Causal action at a distance in a relativistic system of two bound charged spinless particles: Hydrogenlike models

C. Dewdney, P. R. Holland, A. Kyprianidis, and J. P. Vigier
Institut Henri Poincaré, Laboratoire de Physique Théorique 11, rue Pierre et Marie Curie, 75231 Paris Cedex 05 France
(Received 13 November 1984)

We analyze the causal quantum behavior of two charged spinless particles having different masses interacting via an action-at-a-distance relativistic scalar Coulomb potential. Developing de Broglie's idea of splitting the potential energy between the particles, we show that the predictivity constraints of relativistic predictive mechanics established by Droz-Vincent and Sudarshan are satisfied. In this way we develop a relativistic theory of hydrogenlike atoms in the framework of the causal interpretation of quantum mechanics.

I. INTRODUCTION

In a set of preceding papers the problem of the relativistic causal behavior of two correlated spinless particles has been quite generally solved by Droz-Vincent *et al.*[1-3] both in the classical and quantum cases in the frame of predictive mechanics. For example, in two recent papers[4,5] Droz-Vincent has shown that one can solve causally this problem for typical scattering processes and for the Klein-Gordon system of two particles interacting through a nonlocal potential (provided this interaction potential satisfies the predictivity constraints) such as, for example, Gunion and Li's potential[6] or the harmonic-oscillator potential.[7] In the case of two quantum correlated but noninteracting spin-0 and spin-1 particles the solution also yields a causal action-at-a-distance interpretation of the Einstein-Podolsky-Rosen (EPR) paradox and interprets[8] the recent experimental results of Aspect *et al.*[9,10]

The philosophy behind this treatment is simple: N interacting particles (labeled $i=1,\ldots,N$) moving in space-time each have a proper time τ_i so that they have in the relativistic sense not one but N Hamiltonians H_i each providing by quantization the left-hand side of a wave equation. The Hamiltonians yield generating functions leading to equations of motion in a Poisson-bracket form involving the N independent parameters τ_i. The predictivity conditions can be written in the general form

$$\{H_i,H_j\}=0\ ,$$

where $\{\ ,\ \}$ denotes the usual Poisson brackets built from couples of canonical variables. With the N equations

$$H_i=\tfrac{1}{2}m_i^2\ ,$$

where H_i are constants of the motion we obtain by summation the general master equation in configuration space,

$$\sum_i (H_i-\tfrac{1}{2}m_i^2)=0$$

plus $(N-1)!$ causality constraints,

$$H_i-H_j=\tfrac{1}{2}(m_i^2-m_j^2)\ .$$

As one knows, the problem simplifies if one splits the variables into external and internal variables, i.e., separates coordinates of the center of mass,

$$X=\sum_i m_i x_i \Big/ \sum_i m_i \quad \text{and} \quad P=\sum_i P_i$$

from the relative coordinates,

$$z_{ij}=x_i-x_j \quad \text{and} \quad y_{ij}=\tfrac{1}{2}(P_i-P_j)\ .$$

As usual we write the scalar products in compact form $p^2=p\cdot p=p_\mu p^\mu$, etc., and take $\hbar=c=1$ and $\partial_{i\mu}=\partial/\partial x_i^\mu$. We also use the projection operator $\Pi^\alpha{}_\beta=\delta^\alpha{}_\beta-P^\alpha P_\beta/P^2$ which transforms any vector R^μ into its projection $\tilde{R}^\mu=\Pi^\mu{}_\nu R^\nu$ in the systems' rest frame.

The aim of the present work is to extend the preceding causal analysis to the particular case of two oppositely charged particles of different masses (which do not radiate) utilizing an action-at-a-distance Lorentz scalar electromagnetic interaction (Coulomb-like force).

We are conscious, of course, that this particular case does not solve the general problem of the behavior of the causal predictive treatment of two charged particles which

(i) move in an external electromagnetic field,

(ii) interact (for example, through Lienard-Wiechert potentials),

(iii) radiate (due to their acceleration), and

(iv) possess self-energy. This problem will be discussed in a subsequent publication.

In any case we are able to apply our causal analysis to the study of hydrogenlike atoms in which the nucleus and the electron both have real spacetime trajectories. In this way we answer objections raised to the de Broglie-Bohm causal interpretation of quantum mechanics.

