Effective actions and conformal transformations

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We give the conformal transformation law for the effective action for conformally invariant field theories on general curved space-times. It is shown that this law can be used to obtain almost all known renormalized stress tensors for such theories. It can also be used to obtain approximate renormalized stress tensors; for example, we derive as a special case Page's approximation for static Einstein space-times.

I. INTRODUCTION

Two years ago Page[1] introduced a scheme for approximating the renormalized stress tensor associated with a thermal quantum state of a conformally invariant field theory in a static Einstein space-time. In recent months interest has been revived in this scheme since it was shown by the numerical work of Howard[2] that this approximation was extremely (perhaps unreasonably) good for the Hartle-Hawking[3] vacuum state of a Schwarzschild black hole.

Page's approximation scheme relied on making a conformal transformation to the optical (ultrastatic) metric on the space-time. This metric has the important property that the coincidence limit of the second DeWitt coefficient,[4] $a_2(x,x)$, the so-called trace anomaly, vanishes. In this paper we shall show how Page's work can be obtained, as a special case, from a knowledge of the general conformal transformation law of the one-loop effective action. This transformation law is derived in Sec. II and it is shown that it can take a particularly simple form for space-times which are conformal to those where a_2 vanishes. In Sec. III we present a number of examples and show that in many simple cases our analysis enables one to obtain the exact renormalized stress tensor. Section IV provides the connection between this effective-action formalism and the states and Green's functions of the theory.

The work presented here is largely based on work performed by one of us (M.R.B.) about five years ago. None of it was published at that time; then its relevance was unclear, in particular its accuracy as an approximation scheme was unknown. Time has shown that, even in nontrivial cases, it is remarkably good. Why this should be so remains an open question.

II. THE GENERAL SCHEME

We shall deal entirely with free conformally invariant field theories with spin $s=0$, $\frac{1}{2}$, or 1. In anticipation of our use of dimensional regularization we shall work with an n-dimensional space-time manifold.

If $S[\Phi]$ denotes the classical action functional of the theory then the vacuum-to-vacuum amplitude is defined by the functional integral

$$\exp\left[\frac{i}{\hbar}W[\Phi]\right]=\langle\text{out}\,|\,\text{in}\rangle =N^{-1}\int d\phi\,\exp\left[\frac{i}{\hbar}S[\Phi+\phi]\right], \tag{2.1}$$

where N is a metric-independent normalization constant. The $|\text{in}\rangle$ and $|\text{out}\rangle$ vacuum states are defined relative to the background field Φ.

We shall restrict ourselves to the one-loop $O(\hbar)$ approximation to the vacuum-to-vacuum amplitude, in which one keeps only the quadratic fluctuations about Φ. Performing the Gaussian functional integral one finds $W=S+\hbar W^{(1)}+\cdots$, where

$$W^{(1)}=-\frac{i}{2}\,\mathrm{Tr}(\ln G_F) \tag{2.2}$$

with $G_F(x,x')$ being the Schwinger-Feynman Green's function for the theory. $W^{(1)}$ is equal to the one-loop effective action for the theory.

As it stands Eq. (2.2) is meaningless and it must be given a meaning through renormalization theory. Using dimensional regularization[5] we define the renormalized one-loop effective action $W_R^{(1)}$ by adding counterterms to cancel precisely those terms in $W^{(1)}$ that appears as poles at $n=4$. To be specific, we define

$$W_R^{(1)}=(W^{(1)}+\Delta W^{(1)})_{n=4}\,, \tag{2.3}$$

where the counterterms $\Delta W^{(1)}$, which are independent of Φ, are given by[6]

$$\Delta W^{(1)}=\frac{a(s)}{(n-4)}\int d^nx\,g^{1/2}(R_{abcd}R^{abcd}-2R_{ab}R^{ab}+\tfrac{1}{3}R^2) +\frac{b(s)}{(n-4)}\int d^nx\,g^{1/2}(R_{abcd}R^{abcd}-4R_{ab}R^{ab}+R^2) \tag{2.4}$$

with the coefficients $a(s)$ and $b(s)$ given in Table I.

We now wish to consider the properties of W_R under conformal transformations. [We shall henceforth drop the superscript (1) for convenience but remind the reader that we will always be dealing with the one-loop effective action.] We write $\tilde{g}_{ab}=e^{-2\omega}g_{ab}$ and generally distinguish

TABLE I. The coefficients $a(s)$ and $b(s)$ for Eq. (2.4) according to dimensional regularization.

