all reactions.

IX. CONCLUSIONS AND DISCUSSION

The formalism developed in I and this paper represents an unusual approach to describing particle interactions. To be sure, there is already a long history of investigation of relativistic action-at-a-distance theories.[7] In recent years, a considerable effort has been concentrated on the development of Hamiltonian formulations based upon Dirac's theory of constraints,[8] since this formalism provides a direct prescription for quantization. The work here takes an alternative approach to the application of constraints, and is carried out within the Lagrangian formalism. Further, we do not attempt to associate the evolution parameter s with the physical clock, and regard it as an unmeasurable quantity. That is not to say there is not a connection with the foregoing Hamiltonian approaches, but it needs investigation.

Especially because of the novelty of the present approach, it may be helpful to spell out what has gone into the formalism in the way of assumptions, and what has so far come out. This is summarized below.

Assumptions.

(1) Lorentz-invariant Lagrangian which is the square root of the product of the potential and kinematic terms. L is a function of the single unmeasurable evolution parameter s.

(2) Attractive and repulsive harmonic-oscillator forces acting pairwise between constituent particles.

(3) Coupling matrices can be diagonalized.

(4) Asymptotic selection rules (which includes constituent confinement) and the frequency condition.

(5) System has intrinsic properties which are independent of composite momenta and energies.

(6) Physical constituents as linear combinations of harmonic-oscillator solutions.

(7) Opposite X number for particles and their antiparticles.

(8) Superposition of physically indistinguishable scattering systems.

Results.

(1) Relativistic harmonic-oscillator solutions which are quite different from the nonrelativistic case. Frequency depends on amplitudes, linear combinations of solutions are not solutions of harmonic-oscillator equations.

(2) Normal modes of oscillation analogous to internal symmetries.

(3) Definition of physical constituents leads to p and x composites characterized by different intrinsic numbers.

(4) Interactions between composite particles by means of constituent exchange.

(5) Constituent confinement and harmonic-oscillator potential imply conservation of energy, momentum, and angular momentum (Lorentz-invariance leads to conservation laws in s, not $t_{\rm OB}$).

(6) Analog to particle-antiparticle pair creation and annihilation.

(7) Composite particles are composed of either (a) two constituents of the same tint, but opposite B number, or (b) three constituents of all different tint, but same B number.

(8) Conservation of intrinsic numbers in composite scattering.

(9) Fixed relative values of internal angular momenta.

(10) Restricted values of angular momentum for scattering composites; namely, zero orbital angular momentum in initial and final states, even with arbitrary scattering angle.

(11) Although space-time conservation laws hold for composite scattering, *physical* initial conditions are not enough to determine the final state. Knowledge of intrinsic parameter necessary.

(12) Many-particle systems with (a) virtual particles, (b) vacuum states.

From this, it appears that the approach is potentially interesting, but there are problems to be understood, especially the determination of the intrinsic properties of the system, and thus the determination of the final state from initial conditions.

Finally, as we have seen, the classical theory presented here has many of the aspects generally associated with quantum field theories and quantum mechanics. Thus, it is reasonable to continue the development of the classical theory before trying to adopt rules for quantizing it.

Note added in proof. The frequency condition (54), in principle, allows the total system of constituents to be broken up into two or more separate systems related through the frequency alone. This is accomplished by imposing additional conditions, for example,

$$\sum_{I=1}^{M}\sum_{A=1}^{4} a_{IA}\cdot b_{IA}=0\ ,$$
$$\sum_{I=M+1}^{N}\sum_{A=1}^{4} a_{IA}\cdot b_{IA}=0\ . \tag{127}$$

Thus, one Lagrangian, for arbitrary (large) N, can serve to describe any scattering process.

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APPENDIX: DEFINITION OF INTERACTION INTERVAL

Consider the differential equation

$$\ddot{y}(s)=\alpha^2 y(s)\ , \tag{A1}$$

where in the interval

$$-s_0<s<s_0\ , \tag{A2}$$

the constant α^2 is finite and positive, and outside the interval α^2 is identically equal to zero. Then the solutions can be written

$$\begin{aligned} &y_-(s)=A_-s+B_-,\quad s<-s_0\ ,\\ &y(s)=a\exp(\alpha s)+b\exp(-\alpha s),\quad -s_0<s<s_0\ ,\\ &y_+(s)=A_+s+B_+,\quad s>s_0\ . \end{aligned} \tag{A3}$$

Further, assume that

$$\alpha s_0\gg 1\ , \tag{A4}$$

so that

$$y(s) \sim a \exp(\alpha s) \quad \text{as } s \to s_0 ,$$

and (A5)

$$y(s) \sim b \exp(-\alpha s) \quad \text{as } s \to -s_0 .$$

Matching solutions and their derivatives at the end points of the interval (A2), we are able to write the solutions outside of the interval in terms of the constants a and b:

$$y_-(s) = a \exp(\alpha s_0)[\alpha(s - s_0) + 1] ,$$
$$y_+(s) = -b \exp(\alpha s_0)[\alpha(s + s_0) - 1] . \tag{A6}$$

Alternatively, in the solution $y(s)$, we impose

$$\exp(\pm \alpha s) \sim \pm[\alpha \exp(\alpha s_0)]s - (\alpha s_0 + 1)\exp(\alpha s_0) . \tag{A7}$$

[1]M. J. King, Phys. Rev. D **30**, 399 (1984).

[2]L. P. Horwitz and C. Piron, Helv. Phys. Acta **46**, 316 (1973).

[3]A similar situation occurs when comparing Newtonian and relativistic cosmology. See, for example, A. K. Raychaudhuri, *Theoretical Cosmology* (Clarendon, Oxford, 1979), Chap. 2.

[4]E. C. G. Sudarshan and N. Mukunda, *Classical Mechanics: A Modern Perspective* (Wiley, New York, 1974); G. Feldman and P. T. Matthews, Ann. Phys. (N.Y.) **40**, 19 (1966).

[5]In the limit $s \to \pm s_0$, the exponential terms in s are replaced by terms linear in s. See the Appendix.

[6]Setting $A = B = 0$ selects the space-time frame of reference in which the total $4N$ system is "at rest" relative to s.

[7]L. Lusanna, Nuovo Cimento **65B**, 135 (1981), and references contained therein.

[8]P. A. M. Dirac, Rev. Mod. Phys. **21**, 392 (1949). See also E. C. G. Sudarshan and N. Mukunda, Ref. 4.