II. THE COULOMB FIELD AS A RELATIVISTIC SCALAR POTENTIAL AND THE DE BROGLIE PARTITION

The standard treatment of the motion of a relativistic spin-0 particle in an external Coulomb-like field, which inserts the fourth component of the electromagnetic po-

tentials A_μ in the Klein-Gordon equation, does not yield an equation automatically in a form appropriate to an action-at-a-distance theory since it involves a coupling between the potential and the first time derivative of the wave function. In the usual approach to the hydrogen atom one assumes that the proton has infinite mass so that in the rest frame of the proton $\mathbf{A}=0$ and A_0 is independent of time. In this way one obtains a time-independent wave equation where all the potential energy of interaction A_0 is ascribed to the electron. In this paper we intend to develop an alternative theory of Coulomb interaction which does not rely on working in a particular frame, i.e., which is Lorentz covariant, and which does not require the assumption of infinite proton mass. This we do by using a scalar Coulomb field. Our wave equations take a form which is immediately relevant to predictive mechanics and moreover, as we argue in Sec. IV, we feel we are able to give a more coherent account of H-like atoms. First of all we justify treating the Coulomb field as a Lorentz scalar.

In the theory of photons of nonzero mass μ_γ (Refs. 11 and 12) a third photon state appears with helicity $J_3=0$ associated with longitudinal waves (having quanta γ_L), in addition to the helicities $J_3=\pm 1$ associated with transverse waves (with quanta γ_T). The Lagrangian density $L=-\frac{1}{4}F^*_{\mu\nu}F^{\mu\nu}-\frac{1}{2}\mu_\gamma{}^2A^*_\mu A^\mu$ is no longer gauge invariant, and imposing the Coulomb gauge $\partial_\mu \mathscr{A}^\mu=0$, one obtains the wave equation $\partial_\mu F^{\mu\nu}=\mu_\gamma{}^2\mathscr{A}^\nu$, where $\mathscr{A}^\nu=A^\nu\exp(iS)$. The main result of this approach is that the physical solutions of this equation correspond to independent fields ($\mathbf{E},\mathbf{H}$ and $\mathbf{A},V$, respectively) which are weakly coupled by μ_γ ($\mu_\gamma\leq 10^{-48}$ g). Two sets of distinct plane-wave solutions are now possible: transverse and longitudinal. The longitudinal solutions in particular essentially involve a wave $\mathbf{A}^L,V^L$ since $\mathbf{E}^L,\mathbf{H}^L$ almost vanish for $\mu_\gamma\to 0$, and $\mathbf{A}^L\|\mathbf{K}$, $V^L=|\mathbf{K}|\,|\mathbf{A}^L|/\nu$, where $\nu^2=\mathbf{K}^2+\mu_\gamma{}^2$. Since $\partial_\mu A^L_\gamma-\partial_\gamma A^L_\mu\sim 0$ we can put $A^L_\mu\sim\partial_\mu F(x^\nu)$, where $F(x^\mu)$ is a scalar function.

It has been shown by Deser[13] that the first-order action describing the Proca field splits into a transverse part I_T, which differs from the usual Maxwell action not only through the mass term but also through the absence of an instantaneous Coulomb interaction, and a longitudinal and contact part I_L+I_0. The part I_L contains a contribution from a massive free Lorentz scalar field and a term coupling to the longitudinal current j^μ. Since the latter is scaled by the photon mass it is negligible with respect to the Lorentz scalar field, and hence the plane-wave longitudinal solution behaves like a scalar particle of spin zero. For the assumed magnitude of μ_γ, the scalar field reduces to an unscreened Coulomb field.

Furthermore, one can show[14] that the two sets of solutions, longitudinal and transverse, are effectively decoupled [coupling $\sim(\mu_\gamma/\nu)^2$] and constitute two distinct phase spaces when the energy of γ_L is negligible in comparison with the energy of γ_T. For $\mu_\gamma\neq 0$ the distinction between the transverse and longitudinal modes is not a covariant one, but in the zero-mass limit the longitudinal wave essentially describes a spin-0 scalar field.

For the above reason we shall assume that the Coulomb interaction between two charged particles may be treated as a Lorentz scalar field. In order to write down a wave equation for each particle in such a system, however, we need to know how to partition the Coulomb potential energy V in some way so that we may treat each particle as moving in an "external field." We do this in accordance with a method originally developed by de Broglie.[15]

Consider in classical relativistic mechanics a particle of mass m placed in an external scalar potential W. Then, as is well known, its momentum P_μ satisfies

$$P_\mu P^\mu=(m+W)^2\ .$$

If now we treat this particle as one component (of mass m_1) of a Coulomb-interacting two-body system (the other particle having mass m_2) we can write, following the ideas of de Broglie, the following system of equations:

$$\begin{aligned} P_{1\mu}P_1^\mu&=(m_1+V_1)^2\ ,\\ P_{2\mu}P_2^\mu&=(m_2+V_2)^2\ . \end{aligned}\tag{2.1}$$

We intend to relate the potentials V_1,V_2 to V.