	$s=0$	$s=\frac{1}{2}$	$s=1$
$(5760\pi^2)a(s)$	3	18	36
$(5760\pi^2)b(s)$	-1	-11	-62

quantities in the conformal space by a tilde. The bare effective action preserves the symmetries of the field theory and so is invariant under the conformal transformation, whence

$$W_R[g]-W_R[\tilde{g}]=-(\Delta W[g]-\Delta W[\tilde{g}])_{n=4}\,. \qquad (2.5)$$

Using the n-dimensional conformal transformation laws of the Appendix, one can show that

$$(\Delta W[g]-\Delta W[\tilde{g}])_{n=4}=a(s)A[\omega;g]+b(s)B[\omega;g]\,, \qquad (2.6)$$

where

$$A[\omega;g_{ab}]=\int d^4x\, g^{1/2}\{(R_{abcd}R^{abcd}-2R_{ab}R^{ab}+\tfrac{1}{3}R^2)\omega +\tfrac{2}{3}[R+3(\Box\omega-\omega_{;c}\omega^{;c})]\times[\Box\omega-\omega_{;d}\omega^{;d}]\} \qquad (2.7)$$

and

$$B[\omega;g_{ab}]=\int d^4x\, g^{1/2}\{(R_{abcd}R^{abcd}-4R_{ab}R^{ab}+R^2)\omega +4R_{ab}\omega^{;a}\omega^{;b}-2R\omega_{;c}\omega^{;c} +2(\omega_{;c}\omega^{;c})^2-4\omega_{;c}\omega^{;c}\Box\omega\}\,. \qquad (2.8)$$

In writing Eq. (2.6) we have discarded total divergences. From Eqs. (2.5) and (2.6) we find that under conformal transformation $W_R[g]$ transforms according to the rule

$$W_R[\tilde{g}]=W_R[g]-a(s)A[\omega;g]-b(s)B[\omega;g]\,. \qquad (2.9)$$

By employing the identity

$$\left.\frac{\delta}{\delta\omega}F[e^{-2\omega}g_{cd}]\right|_{\omega=0}=-2g_{ab}\frac{\delta}{\delta g_{ab}}F[g_{cd}] \qquad (2.10)$$

we find

$$T_R{}^a{}_a=a(s)(R_{abcd}R^{abcd}-2R_{ab}R^{ab}+\tfrac{1}{3}R^2+\tfrac{2}{3}\Box R) +b(s)(R_{abcd}R^{abcd}-4R_{ab}R^{ab}+R^2)\,, \qquad (2.11)$$

where the renormalized effective stress tensor $T_R{}^{ab}$ is defined by

$$T_R{}^{ab}=2g^{-1/2}\frac{\delta}{\delta g_{ab}}W_R\,. \qquad (2.12)$$

Equation (2.11) is in agreement with the usual trace anomaly.[6]

The trace anomaly is proportional to the coincidence limit of the second DeWitt coefficient, $a_2(x,x)$. Spaces in which $a_2(x,x)$ vanishes have two important properties. First, there exist states in which the renormalized stress tensor vanishes identically.[7] Secondly, the Gaussian approximation is especially good in these spaces and Page[1] has argued that the stress tensor should be approximately zero in many cases of interest. Both these points will be discussed further in Sec. IV.

These properties suggest that it should be of interest to look for conformal factors ω such that

$$T_R{}^a{}_a[\tilde{g}]=0\,. \qquad (2.13)$$

In such a space we may then set $T_R{}^{ab}[\tilde{g}]=0$, as an exact statement for the first class of states or as an approximation for the second class. Equation (2.9) then enables us to obtain an expression for $T_R{}^{ab}[g]$. To this end we first note that

$$\left.\frac{\delta}{\delta\omega}\right|_{g\text{ fixed}}\{a(s)A[\omega;g]+b(s)B[\omega;g]\}=\tilde{g}^{1/2}T_R{}^a{}_a[\tilde{g}]\,. \qquad (2.14)$$

