Compatibility between the Brownian metric and the kinetic metric in Nelson stochastic quantization

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In the frame of Nelson stochastic quantization for dynamical systems on a manifold, we consider diffusion processes with Brownian covariance given by a Riemannian metric on the manifold. The dynamics is specified through a stochastic variational principle for a generalization of the classical action, with a given kinetic metric. The resulting programming equation, of the Hamilton-Jacobi type, depends on both metrics, the Brownian one and the kinetic one. We introduce a simple notion of compatibility between the two metrics, such that the programming equation and the continuity equation lead to the Schrödinger equation on the manifold.

I. INTRODUCTION

It is very well known that Nelson stochastic mechanics[1,2] provides an approach to quantization of dynamical systems based on the theory of stochastic processes and physically equivalent to the operator approach. Here we consider dynamical systems on a manifold M, whose dynamics is ruled, at the classical level, by a Lagrangian of the type

$$\mathscr{L}(q,\dot{q})=\tfrac{1}{2}mg_{ij}(q(t))\dot{q}^{\,i}(t)\dot{q}^{\,j}(t)-V(q(t)) , \tag{1}$$

for a given kinetic metric $g=\{g_{ij}\}$. We consider the general case, where g can be also not positive definite, as, for example, in the case of a particle moving on a space-time with the Einstein metric, where t has the meaning of an auxiliary invariant parameter (proper time).

In the stochastic quantization procedure, $q(t)$ is promoted to a stochastic process on the manifold, with specified kinematical properties.

In particular we assume that $q(t)$ performs a diffusion on M, with a given Brownian metric $\eta=\{\eta^{ij}\}$ (see Sec. I). It is important to remark explicitly here that the two metrics g and η play a completely different role. The kinetic metric g rules the dynamics and could not be positive definite. The Brownian metric η describes the noise acting on the trajectories $q(t)$ and must be always positive definite. The aim of this paper is to investigate the relations between the two metrics, necessary for a complete consistency of the stochastic quantization procedure.

The dynamics for the stochastic process $q(t)$ can be specified through an appropriate form of Newton's second principle of dynamics (see Refs. 1–4). In Ref. 4 we have shown that the notion of geodesic correction to stochastic parallel displacement plays a very important role in the formulation of the second principle of dynamics. Here we introduce dynamical laws through a stochastic variational action principle, following the strategy advocated in Ref. 5 (see also Refs. 1 and 6). Then the resulting programming equation, of the Hamilton-Jacobi type, depends on both metrics. In order to obtain complete consistency of the stochastic quantization procedure, it is necessary to assume a kind of compatibility condition between the two metrics, simply expressed as follows. Starting from the kinetic metric $g=\{g_{ij}\}$ and the Brownian metric $\eta=\{\eta^{ij}\}$, let us introduce the inverse matrices g^{ij} and η_{ij}, defined by

$$g_{ij}g^{jk}=\delta_i^k,\quad \eta^{ij}\eta_{jk}=\delta_k^i . \tag{2}$$

Then the compatibility condition is expressed as

$$g_{ij}\eta^{jk}\eta^{il}=g^{kl} . \tag{3}$$

When g is positive definite, then (3) requires $\eta\equiv g$, but when g is not positive definite (as, for example, in the case of a particle moving in a gravitational background with Einstein metric g), then the relation (3) can have many different solutions. If the compatibility condition is satisfied, then the programming equation and the continuity equation lead, through a standard procedure, to the Schrödinger equation on the manifold, so that the consistency of the stochastic quantization procedure is assured.

The paper is organized as follows. In Sec. II we recall all kinematical and dynamical structural properties of the processes employed in the stochastic quantization procedure. In Sec. III we derive the programming equation from the stochastic variational principle. In Sec. IV we introduce the compatibility condition and verify that it leads to the Schrödinger equation. Finally Sec. V is devoted to some conclusions and outlook for future developments of the theory.