Now, we know that the total scalar Hamiltonian of the system must have the form $\frac{1}{2}P_1{}^2+\frac{1}{2}P_2{}^2-2cV=\frac{1}{2}(m_1{}^2+m_2{}^2)$ where V is the scalar potential energy of interaction, and c is a constant having the dimension of mass which is to be determined. Let us suppose that the Hamiltonians of the individual particles are given by

$$\begin{aligned} \tfrac{1}{2}P_1{}^2-c_1V&=\tfrac{1}{2}m_1{}^2\ ,\\ \tfrac{1}{2}P_2{}^2-c_2V&=\tfrac{1}{2}m_2{}^2\ , \end{aligned}\tag{2.2}$$

where c_1,c_2 are constants with $2c=c_1+c_2$. Comparing the expressions given for the momenta squared by (2.1) and (2.2) we immediately find that

$$\begin{aligned} V_1&=-m_1+(m_1{}^2+2c_1V)^{1/2}\ ,\\ V_2&=-m_2+(m_2{}^2+2c_2V)^{1/2}\ . \end{aligned}\tag{2.3}$$

If we now consider the case where V is small in relation to m_1 and m_2 so that we may neglect V^2, then expanding the square root terms in (2.3) yields $V_1=(c_1/m_1)V$, $V_2=(c_2/m_2)V$. Comparing these expressions with those given by de Broglie[15] for nonrelativistic interacting particles shows that $c_1=c_2=c=\mu=m_1m_2/(m_1+m_2)$, the reduced mass. Finally, then, our energy relations in the exact relativistic case are

$$\begin{aligned} P_1{}^2&=(m_1+V_1)^2=m_1{}^2+2\mu V\ ,\\ V_1&=-m_1+(m_1{}^2+2\mu V)^{1/2}\ ,\\ P_2{}^2&=(m_2+V_2)^2=m_2{}^2+2\mu V\ ,\\ V_2&=-m_2+(m_2{}^2+2\mu V)^{1/2}\ . \end{aligned}\tag{2.4}$$

Our demonstration of these formulas evidently rests on the assumption of the partition of potential energy proposed by de Broglie for nonrelativistic systems. Such a partition is not arbitrary and as shown by Lucas[16] follows from the law of action and reaction. Clearly, de Broglie's expressions for the partition are modified in the relativistic regime. In the case where $m_1\gg m_2$ we see from (2.4)

that particle m_1 effectively becomes free and all the interaction energy is associated with particle m_2.

We may now quantize our interacting system by replacing $P_{i\mu}\rightarrow i\partial_{i\mu}$, $i=1,2$, in (2.4) to obtain

$$(\Box_1+m_1{}^2+2\mu V)\psi(x_1,x_2)=0\ ,$$
$$(\Box_2+m_2{}^2+2\mu V)\psi(x_1,x_2)=0\ . \tag{2.5}$$

Here ψ is a wave function in the configuration space of the system and

$$V=\frac{Ze^2}{(\tilde{z}^\mu\tilde{z}_\mu)^{1/2}}\ , \tag{2.6}$$

where $\tilde{z}^\mu=z^\mu-(z_\nu P^\nu)P^\mu/P^2$, P^μ being the momentum of the center of mass. In the c.m. rest frame $\tilde{z}^0=0$, $\tilde{z}^i=z^i$, $i=1,2,3$, and the potential acts instantaneously.

The equations (2.5) form our starting point for a causal, relativistic theory of two interacting quantum particles, each having trajectories in spacetime. Of course we must show that these relations, involving as they do action at a distance, do not conflict with relativistic causality. This we shall do in the next section.

Finally, we point out the relation of our approach to the usual theory where V is the fourth component of a vector. For a particle of mass m in an external field A_μ the minimal-substitution Klein-Gordon equation is

$$(i\partial_\mu-A_\mu)(i\partial^\mu-A^\mu)\psi=m^2\psi\ .$$

When $A_\mu=(V,\mathbf{0})$ where V is independent of time, we write $\psi=e^{-iEt}\phi(\mathbf{x})$ to obtain

$$(-\Delta+m^2-E^2+2VE+V^2)\phi=0\ . \tag{2.7}$$

This equation is to be compared with our system of equations (2.5) or (4.5) in the case where $m_2\gg m_1=m$. We find that m_1 behaves as a free particle and that m satisfies

$$(-\Delta+m^2-E^2+2Vm)\phi=0\ . \tag{2.8}$$

In (2.7) and (2.8) let us write $E=m+\epsilon$. Then a comparison of these relations shows that (2.7) is identical with (2.8) in the approximation where V^2 and $V\epsilon$ may be neglected. Essentially, then, it is only in the nonrelativistic limit that the two approaches coincide. We show this explicitly by calculating energy levels in Sec. IV.