Thus when $T_R{}^a{}_a[\tilde{g}]=0$ we can write

$$T_R{}^{ab}[g]=2g^{-1/2}\frac{\delta W_R}{\delta g_{ab}} =2g^{-1/2}\left.\frac{\delta}{\delta g_{ab}}\right|_{\omega\text{ fixed}}\{a(s)A[\omega;g]+b(s)B[\omega;g]\} =a(s)T_A{}^{ab}+b(s)T_B{}^{ab}\,, \qquad (2.15)$$

where

$$T_A{}^{ab}=8R^{cabd}\omega_{;cd}-\tfrac{4}{3}(\omega_{;c}\omega^{;c})^{;ab} +2g^{ab}[2\omega^{;c}(\omega_{;d}\omega^{;d})_{;c}+(\omega_{;c}\omega^{;c})^2+\tfrac{2}{3}\Box(\omega_{;c}\omega^{;c})] -8(\omega_{;c}\omega^{;c})^{;(a}\omega^{;b)}-8\omega^{;a}\omega^{;b}(\omega_{;c}\omega^{;c}) \qquad (2.16)$$

and

$$T_B{}^{ab}=8R^{cabd}\omega_{;cd}+8R^{cabd}\omega_{;c}\omega_{;d}-8\omega^{;ac}\omega_{;c}{}^{;b} -8(\omega_{;c}\omega^{;c})^{;(a}\omega^{;b)}-8\omega^{;a}\omega^{;b}(\omega_{;c}\omega^{;c}) +4g^{ab}[\omega_{;cd}\omega^{;cd}+(\omega_{;c}\omega^{;c})_{;d}\omega^{;d}+\tfrac{1}{2}(\omega_{;c}\omega^{;c})^2]\,. \qquad (2.17)$$

It can be shown that the tensors T_A and T_B have the following properties when Eq. (2.13) is satisfied:

$$T_A{}^a{}_a=R_{abcd}R^{abcd}-2R_{ab}R^{ab}+\tfrac{1}{3}R^2+\tfrac{2}{3}\Box R\,, \qquad (2.18a)$$

$$T_B{}^a{}_a=R_{abcd}R^{abcd}-4R_{ab}R^{ab}+R^2\,, \qquad (2.18b)$$

$$T_A{}^{ab}{}_{;b}=T_B{}^{ab}{}_{;b}=0\,, \qquad (2.19)$$

$$T_A{}^{ab}[\tilde{g}]=T_B{}^{ab}[\tilde{g}]=0\,. \qquad (2.20)$$

It is convenient to record the expanded version of Eq.

(2.13). This is given by

$$T_R{}^a{}_a[\tilde{g}]=a(s)T_A{}^a{}_a[\tilde{g}]+b(s)T_B{}^a{}_a[\tilde{g}]=0\ , \qquad (2.21)$$

where

$$T_A{}^a{}_a[\tilde{g}]=e^{4\omega}\{C_{abcd}C^{abcd}+\tfrac{2}{3}\Box R+\tfrac{4}{3}R\Box\omega+\tfrac{4}{3}R_{;a}\omega^{;a}+4\Box(\Box\omega)+8[(\Box\omega)^2-\omega_{;ab}\omega^{;ab}-R_{ab}\omega^{;a}\omega^{;b}-\omega^{;c}\omega_{;c}\Box\omega-(\omega_{;a}\omega^{;a})_{;b}\omega^{;b}]\} \qquad (2.22)$$

and

$$T_B{}^a{}_a[\tilde{g}]=e^{4\omega}\{R_{abcd}R^{abcd}-4R_{ab}R^{ab}+R^2+8[(\Box\omega)^2-\omega_{;ab}\omega^{;ab}-R_{ab}\omega^{;a}\omega^{;b}-\omega^{;c}\omega_{;c}\Box\omega-(\omega_{;a}\omega^{;a})_{;b}\omega^{;b}]\}\ . \qquad (2.23)$$

In the cases where we solve Eq. (2.21) we exactly solve the equations

$$T_A{}^a{}_a[\tilde{g}]=0 \qquad (2.24)$$

and

$$T_B{}^a{}_a[\tilde{g}]=0\ . \qquad (2.25)$$

This allows the resulting solutions to be used to obtain the stress tensors for any spin without extra work.

Although for simplicity we have used dimensional regularization to obtain our results we wish to stress that the crucial result is independent of the regularization scheme used: Eqs. (2.7) and (2.8) define four-dimensional actions possessing the property that the tensors formed by taking their variation with respect to the metric have traces proportional to the trace anomaly.