II. THE KINEMATICAL AND DYNAMICAL FRAME

Let us consider the configuration-space manifold M, with a given positive-definite $\eta\equiv\{\eta^{ij}\}$. We introduce a class of trial diffusion processes $[t_0,t_1]\in t\to q(t)\in M$, with density $\rho(\cdot,t)$ with respect to the invariant measure on M given by $\sqrt{\eta}\,dx$, where η here denotes the determinant of η_{ij} and dx is the Lebesgue measure. We assume that the kinematics of the process is ruled by the given conditional expectations

$$E(\Delta q^i(t)\mid q(t)=x)=[v^i_{(+)}(x,t)+m^i(x)]\Delta t+O(\Delta t^2)\ , \tag{4}$$

$$E(\Delta q^i(t)\Delta q^j(t)\mid q(t)=x)=2\nu\eta^{ij}(x)\Delta t+O(\Delta t^2)\ . \tag{5}$$

In (4) and (5), $\Delta q^i(t)=q^i(t+\Delta t)-q^i(t)$ in a generic local chart, $\Delta t>0$ and $v_{(+)}(\cdot,t)$ is a given controlling field, belonging to the tangent space TM_x. Moreover

$$m^i(x)=-\nu\eta^{jk}\Gamma^i_{jk}\ , \tag{6}$$

where Γ are the Christoffel symbols associated to η and ν is a diffusion coefficient, to be specified later. Notice[1,4] that m^i *does not* transform as a vector under a change of local charts, in agreement with the fact that Δq^i is a vector only up to the order $\Delta t^{1/2}$ (see Ref. 6). The density ρ satisfies the Fokker-Planck equation

$$\partial_t\rho=-\nabla_i(\rho v^i_{(+)})+\nu\Delta\rho\ , \tag{7}$$

where ∇ is the gradient with respect to the connection given by Γ and Δ is the associated Laplace-Beltrami operator.

Through standard methods,[1,2,4] one can also introduce the drift field $v_{(-)}(\cdot,t)$ for the time-inverted process, given by

$$v^i_{(-)}(x,t)=v^i_{(+)}(x,t)-2\nu\eta^{ij}\nabla_j\rho/\rho\ , \tag{8}$$

and the current and osmotic velocities

$$v^i(x,t)=\tfrac{1}{2}(v^i_{(+)}+v^i_{(-)})\ ,$$
$$u^i(x,t)=\tfrac{1}{2}(v^i_{(+)}-v^i_{(-)})=\nu\eta^{ij}\nabla_j\rho/\rho\ . \tag{9}$$

In terms of the current velocity v, Eq. (7) assumes the form of a continuity equation

$$\partial_t\rho+\nabla_i(\rho v^i)=0\ . \tag{10}$$

We assume $\rho(\cdot,t_0)$ and $v_{(+)}(\cdot,t), t_0\le t\le t_1$ as controlling fields. The associated process, defined by (4) and (5), plays the role of trial process in the stochastic variational principle, introduced later. The physical processes will be selected as the critical processes, making stationary the action.

For the sake of simplicity we consider smooth processes, where ρ is smooth and nowhere zero and $v_{(+)}$ is smooth and bounded. But our considerations can be easily extended to a class of much more singular processes, as introduced by Carlen in Ref. 7 (see also Ref. 6).

According to the general strategy of Ref. 5, let us introduce the stochastic Lagrangian, associated to a generic process $q(t)$, in the time-invariant form

$$\mathscr{L}(x,t)=\tfrac{1}{2}mg_{ij}v^i_{(+)}v^j_{(-)}-V\ , \tag{11}$$

strictly related to the expression (1) for the classical Lagrangian (see also Refs. 1 and 6). Then the action is given by

$$A=\int_{t_0}^{t_1}E(\mathscr{L}(q(t),t))dt\ . \tag{12}$$

Critical processes make A stationary ($\delta A=0$), under appropriate boundary conditions, when $\rho(\cdot,t_0)$ and $v_{(+)}(\cdot,t)$ undergo generic variations $\delta\rho(\cdot,t_0)$ and $\delta v_{(+)}(\cdot,t)$. Notice that here the Brownian metric η is kept fixed. The more general case, where η is also varied, is analyzed in Ref. 8.