III. CAUSAL INTERACTING KLEIN-GORDON SYSTEM

In order to demonstrate that the system (2.5) is causal we shall, following Cufaro-Petroni *et al.*,[17] work with the quantum potential formalism since this is relevant to the subsequent discussion. Writing $\psi=\exp[R(x_1,x_2)+iW(x_1,x_2)]$ the real parts of Eqs. (2.5) are the Hamilton-Jacobi-like equations:

$$H_1=\tfrac{1}{2}\partial_{1\mu}W\partial_1^\mu W+U_1-\mu V=\tfrac{1}{2}m_1{}^2\ ,$$
$$H_2=\tfrac{1}{2}\partial_{2\mu}W\partial_2^\mu W+U_2-\mu V=\tfrac{1}{2}m_2{}^2\ , \tag{3.1}$$

where $\mu=m_1m_2/(m_1+m_2)$ is the reduced mass and U_1,U_2 are quantum potentials,

$$U_1=-\tfrac{1}{2}(\Box_1R+\partial_{1\mu}R\partial_1^\mu R)\ ,$$
$$U_2=-\tfrac{1}{2}(\Box_2R+\partial_{2\mu}R\partial_2^\mu R)\ . \tag{3.2}$$

The imaginary parts of (2.5) give the conservation equations

$$\partial_{1\mu}(e^{2R}\partial_1^\mu W)=0\ ,$$
$$\partial_{2\mu}(e^{2R}\partial_2^\mu W)=0\ . \tag{3.3}$$

To prove causality we must first of all study the structure of the Hamiltonians (left-hand sides) in (3.1). Let us restrict ourselves to solutions which are eigenstates of the center-of-mass momentum operator $-i\partial/\partial X^\mu=-i(\partial_{1\mu}+\partial_{2\mu})$, that is,

$$\psi(x_1,x_2)=\exp(iP_\mu X^\mu)\phi(z^\mu)\ ,$$
$$X^\mu=\frac{m_1x_1^\mu+m_2x_2^\mu}{m_1+m_2}\ ,\quad z^\mu=x_2^\mu-x_1^\mu\ , \tag{3.4}$$

where P_μ is a constant timelike vector. Then

$$(\partial_{1\mu}+\partial_{2\mu})R=0 \tag{3.5}$$

and

$$(\partial_{1\mu}+\partial_{2\mu})W=P_\mu\ . \tag{3.6}$$

Using (3.5) we immediately deduce from (3.2) that $U_1=U_2=U$, say, and moreover that $U=U(z^\mu)$. Taking the difference of Eqs. (3.3) and noting from (3.6) that $\Box_1W=\Box_2W$, we find using Eqs. (3.5) and (3.6) that $P^\mu\partial_{1\mu}R=P^\mu\partial_{2\mu}R=0$. In terms of the coordinate-difference operator $(m_1+m_2)\partial/\partial z^\mu=m_1\partial_{2\mu}-m_2\partial_{1\mu}$ this yields

$$P^\mu\frac{\partial R}{\partial z^\mu}=0\ .$$

It follows that the quantum potential U is a function only of $\tilde{z}^\mu=z^\mu-(z_\nu P^\nu)P^\mu/P^2$ since $R=R(\tilde{z}^\mu)$, the same functional dependence as the Coulomb potential (2.6).

Bearing in mind the above considerations we now construct a Hamiltonian system corresponding to Eq. (3.1), our intention being to map this fictitious Hamilton-Jacobi system into the Klein-Gordon system under consideration. We thus define in analogy to Eq. (3.1), using the Hamilton-Jacobi characteristic function to define by means of a canonical transformation momenta $P_{i\mu}=\partial_{i\mu}W$, the following two Hamiltonians:

$$H_1=\tfrac{1}{2}P_1{}^2+U(\tilde{z})-\mu V(\tilde{z})\ ,$$
$$H_2=\tfrac{1}{2}P_2{}^2+U(\tilde{z})-\mu V(\tilde{z})\ . \tag{3.7}$$