Finally we note that for regularization schemes other than dimensional regularization the trace anomaly can take the form

$$T_R{}^a{}_a=a(s)(R_{abcd}R^{abcd}-2R_{ab}R^{ab}+\tfrac{1}{3}R^2+\tfrac{2}{3}\Box R)+b(s)(R_{abcd}R^{abcd}-4R_{ab}R^{ab}+R^2)+c(s)\Box R\ , \qquad (2.26)$$

where $c(s)$ can be nonzero for spin-1 fields. This coefficient may be adjusted by adding a term proportional to R^2 to the effective Lagrangian if that is thought to be desirable. Here we shall not discuss it further and we take $c(s)$ to be zero for all spins.

III. EXAMPLES

We shall list below a number of examples where the formalism of the previous section may be used to obtain an effective stress tensor. We shall for brevity only give the results for scalar fields although, of course, one can in all cases find corresponding results for spin $\frac{1}{2}$ and 1.

A. Conformally flat spaces

Consider a space which is conformally related to the whole of Minkowski space and choose coordinates in which the metric takes the form $g_{ab}=e^{-2\chi}\eta_{ab}$. Equation (2.21) is trivially satisfied if we choose $\omega=-\chi$. Using the identity $R_{ab}[e^{-2\omega}g]=0$ we find

$$\omega_{;ab}+\omega_{;a}\omega_{;b}-\tfrac{1}{2}g_{ab}\omega_{;c}\omega^{;c}=-\tfrac{1}{2}(R_{ab}-\tfrac{1}{6}Rg_{ab}) \qquad (3.1)$$

which enables us to write

$$T_A{}^{ab}=\tfrac{1}{18}[g^{ab}(4\Box R-R^2)+4RR^{ab}-4R^{;ab}] \qquad (3.2)$$

and

$$T_B{}^{ab}=2[g^{ab}(\tfrac{1}{4}R^2-\tfrac{1}{2}R_{cd}R^{cd})+R^{ac}R^b{}_c-\tfrac{2}{3}RR^{ab}] \qquad (3.3)$$

and Eq. (2.15) yields $T_R{}^{ab}$ for arbitrary spin.

In this case the conformal space is just Minkowski space and the appropriate state there is the Minkowski vacuum which has vanishing stress tensor. Equations (3.2) and (3.3) thus define the exact stress tensor in the conformal Minkowski vacuum on the original space-time, in agreement with the results of Ref. 8.

B. Static Einstein space-times

The second general class of space-times where one can find a solution to Eqs. (2.24) and (2.25) is the set of static Einstein space-times, that is, space-times where $R_{ab}=0$ and there exists a curl-free Killing vector field K. In such a case the following equations hold:

$$K_{(a;b)}=0\ , \qquad (3.4)$$

$$K_{a;bc}=C_{abcd}K^d\ , \qquad (3.5)$$

and

$$\eta^{abcd}K_{b;c}K_{;d}=0\ . \qquad (3.6)$$

We can introduce a time coordinate t associated with K in the sense that $K^a=(\partial/\partial t)^a$. Equations (3.4), (3.5), and (3.6) are then sufficient to show that

$$\omega[T]\equiv(2\pi T)t+\tfrac{1}{2}\ln(-K^2)$$

satisfies the equations

$$\Box\omega=0 \qquad (3.7)$$

and

$$2\omega^{;a}(\omega_{;b}\omega^{;b})_{;a}+\Box(\omega_{;a}\omega^{;a})=\tfrac{1}{4}C_{abcd}C^{abcd}\ . \qquad (3.8)$$

In turn, these equations imply that ω satisfies Eqs. (2.24) and (2.25).

Below we shall see that $\omega[T]$ yields the approximate renormalized stress tensor for a thermal state with temperature T. The vacuum stress tensor $T_R{}^{ab}[0]$ defined by $\omega[0]$ is a rather complicated function of K^a, however the finite-temperature corrections are relatively simple: one can write

$$T_R{}^{ab}[T]=T_R{}^{ab}[0]+T^2[\tfrac{1}{3}a(s)+b(s)]\frac{1}{K^4}\times\left[\Box K^2\left[g^{ab}-\frac{4K^aK^b}{K^2}\right]+8C^{cabd}K_cK_d\right]+2T^4[a(s)+b(s)]\frac{1}{K^4}\left[g^{ab}-\frac{4K^aK^b}{K^2}\right]. \quad (3.9)$$

It is interesting to note that the second term vanishes for scalar fields leaving the thermal part of the stress tensor isotropic.