III. THE PROGRAMMING EQUATION

In order to give a very handy expression for δA, by following the same method as in Ref. 5, it is convenient to introduce the forward Lagrangian

$$\mathscr{L}_{(+)}(x,t)=\tfrac{1}{2}mg_{ij}v^i_{(+)}v^j_{(+)}+m\nu\nabla_k(g_{ij}\eta^{jk}v^i_{(+)})-V \tag{13}$$

and the analogous time-inverted backward one $\mathscr{L}_{(-)}$. Notice that $\mathscr{L}_{(+)}$ is a function only of $v_{(+)}$. Moreover, a simple integration by parts shows that

$$E(\mathscr{L}(q(t),t))=E(\mathscr{L}_{(\pm)}(q(t),t))\ , \tag{14}$$

so that $\mathscr{L}$ and $\mathscr{L}_\pm$ are equivalent in the definition of the action given by (12).

For a given smooth function $S_1(\cdot)$ on M, let us also introduce the auxiliary function $S(\cdot,t)$, defined by

$$S(x,t)=-\int_t^{t_1}E(\mathscr{L}_{(+)}(q(t'),t')\mid q(t)=x)dt' +E(S_1(q(t_1))\mid q(t)=x)\ . \tag{15}$$

Clearly one has $S(\cdot,t_1)=S_1(\cdot)$ and

$$(D_{(+)}S)(x,t)=\mathscr{L}_{(+)}(x,t)\ , \tag{16}$$

where the forward stochastic derivative $D_{(+)}$ is given by[4]

$$D_{(+)}=\partial_t+v^i_{(+)}\nabla_i+\nu\Delta\ . \tag{17}$$

From (12), (14), and (15) one immediately finds

$$A=E(S_1(q(t_1)))-E(S(q(t_0),t_0))\ . \tag{18}$$

As a consequence of (16), under the stated variation $\delta v_{(+)}$, we have

$$\delta D_{(+)}S+D_{(+)}\delta S=\delta\mathscr{L}_{(+)}\ , \tag{19}$$

where $\delta D_{(+)}=\delta v^i_{(+)}\nabla_i$.

Let us now take the expectation in (19) and integrate on $[t_0,t_1]$. By exploiting (13) and (9) we find, as in the flat case,[5]

$$\int_{t_0}^{t_1}dt\,E((\nabla_iS-mg_{ij}v^j)\delta v^{(i)}_{(+)})=E(\delta S(q(t_0),t_0))\ . \tag{20}$$

Let us now take the variation δA in (18). Taking into account that averages are given by

$$E(F(q(t),t))=\int F(x,t)\rho(x,t)\sqrt{\eta}\,dx\ ,$$

and that $\delta S_1=0$, exploiting also (20), we finally have the basic variation formula

$$\delta A=\int S_1(x)\delta\rho(x,t_1)\sqrt{\eta}\,dx-\int S(x,t_0)\delta\rho(x,t_0)\sqrt{\eta}\,dx+\int_{t_0}^{t_1}dt\int\rho(x,t)[mg_{ij}(x)v^j(x,t)-\nabla_i S(x,t)]\delta v^i_{(+)}(x,t)\sqrt{\eta}\,dx\ . \tag{21}$$

Now we consider the critical processes for which $\delta A=0$ under variations $\delta v_{(+)}$ such that the boundary terms at t_0 and t_1 in (21) cancel each other. For these critical processes, since $\delta v_{(+)}$ is otherwise arbitrary, we must have the following relation between v and S:

$$mg_{ij}v^j=\nabla_i S,\quad v^i=g^{ij}\nabla_j S/m\ . \tag{22}$$

Therefore, the continuity equation (10) becomes

$$\partial_t\rho+\nabla_i(\rho g^{ij}\nabla_j S)/m=0\ , \tag{23}$$

while (16) with (22) leads to the programming equation of the Hamilton-Jacobi type

$$\partial_t S+\frac{1}{2m}g^{ij}\nabla_i S\nabla_j S+V-2m\nu^2[\bar{g}^{ij}\nabla_i R\nabla_j R+\nabla_i(\bar{g}^{ij}\nabla_j R)]=0\ , \tag{24}$$