As regards the predictivity constraint $\{H_1,H_2\}=0$, it is straightforward to show that the Poisson brackets of two Hamiltonians having the form $H_1=\frac{1}{2}P_1{}^2+W_1$, $H_2=\frac{1}{2}P_2{}^2+W_2$ reduce to

$$\{H_1,H_2\}=P_2^\mu\frac{\partial W_1}{\partial x_2^\mu}-P_1^\mu\frac{\partial W_2}{\partial x_1^\mu}+\{W_1,W_2\}\ .$$

When the potentials have the form of those appearing in

(3.1) we immediately deduce [by writing $\partial W_1/\partial x_2^\mu = (\partial \tilde{z}^\nu/\partial x_2^\mu)(\partial W_1/\partial \tilde{z}^\nu)$, for example] that the Poisson brackets vanish everywhere in eight-dimensional relativistic phase space. This ensures that the position coordinates x_i^μ, $i=1,2$, of each particle depend only on the proper time associated with that particle.

One finds from (3.7) that

$$\{P_\mu, H_1\} = \{P_\mu, H_2\} = 0$$

so that the c.m. momentum is a constant of the motion. It follows that we can slice spacetime with three-planes orthogonal to P_μ and connect the two particles by spacelike lines lying in these hyperplanes.

Note that the positions x_1, x_2 are coordinates canonical to the momenta P_1, P_2 in the rest frame of the c.m., i.e., when $\mathbf{P}=0$ where $P_\mu = (P_0, \mathbf{P})$. In this frame the potential acts instantaneously.

We have thus shown that when the c.m. is represented by a plane wave, the relativistic Klein-Gordon system of two charged bodies of different mass interacting via a Coulomb potential is causal, i.e., the paths of the particles are timelike, the Cauchy problem is solvable, and the theory is Poincaré invariant.

Although the effective potential acting on each particle depends only on $\tilde{z}$, that is, it does not depend on the relative time in the c.m. rest frame, the wave function

$$\psi = \exp[R(\tilde{z}) + iP_\mu X^\mu + iW'(z)]$$

is not so restricted: $P^\mu \partial W'/\partial z^\mu \neq 0$. In fact, taking the difference of Eqs. (3.1) and substituting (3.6), we find

$$2P^\mu \frac{\partial W'}{\partial z^\mu} = \left[\frac{m_1 - m_2}{m_1 + m_2}\right] P^2 + {m_2}^2 - {m_1}^2 . \tag{3.8}$$

IV. CAUSAL NONLOCAL THEORY OF HYDROGENLIKE ATOMS

In textbook discussions of the hydrogen atom, which reduce the configuration space motion to the motion of a fictitious reduced mass particle in real space, it is sometimes stated that the emission of a photon is associated with the transition of an electron between stationary states. In fact, there is nothing in the mathematical formulation which enables one to draw this conclusion—rather all that can be said is that the relative energy of proton and electron changes on emission and no more detailed account is possible. To make the claim that it is the electron alone which possesses energy levels consistent, a further assumption is introduced that the nucleus is fixed so that the electron is effectively a test charge moving in an external field. As a consequence no partitioning of the potential energy of the field is necessary, it all being assumed to be associated with the electron. This step is justified[18] by the fact that the effect of the motion of the nucleus is of a higher order of smallness even than relativistic corrections to the energy levels calculated from the Schrödinger equation.

While such a procedure may be mathematically consistent when there is a large disparity in the masses of the particles, it actually masks, under the guise of an approximation technique, a crucial methodological step, viz., the representation of configuration-space motion in real space (we ignore here the fact that the orthodox interpretation of quantum mechanics has no concept of particle trajectory and think in terms of the imagery of the causal interpretation which has). The validity of leaving out of account the correlated nature of electron and proton motions is by no means clear, in particular in relation to photon emission. After all, for systems such as deuteron or positronium where the masses are nearly the same, one could not make the assumption that the motions may be reduced to that of one of the particles in real spacetime having energy levels independent of the other particle. Similarly, a complete (and correct) theory of H-like atoms should take into account from the outset that we are dealing with a correlated two-body system, regardless of whether or not the magnitudes of the contributions to the energy levels are similar for each particle.

This is the approach we shall develop below. We shall indeed represent configuration-space motions in spacetime but it is the motion of two correlated particles, rather than one, that we are studying.