As a particularly simple example we can consider Rindler space with its associated Killing vector field K. For scalar fields we find

$$T_R{}^{ab}[T]=\frac{1}{1440\pi^2}[(2\pi T)^4-1]\frac{1}{K^4}\left[g^{ab}-\frac{4K^aK^b}{K^2}\right]. \quad (3.10)$$

Clearly we have $T_R{}^{ab}[1/2\pi]=0$, in agreement with the fact that the thermal Rindler state with temperature $1/2\pi$ is identical to the Minkowski vacuum.

A more interesting example of the use of the above results is provided by Schwarzschild space-time. The result for the renormalized expectation value of the stress tensor in a thermal state with the Hawking[9] temperature $T_H=1/8\pi M$ can be expressed through the equations

$$T_A{}^{ab}[T_H]=\frac{1}{24r^6R^4}[(3r^6+6r^5R+9r^4R^2+12r^3R^3+13r^2R^4+14rR^5-41R^6)U^{ab}+8R^3(r^3+r^2R+rR^2+2R^3)V^{ab}+72R^6W^{ab}] \quad (3.11)$$

and

$$T_B{}^{ab}[T_H]=\frac{1}{8r^6R^4}[(r^6+2r^5R+3r^4R^2+4r^3R^3+3r^2R^4+2rR^5-23R^6)U^{ab}+8R^3(r^3+r^2R+rR^2+2R^3)V^{ab}+24R^6W^{ab}], \quad (3.12)$$

where $R\equiv 2M$ and the tensors U^{ab}, V^{ab}, and W^{ab} are given in Schwarzschild coordinates (t,r,θ,ϕ) by

$$U^a{}_b=\text{diag}(-3,1,1,1)^a{}_b\ , \quad (3.13)$$

$$V^a{}_b=\text{diag}(0,2,-1,-1)^a{}_b\ , \quad (3.14)$$

and

$$W^a{}_b=\text{diag}(3,1,0,0)^a{}_b\ . \quad (3.15)$$

Equations (3.11) and (3.12) are in agreement with the results of Page,[1] and correspond to the approximate renormalized stress tensor in the Hartle-Hawking vacuum.

The stress tensor for a state with arbitrary temperature T can be obtained with the help of Eq. (3.9) which yields

$$T_A{}^{ab}[T]=T_A{}^{ab}[T_H]+\frac{4\pi^2}{3r^2(r-R)^2}[24\pi^2(T^4-T_H{}^4)r^4-(T^2-T_H{}^2)R^2]U^{ab}+\frac{16\pi^2}{3}\frac{R}{r^2(r-R)}(T^2-T_H{}^2)V^{ab} \quad (3.16)$$

and

$$T_B{}^{ab}[T]=T_B{}^{ab}[T_H]+\frac{4\pi^2}{r^2(r-R)^2}[8\pi^2(T^4-T_H{}^4)r^4-(T^2-T_H{}^2)R^2]U^{ab}+16\pi^2\frac{R}{r^2(r-R)}(T^2-T_H{}^2)V^{ab}\ . \quad (3.17)$$

In particular, if we set $T=0$ we obtain an approximate expression for the renormalized stress tensor in the Boulware vacuum[10] which agrees with all known properties of the exact stress tensor.

C. Flat-space curl-free conformal Killing vector fields

The quantum field theory associated with the curl-free conformal Killing vector fields in flat space is discussed in Ref. 11. Choosing $\omega=\frac{1}{2}\ln(-K^2)$ one finds that

$$\Box\Box\omega=0 \quad (3.18)$$

and

$$(\Box\omega)^2-\omega_{;ab}\omega^{;ab}-\omega^{;c}\omega_{;c}\Box\omega-(\omega_{;a}\omega^{;a})_{;b}\omega^{;b}=0 \quad (3.19)$$

so that Eqs. (2.24) and (2.25) are satisfied. Then for scalar fields Eq. (2.15) yields

$$T_R{}^{ab} = -\frac{1}{1440\pi^2}\frac{(\Box K^2 - 3K^2 \Box \ln K^2)^2}{16K^4}\left[g^{ab} - \frac{4K^aK^b}{K^2}\right] \tag{3.20}$$

in agreement with the exact results of Ref. 11. This exact agreement is a consequence of the vanishing stress in the appropriate conformal state in the related ultrastatic space.[11]