where $\bar{g}$ is defined by

$$\bar{g}^{ij}=\eta^{ik}\eta^{jl}g_{kl} \tag{25}$$

and R is related to ρ by

$$\rho(x,t)=\exp[2R(x,t)]\ . \tag{26}$$

In the flat case, Eqs. (23) and (24) lead immediately to the Schrödinger equation through the canonical[9] ansatz for the wave function

$$\psi(x,t)=\exp[R(x,t)+iS(x,t)/\hbar]\ , \tag{27}$$

where ν and $\hbar$ are related by

$$\hbar=2m\nu\ . \tag{28}$$

In the nonflat case, considered here, some additional condition, relating g_{ij} and η^{ij}, must be introduced.

IV. THE COMPATIBILITY CONDITION

Let us assume that g_{ij} and μ^{ij} are related through the basic compatibility condition (3), which can be also stated as $\bar{g}^{ij}=g^{ij}$, on the basis of (25). If this condition is satisfied, then a very straightforward calculation, as in the flat case, shows that the wave function ψ, defined in (27), as a consequence of (23) and (24), satisfies the Schrödinger equation

$$i\hbar\partial_t\psi=-\frac{\hbar^2}{2m}\Delta_g\psi+V\psi\ , \tag{29}$$

where Δ_g is the Laplace-Beltrami operator associated to the kinetic metric g. If g is of Lorentz $(-1,-1,-1,1)$ signature then Δ_g is in fact a d'Alembertian. Vice versa, if (29) is enforced to be equivalent to (23) and (24), then the compatibility condition (3) must necessarily hold.

Let us notice a very important fact. In the programming equation (24), the term in the square brackets depends both on the kinetic metric g and the Brownian metric η. But if the compatibility condition is satisfied, then any dependence on η will in fact disappear, in agreement with (29), where nothing about η is left.

Let us also remark that as a consequence of (3) we have

$$\det(\eta_{ij})=|\det(g_{ij})|\ , \tag{30}$$

therefore any divergence $\nabla_i f^i$ can be calculated by taking ∇_i according to the connection associated to η or to g, since

$$\nabla_i f^i=\partial_i\{f^i[\det(\eta_{ij})]^{1/2}\}/[\det(\eta_{ij})]^{1/2}\ . \tag{31}$$

It is very simple to verify that, if g is positive definite, then the compatibility condition is in fact equivalent to the equality $\eta=g$. This provides an independent justification to the fact that in the standard treatment[1–4] the covariance of the Brownian noise is taken in agreement with the positive definite kinetic metric.

On the other hand, Lagrangian variational principles have a meaning also in the case where g is not positive definite. In these cases the compatibility condition allows to employ a Brownian metric different from the metric appearing in the Lagrangian. For example, in the Lorentzian case, one can immediately see that the metric

$$\eta^{\mu\nu}=2u^\mu u^\nu-g^{\mu\nu},\quad \mu,\nu=0,1,2,3\ , \tag{32}$$

is positive definite and compatible with g, provided $u=\{u^\mu(x)\}$ is a smooth field of time oriented velocities, i.e., for example,

$$u^\mu u_\mu=1,\quad u^0\geq 1\ . \tag{33}$$

We refer to Ref. 6 for a more detailed analysis of this example, of interest for the quantization of dynamical systems subject to gravitational forces.

V. CONCLUSIONS AND OUTLOOK

We have considered the stochastic quantization procedure by keeping separated the kinetic metric, appearing in the Lagrangian, and the Brownian metric, appearing in the covariance (5) of the trial processes. We have found that a consistent formulation of the procedure, leading to Schrödinger equation, enforces a kind of compatibility relation between the two metrics. If also the kinetic metric is positive definite, then compatibility implies that the two metrics must coincide. If the kinetic metric is not positive definite, then our procedure allows to apply also in this case the stochastic quantization method, which must employ a positive-definite Brownian metric, necessarily different from (but compatible with) the kinetic metric. In order to get a better possible physical understanding of the underlying Brownian metric, it would be useful to consider stochastic variational principles where also the Brownian metric η is subject to variations, together with the drift $v_{(+)}$. Work on this subject is in progress and will be reported elsewhere.[8]

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