A. Separation into c.m. and relative motions

Transforming to c.m. and relative coordinates (X,z) the system (2.5) becomes

$$\left[\left(\frac{m_1}{m_1+m_2}\right)^2 \Box_X + \Box_z - \frac{2m_1}{m_1+m_2}\partial_X\cdot\partial_z + {m_1}^2 + 2\mu V\right]\psi(X,z) = 0 , \tag{4.1}$$

$$\left[\left(\frac{m_2}{m_1+m_2}\right)^2 \Box_X + \Box_z + \frac{2m_2}{m_1+m_2}\partial_X\cdot\partial_z + {m_2}^2 + 2\mu V\right]\psi(X,z) = 0 . \tag{4.2}$$

Assuming the wave function may be expressed in the form $\psi = \xi(X)\phi(z)$, adding $(1/m_1)\times(4.1)$ to $(1/m_2)\times(4.2)$ and dividing by ψ yields

$$\frac{m_1 m_2}{(m_1+m_2)^2}\frac{\Box_X \xi}{\xi} + \frac{\Box_z \phi}{\phi} + \mu(m_1+m_2+2V) = 0 . \tag{4.3}$$

The difference of (4.1) and (4.2) gives, after dividing by ψ,

$$\left[\frac{m_1-m_2}{m_1+m_2}\right]\frac{\Box_X \xi}{\xi} - \frac{2\partial_X \xi}{\xi}\cdot\frac{\partial_z \phi}{\phi} + {m_1}^2 - {m_2}^2 = 0 . \tag{4.4}$$

Since (4.3) is true for arbitrary X and z, we must have

$$\frac{\Box_X \xi}{\xi} = c ,$$

where c is a constant and so the c.m. and relative motions have been separated, subject to (4.4).

Restricting ourselves to the ansatz (3.4), i.e., $\xi = \exp(iP_\mu X^\mu)$, we see that $c = -P^2$. In this case (4.3) and (4.4) reduce to

$$\left[\Box_z-\frac{P^2m_1m_2}{(m_1+m_2)^2}+\mu(m_1+m_2+2V)\right]\phi(z)=0\ , \tag{4.5}$$

$$2iP^\mu\frac{\partial}{\partial z^\mu}\ln\phi=P^2\left[\frac{m_2-m_1}{m_1+m_2}\right]+{m_1}^2-{m_2}^2\ . \tag{4.6}$$

Equation (4.6) determines how ϕ depends on the relative time and will be recognized as (3.8). Given that ψ has the form (3.4), (4.5) and (4.6) are equivalent to (2.5) and form the basis of our subsequent discussion.

When $m_1=m_2=m$ we recover from these last two relations equations similar to those proposed by Gunion and Li[6] to describe a system of interacting spinless quarks:

$$[\Box_z-\tfrac{1}{4}P^2+m^2+mV]\phi=0 \quad\text{and}\quad P^\mu\frac{\partial\phi}{\partial z^\mu}=0\ .$$

The first equation here differs from that of Gunion and Li in that we have taken into account the de Broglie partition of potential energy introduced in order to ensure the validity of the law of action and reaction.

B. Nonrelativistic limit

From (4.6) we can derive the relative-time dependence of ϕ in the c.m. rest frame. In this frame,

$$\frac{\partial W'}{\partial z^0}=\frac{(m_1-m_2)[{P_0}^2-(m_1+m_2)^2]}{2P_0(m_1+m_2)}\equiv a\ ,$$

where a is a constant, so that the phase of the wave function ψ is given by

$$W=P_0X^0+az^0+S(\tilde{z})\ , \tag{4.7}$$

since $\tilde{z}^\mu=(\tilde{z}^0,\tilde{z}^i)=(0,z^i)$, and so $S(z^i)=S(\tilde{z}^i)$.

The relative wave function therefore has the form

$$\phi=\exp\{R(\tilde{z})+i[az^0+S(\tilde{z})]\}\ .$$

Substituting this into (4.5) we find, noting that $\Delta_z=\Delta_{\tilde{z}}$ in the c.m. rest frame,

$$\left[-\frac{1}{2\mu}\Delta_{\tilde{z}}+V\right]e^{R+iS}=Ee^{R+iS}\ , \tag{4.8}$$

where

$$E=[{P_0}^2-(m_1+m_2)^2][{P_0}^2-(m_2-m_1)^2]/8{P_0}^2\mu\ , \tag{4.9}$$

This of course has the form of the well-known reduced mass Schrödinger equation for a two-body system interacting via the potential V.[19] Solving (4.9) for P_0 we obtain

$${P_0}^2={m_1}^2+{m_2}^2+4\mu E+2({m_1}^2+2\mu E)^{1/2}({m_2}^2+2\mu E)^{1/2}\ .$$