D. Flat-space Killing vector fields

Since $K_{a;bc}=0$ for a Killing vector field in flat space there are only two independent scalar invariants that one can form from K^a, these one can choose to be K^2 and $K^2{}_{;a}K^{2;a}$. Seeking solutions to Eqs. (2.24) and (2.25) of the form $\omega = \omega(K^2, K^2{}_{;a}K^{2;a})$ one finds the solutions

$$\omega = \tfrac{1}{2}\ln[K^aL_a(\nu_1)](+)\tfrac{1}{2}\ln[K^aL_a(\nu_2)] , \tag{3.21}$$

where

$$L_a(\nu) \equiv (K_{a;c}K^{c;b} - \nu\delta_a{}^b)K_b \tag{3.22}$$

and $(+)$ means that either logarithm is a solution on its own and also their sum is a solution. Here ν_1 and ν_2 are the roots of the quadratic equation

$$\nu^2 + \tfrac{1}{4}(\Box K^2)\nu + \det(K_a{}^{;b}) = 0 \tag{3.23}$$

which is the characteristic equation of the covariantly constant matrix $K_a{}^{;b}$.

The corresponding stress tensor can be written

$$T_R{}^{ab} = -\frac{(\nu_1-\nu_2)^2}{3840\pi^2}\left[\frac{\nu_2{}^2}{[K^cL_c(\nu_1)]^2}\left[g_{ab} + \frac{4L_a(\nu_1)L_b(\nu_1)}{(\nu_1-\nu_2)[K^cL_c(\nu_1)]}\right](+)\frac{\nu_1{}^2}{[K^cL_c(\nu_2)]^2}\left[g_{ab} + \frac{4L_a(\nu_2)L_b(\nu_2)}{(\nu_2-\nu_1)[K^aL_a(\nu_2)]}\right]\right] . \tag{3.24}$$

In the particular case of the Rindler Killing vector field $\Box K^2 = -4$ and one has $\nu_1 = 0$ and $\nu_2 = 1$. Choosing the nontrivial solution gives

$$T_R{}^{ab} = -\frac{1}{1440\pi^2}\frac{(\Box K^2)^2}{16K^4}\left[g^{ab} - \frac{4K^aK^b}{K^2}\right] , \tag{3.25}$$

or, in Rindler coordinates (τ, ξ, y, z),

$$T_R{}^a{}_b = -\frac{1}{1440\pi^2\xi^4}\mathrm{diag}(-3,1,1,1)^a{}_b \tag{3.26}$$

in agreement with Ref. 12.

The final two examples we give in this section are flat-space Casimir effects. In both cases we look for solutions to the equation $\Box\omega = 0$ which respect the symmetries of the background space-time. For flat-space solutions to $\Box\omega = 0$ to satisfy Eqs. (2.24) and (2.25) they must also satisfy the equation

$$\Box(\omega_{;a}\omega^{;a}) + 2\omega^{;a}(\omega_{;b}\omega^{;b})_{;a} = 0 . \tag{3.27}$$

E. Casimir effect for parallel plates

For flat space periodically identified in the x direction with period a we seek solutions $\omega = \omega(x)$ such that $e^{-\omega(x)} = e^{-\omega(x+a)}$. The solution (ignoring the trivial additive constant) is $\omega = -2\pi i x/a$. The corresponding stress tensor is given by

$$T_R{}^a{}_b = \frac{2^4\pi^2}{1440a^4}\mathrm{diag}(1,-3,1,1)^a{}_b \tag{3.28}$$

in agreement with Ref. 13.

The solution $\omega = -\pi i x/a$ gives the stress tensor for the scalar field satisfying Dirichlet boundary conditions; this is obtained by replacing a by $2a$ in Eq. (3.28).

F. Casimir effect for the wedge

We use cylindrical polar coordinates for which the line element is

$$ds^2 = -dt^2 + dr^2 + r^2d\theta^2 + dz^2 \quad (0 \le \theta \le \alpha) .$$

We seek a solution of the form $\omega = \omega(r,\theta)$ where $e^{-\omega(r,\theta)} = e^{-\omega(r,\theta+\alpha)}$. The solution is given by $\omega = \ln r - 2\pi i(\theta/\alpha)$ and the corresponding stress tensor is

$$T_R{}^a{}_b = \frac{1}{1440\alpha^2r^4}\left[\frac{4\pi^2}{\alpha^2} - \frac{\alpha^2}{4\pi^2}\right]\mathrm{diag}(1,1,-3,1)^a{}_b . \tag{3.29}$$

Equation (3.29) gives the result for periodic boundary conditions; as before, the result for Dirichlet boundary conditions is obtained by replacing α by 2α and agrees with Ref. 14.