We now pass to the nonrelativistic limit by supposing that E is small in relation to m_1 or m_2. We find

$$P_0=m_1+m_2+E-E^2\frac{[{m_1}^4+{m_2}^4+2m_1m_2({m_1}^2+{m_2}^2)]}{4m_1m_2(m_1+m_2)^3}+O(E^3)\ .$$

Using the standard energy levels E for hydrogenlike atoms calculated from (4.8) and assuming that $m_2(\text{nucleus})\gg m_1(\text{electron})$ yields finally,

$$P_0=m_1+m_2\left[1-\frac{(Ze^2)^2}{2n^2}-\frac{(Ze^2)^4}{2n^4}\cdot\tfrac{1}{8}\right]\ ,\quad n\in\mathbb{Z}\ ,$$

neglecting higher-order terms. As expected, we recover the correct nonrelativistic limit. The first relativistic correction differs from that calculated in the usual approach which employs the fourth component of A_μ in the Klein-Gordon or Dirac equations. We note that the energy levels calculated by Komar[20] from wave equations similar to ours differ from those given above since he did not work with a center-of-mass coordinate X^μ.

C. Particle energies

Each particle in our interacting system has an energy given by $P_{i0}=\partial_{i0}W$, $i=1,2$. Working in the c.m. frame ($P_{1i}+P_{2i}=\mathbf{P}=0$, $i=1,2,3$) we find from (4.7),

$$P_{10}=\frac{P_0}{2}-\frac{({m_2}^2-{m_1}^2)}{2P_0}\ ,$$

$$P_{20}=\frac{P_0}{2}+\frac{({m_2}^2-{m_1}^2)}{2P_0}\ ,$$

so that $P_{10}+P_{20}=P_0$. These are exact relativistic formulas. In the nonrelativistic limit where $P_0=m_1+m_2+E$, we have

$$P_{10}=m_1+\alpha E\ ,\quad \alpha=\frac{m_2}{m_1+m_2}$$

$$P_{20}=m_2+\beta E\ ,\quad \beta=\frac{m_1}{m_1+m_2}\ .$$

We see that each particle has well-defined energy levels: nonrelativistically $E_1=\alpha E$, $E_2=\beta E$. However, this does not mean that one of the particles "jumps" on photon emission, independently of the other, so that we could expect two radiation spectra having frequencies in the ratio of the masses. Rather, emission is a process involving the simultaneous action of all the parts of the system correlated by the nonlocal quantum potential. There is just one observed energy E which is a parameter associated with the whole system. No more detailed account of photon emission is possible. We are able, though, to discuss the system in terms of two real correlated trajectories in spacetime without having to neglect the motion of one of the particles. Insofar as parts of correlated systems cannot be simply dropped from consideration, and all physical effects ascribed to the remaining parts, as we believe to be the case, then any particle placed in an external field should be treated as forming a whole with the source of the field (however distant). One cannot talk of the energy levels of *the* particle—they are associated with the system, however disparate the masses may be. Such a notion, derived from the quantum theory, is clearly in line with the tenets of the Wheeler-Feynman theory.

D. Particle velocities

In the relativistic case the three-velocity of each particle is given by the guidance formula,

$$\mathbf{v}_i = -\frac{\nabla_i W}{\partial_{i0} W} = -\frac{\nabla_i S}{P_{i0}} , \quad i = 1,2 ,$$

where we have used (4.7). In the nonrelativistic case this evidently reduces to

$$\mathbf{v}_i = -\frac{1}{m_i}\nabla_i S , \quad i = 1,2 .$$

The standard solution to (4.8) yields for the phase,

$$S = k \tan^{-1}\left[\frac{\tilde{z}^2}{\tilde{z}^1}\right] , \quad k \in \mathbb{Z}$$

from which we find the general expressions for the velocities in the plane $\tilde{z}^3 = 0$,

$$\mathbf{v}_1 = \frac{k}{m_1 \tilde{z}^i \tilde{z}_i}(-\tilde{z}^2, \tilde{z}^1, 0) , \quad \mathbf{v}_2 = -\frac{m_1}{m_2}\mathbf{v}_1 , \quad i = 1,2,3 .$$

Clearly, in the ground state each particle is at rest due to the quantum potential U balancing V. The magnitudes of the velocities are given by

$$|\mathbf{v}_1| = \frac{k}{m_1 (\tilde{z}^i \tilde{z}_i)^{1/2}} .$$

Since $\mathbf{v}_i \cdot \nabla V = 0$, where ∇V is the Coulomb force, we see that the paths of the particles are circles in the c.m. frame.