IV. STATES AND GREEN'S FUNCTIONS

In this section we shall answer the question, what does it all mean? The results of Sec. II were obtained by a sort of magic in n-dimensional space-times. Here we shall work entirely with the physical, four-dimensional world and show how a given renormalized stress tensor is related to a given state of the matter field in the space-time. For simplicity we shall discuss only the scalar field; a similar analysis holds for other spins.

Let us suppose that the scalar field ϕ is in some unit-norm state $|S\rangle$. We define the Feynman Green's function for this state as

$$G(x,x')=i\langle S\,|\,T[\phi(x)\phi(x')]\,|\,S\rangle\ , \tag{4.1}$$

where T denotes time ordering. G must be symmetric in x and x' and must satisfy the inhomogeneous wave equation

$$(\Box-\tfrac{1}{6}R)G(x,x')=-\delta(x,x')\ . \tag{4.2}$$

Different states $|S\rangle$ will correspond to different solutions of Eq. (4.2). We shall consider those states whose two-point functions have the Hadamard form.[15] It is known that symmetric Green's functions of this form must have the following structure:[16]

$$G(x,x')=\frac{i}{8\pi^2}\left[\frac{\Delta^{1/2}}{\sigma+i\epsilon}+v\ln(\sigma+i\epsilon)+w\right]\ , \tag{4.3}$$

where v and w have the covariant Taylor series expansions

$$v(x,x')=\tfrac{1}{2}v_{ab}(x)\sigma^a\sigma^b-\tfrac{1}{4}v_{ab;c}(x)\sigma^a\sigma^b\sigma^c+O(\sigma^2)\ , \tag{4.4}$$

$$w(x,x')=w(x)-\tfrac{1}{2}w_{;a}(x)\sigma^a+\tfrac{1}{2}w_{ab}(x)\sigma^a\sigma^b-\tfrac{1}{4}[w_{ab;c}(x)-\tfrac{1}{6}w_{;abc}(x)]\sigma^a\sigma^b\sigma^c+O(\sigma^2)\ , \tag{4.5}$$

where $\sigma^a\equiv\sigma^{;a}$ and

$$v^{ab}=\tfrac{1}{120}(C^{cabd}R_{cd}+2C^{c(ab)d}{}_{;cd})\ , \tag{4.6}$$

$$w^{ab}=t^{ab}-\tfrac{1}{2}t^c{}_c g^{ab}+\tfrac{1}{6}R^{ab}w+\tfrac{1}{3}(w^{;ab}-\tfrac{1}{4}g^{ab}\Box w)\ , \tag{4.7}$$

$$t^a{}_a=\tfrac{1}{360}(R_{abcd}R^{abcd}-R_{ab}R^{ab}+\Box R)\ , \tag{4.8}$$

$$t^{ab}{}_{;b}=0\ . \tag{4.9}$$

To this order in σ, the state dependence of G is contained in the arbitrary scalar $w(x)$ and the symmetric tensor $t^{ab}(x)$ which is arbitrary except for conditions (4.8) and (4.9).

This result is quite general and is a well-defined statement about the finite function $G(x,x')$—no use is made of any renormalization theory. Now, if we compute the renormalized expectation value in the state $|S\rangle$ of the stress tensor operator of the scalar field, the answer is[7]

$$T_R{}^{ab}=\frac{1}{8\pi^2}(t^{ab}+v^{ab}\ln\mu^2)\ , \tag{4.10}$$

where μ is an arbitrary renormalization mass and the symmetric tensor v^{ab}, Eq. (4.6), is conserved and trace-free—indeed it is the usual local renormalization ambiguity which can be written as

$$v^{ab}=\tfrac{1}{240}g^{-1/2}\frac{\delta}{\delta g_{ab}}\int d^4x\,g^{1/2}C_{cdef}C^{cdef}\ . \tag{4.11}$$

Equation (4.10) makes plain the correspondence between renormalized stress tensors and the Green's functions of the theory—the crucial link is provided by the conserved tensor t^{ab} whose trace is proportional to the curvature invariant a_2—Eq. (4.8). It is worth emphasizing that there is nothing anomalous about the trace of this tensor where it appears in the Taylor series expansion of the Feynman function—it is only the trace of Eq. (4.10) that can be said to be anomalous.