V. TOWARDS A RESOLUTION OF OBJECTIONS TO THE CAUSAL INTERPRETATION OF QUANTUM MECHANICS

The causal representation of quantum phenomena developed by de Broglie[21] and Bohm[22] provides, in the one-body relativistic and nonrelativistic cases (based on the Schrödinger and Klein-Gordon equations), a model which is consistent with all the predictions of orthodox quantum mechanics but which ascribes to a particle real motion in spacetime. The extension of this approach to the many-body case has, however, been problematical for two reasons. First, the many-body wave function evolves in configuration space and a mapping of the individual correlated particle motions into spacetime is required. Second, the many-body quantum potential acts nonlocally at a distance which suggests an inconsistency with the causality requirements of the theory of relativity.

One can overcome the first problem on the basis of the early form of the causal interpretation by associating particle momenta with the gradients of the phase of the configuration-space wave function. In this view the particles each have trajectories in spacetime and are correlated nonlocally by the quantum potential. There is only one Hamilton-Jacobi equation for the whole system, however. The extension of this method to relativistic quantum mechanics is not possible since it provides no way of proving compatibility between nonlocality and causality.

A way of resolving this problem is provided by the methods of predictive mechanics. This scheme, which employs N relativistic equations rather than one, enables one to go beyond the original de Broglie-Bohm theory by demonstrating that causality is preserved even in an action-at-a-distance context. We have shown in detail in this paper how to carry this through for H-like atoms [and more generally for any two-body system interacting via a potential of the form $V = V(\tilde{z})$], taking into account the de Broglie partition of potential energy. Moreover, the causal interpretation gives a sound theoretical basis for the application of predictive mechanics in the quantum domain.

Our present treatment is evidently provisional and in later publications we shall extend it to include spin by using the Feynman—Gell-Mann formalism. In this way a comparison with the results of the Bethe-Salpeter theory will be possible.

The authors (C.D., P.R.H., and A.K.) wish to thank the Royal Society, the Leverhulme Trust, and the French Government, respectively, for support which made this work possible.

[1]Ph. Droz-Vincent, Ann. Inst. Henri Poincaré **27**, 407 (1977).
[2]*Relativistic Action at a distance: Classical and Quantum Aspects,* Vol. 162 of *Lecture Notes in Physics,* edited by J. Llosa (Springer, Berlin, 1982).
[3]E. C. G. Sudarshan and N. Mukunda, Found. Phys. **13**, 385 (1983).
[4]Ph. Droz-Vincent, Phys. Rev. D **29**, 687 (1984).
[5]Ph. Droz-Vincent, Phys. Rev. D **19**, 702 (1979).
[6]J. F. Gunion and L. F. Li, Phys. Rev. D **12**, 3583 (1975).
[7]H. Leutwyler and J. Stern, Nucl. Phys. **B133**, 115 (1978).
[8]N. Cufaro-Petroni and J. P. Vigier, Phys. Lett. **93A**, 383 (1983).
[9]A. Aspect, G. Grangier, and G. Roger, Phys. Rev. Lett. **47**, 460 (1981); **49**, 91 (1982).
[10]A. Aspect, J. Dalibard, and G. Roger, Phys. Rev. Lett. **49**, 1804 (1982).
[11]L. de Broglie, *La Mécanique ondulatoire du photon* (Hermann, Paris, 1940).
[12]L. Bass and E. Schrödinger, Proc. R. Soc. London **A232**, 1 (1955).
[13]S. Deser, Ann. Inst. Henri Poincaré **16**, 79 (1972).
[14]M. Moles and J. P. Vigier, C. R. Acad. Sci. Paris **276**, 697 (1973).
[15]L. de Broglie, C. R. Acad. Sci. Paris **275**, 899 (1972).
[16]R. Lucas, C. R. Acad. Sci. Paris **282**, 43 (1976).
[17]N. Cufaro-Petroni, Ph. Droz-Vincent, and J. P. Vigier, Lett. Nuovo Cimento **31**, 415 (1981).
[18]See, e.g., V. B. Berestetskii, E. M. Lifschitz, and L. P. Pitaevskii, *Relativistic Quantum Theory, Part 1* (Pergamon, Oxford, 1971), p. 105.
[19]See, e.g., L. I. Schiff, *Quantum Mechanics,* 3rd edition (McGraw-Hill, New York, 1968), p. 90.
[20]A. Komar, Phys. Rev. D **18**, 1887 (1978).
[21]L. de Broglie, *Nonlinear Wave Mechanics* (Elsevier, Amsterdam, 1960).
[22]D. Bohm, Phys. Rev. **85**, 166 (1952).