If a space-time is such that a_2 vanishes then there can exist states for which t^{ab} being zero satisfies Eqs. (4.8) and (4.9). For these states the renormalized stress tensor is essentially zero. We shall call such states zero-energy states. If a space-time is conformal to one where a_2 vanishes then Eq. (2.15) gives the renormalized stress tensor, where it is to be understood that the expectation value is taken in a state which is the conformal image of a zero-energy state.

So far all we have said has been exact. However, the status of the use of the above analysis as an approximation scheme is less precise—it is more of an art. For example, we know of no reason (other than the pleasantly vague arguments of Page[1]) why Eqs. (3.11) and (3.12) give such a good approximation to the Hartle-Hawking stress tensor. Our procedure necessarily neglects any vacuum energy in the conformally related, ultrastatic space-time. In Schwarzschild space-time there are states which are the conformal images of zero-energy states in the ultrastatic space-time; they are apparently close to but not equal to the Boulware vacuum.

V. CONCLUSION

We hope to have demonstrated that the conformal transformation law for the renormalized effective action provides a compact and elegant way of studying properties of conformally invariant field theories in curved space-time. It is amusing that one can obtain from this law expressions for almost all known renormalized stress tensors in curved and topologically nontrivial flat space-times—the result for the Casimir wedge is particularly pretty. Even where one does not obtain the exact stress tensor it appears that one gets a good approximation.

For states of matter which are the conformal images of zero-energy states Eq. (2.15) gives a general analytic expression for the renormalized stress tensor. This allows one to phrase the general back-reaction problem as

$$G^{ab}=T_R{}^{ab}\ , \tag{5.1}$$

where $T_R{}^{ab}$ is given by Eq. (2.15) and ω is determined from Eq. (2.21).

Where these states only approximate states of physical interest—for example, in Schwarzschild space-time—this procedure will yield an approximate solution of the physical problem. The recent work of York[17] represents a first step in obtaining an iterative solution of these equations for black holes in boxes. It would be of great interest to find a solution ω of Eq. (2.21) that yields a stress tensor which approximates the renormalized stress tensor for the Unruh vacuum.[18] Then one could look for nonstatic, spherically symmetric solutions of Eq. (5.1) and so discover what happens to an evaporating black hole.

APPENDIX

Our space-time conventions follow those of Ref. 19. The following n-dimensional conformal transformation laws were used in Sec. II:

$$R^{ab}{}_{cd}[\tilde{g}]=e^{2\omega}(R^{ab}{}_{cd}[g]+\delta^{[a}{}_{[c}\omega^{b]}{}_{d]}) , \quad \text{(A1)}$$

$$R^{b}{}_{d}[\tilde{g}]=e^{2\omega}\{R^{b}{}_{d}[g]+\tfrac{1}{4}[(n-2)\omega^{b}{}_{d}+\delta^{b}{}_{d}\omega^{a}{}_{a}]\} , \quad \text{(A2)}$$

$$R[\tilde{g}]=e^{2\omega}[R(g)+\tfrac{1}{2}(n-1)\omega^{a}{}_{a}] , \quad \text{(A3)}$$

$$\tilde{\Box}\phi=e^{2\omega}[\Box\phi-(n-2)\omega^{;a}\phi_{;a}] , \quad \text{(A4)}$$

where $\tilde{g}_{ab}=e^{-2\omega}g_{ab}$ and

$$\omega^{a}{}_{b}\equiv 4(\omega^{;a}{}_{;b}+\omega^{;a}\omega_{;b})-2\delta^{a}{}_{b}\omega^{;c}\omega_{;c} . \quad \text{(A5)}$$

The following four-dimensional identities were used in Sec. III:

$$V_{a;bc}-V_{a;cb}=R^{d}{}_{abc}V_{d} , \quad \text{(A6)}$$

$$C^{a}{}_{bcd;a}=R_{b[d;c]}-\tfrac{1}{6}g_{b[d}R_{;c]} , \quad \text{(A7)}$$

$$C_{abcd}C^{abcd}=R_{abcd}R^{abcd}-2R_{ab}R^{ab}+\tfrac{1}{3}R^{2} , \quad \text{(A8)}$$

$$C_{acde}C_{b}{}^{cde}=\tfrac{1}{4}g_{ab}C_{cdef}C^{cdef} . \quad \text{(A9)}$$

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