$N=4$ remaining supersymmetry in a Kaluza-Klein monopole background in $D=11$ supergravity theory

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Upon seven-torus compactification of eleven-dimensional supergravity, a Kaluza-Klein monopole is embedded into one U(1) group of the isometry group $U(1)^7$. Four independent Killing spinors remain unbroken in this background.

Higher-dimensional general relativity (Kaluza-Klein theories)[1] is a promising candidate to unify gauge theories and gravity. However, dimensions of Kaluza-Klein theories are completely arbitrary unless constrained by supersymmetry. Requirement of the absence of states with spin higher than 2 puts an upper limit to the dimension of $D\leq 11$ for Riemmanian space and $D\leq 24$ for quasi-Riemmanian space.[2] Although the quantum behavior of Kaluza-Klein theories is not well studied yet, the supersymmetry will soften ultraviolet divergences compared to nonsupersymmetric cases. Furthermore, the topologically nontrivial solutions will play an important role in nonperturbative effects.[3]

The finite-energy Kaluza-Klein monopole solutions have been studied recently.[4-7] They are spherically symmetric and static, and are the usual magnetic monopoles in the asymptotic region of four space-time dimensions. This monopole solution is regular at the origin and the space-time geometry is intrinsically interwoven with the internal space in which direction the monopole is embedded.

It is interesting to examine the remaining supersymmetry in the background of Kaluza-Klein monopoles. In eleven-dimensional supergravity which is maximal in the pseudo-Riemmanian Kaluza-Klein theory, the fate of supersymmetry upon compactifications has been rather extensively studied. Some of them are the seven-torus[8] ($N=8$), round seven-sphere[9] ($N=8$), left squashed ($N=1$) and right squashed ($N=0$) seven-sphere,[10] M^{ppr} ($N=2$) and M^{pqr} ($N=0, p\neq q$) manifold,[11] where the number of remaining supersymmetries is given inside the parentheses. A table of known compactifications and surviving supersymmetry can be found in Ref. 12. Most interestingly, the $K_3\times T_3$ solution[13] is an example with four surviving supersymmetries, but without isometry group corresponding to K^3. It turns out that there is an $N=4$ supersymmetric solution in the seven-torus compactification with a Kaluza-Klein monopole embedded into one U(1) group of the isometry $U(1)^7$ group. It is noteworthy that the Kaluza-Klein monopole solution is a unique example of $N=4$ supersymmetry with an isometry group.

The bosonic parts in $D=11$, $N=1$ supergravity relevant for the background are

$$R_{MN}-\tfrac{1}{2}g_{MN}R=\tfrac{1}{3}(F_{MPQR}F_N{}^{PQR}-\tfrac{1}{8}g_{MN}F_{PQRS}F^{PQRS}) ,$$
$$\nabla_M F^{MPQR}=-\tfrac{1}{576}\epsilon^{M_1\cdots M_8 PQR}F_{M_1\cdots M_4}F_{M_5\cdots M_8} . \qquad (1)$$

The background solutions with Kaluza-Klein monopole are obtained by the following vacuum expectation values (VEV's):

$$\langle F_{\mu\nu\rho\sigma}\rangle=\langle F_{mnpq}\rangle=0 ,$$
$$\langle \Psi_\mu\rangle=\langle\Psi_m\rangle=0 . \qquad (2)$$

The VEV's for the elfbein are[4]

$$e^{(t)}=dt ,$$
$$e^{(r)}=e^{b/2}dr ,$$
$$e^{(\theta)}=e^{b/2}r\,d\theta ,$$
$$e^{(\phi)}=e^{b/2}r\sin\theta\,d\phi , \qquad (3)$$
$$e^{(5)}=e^{-b/2}[dx^5+n\kappa g(\cos\theta-1)d\phi] ,$$
$$e^{(6)}=dx^6,\ \ldots,\ e^{(11)}=dx^{11} .$$

Here $e^b=1+|n|R/2r$, $R=2\kappa g$, and n is the monopole charge. The indices inside the parentheses are for the frame indices. The spherically symmetric and static monopole solutions carrying the magnetic charge in more than one U(1) direction of the isometry group $U(1)^7$ simultaneously are shown to be absent.[6]

The number of independent supersymmetries will be determined by the Killing spinor equation[8-12]

$$D_M\eta=(\partial_M-\tfrac{1}{4}\omega_M{}^{AB}\Gamma_{AB})\eta=0 , \qquad (4)$$

where

$$\Gamma_{AB}=\tfrac{1}{2}(\Gamma_A\Gamma_B-\Gamma_B\Gamma_A) .$$

Here early alphabet letters are used as frame labels, while mid-alphabet letters are employed for world indices.

The relevant spin connections are

$$
\begin{aligned}
&\omega_\theta{}^{(r)}{}_{(\theta)}=\frac{-(4r+|n|R)}{2(2r+|n|R)}\,,\\
&\omega_\phi{}^{(r)}{}_{(\phi)}=\frac{-(4r+|n|R)}{2(2r+|n|R)}\sin\theta\,,\\
&\omega_\phi{}^{(r)}{}_{(5)}=\frac{-n|n|R^2}{2(2r+|n|R)^2}(\cos\theta-1)\,,\\
&\omega_5{}^{(r)}{}_{(5)}=-\frac{|n|R}{(2r+|n|R)^2}\,,\\
&\omega_\phi{}^{(\theta)}{}_{(\phi)}=-\cos\theta+\frac{n^2R^2(\cos\theta-1)}{2(2r+|n|R)^2}\,,\\
&\omega_5{}^{(\theta)}{}_{(\phi)}=\frac{nR}{(2r+|n|R)^2}\,,\\
&\omega_\phi{}^{(\theta)}{}_{(5)}=\frac{nR}{2(2r+|n|R)}\sin\theta\,,\\
&\omega_\theta{}^{(\phi)}{}_{(5)}=-\frac{nR}{2(2r+|n|R)}\,.
\end{aligned}
\tag{5}
$$

The flat indices are raised and lowered with the metric $\eta_{AB}=\mathrm{diag}(-1,+1,\ldots,+1)$, and $\omega_M{}^{AB}=-\omega_M{}^{BA}$. Of course, the flat-space limit without monopoles is obtained by putting $n=0$. Also since the Kaluza-Klein monopole solutions [equivalently Taub-Nut (Newman-Unti-Tamburino) solutions[14]] are regular at the origin, all spin connections are also regular.

The convenient choices for the Γ matrix with the Clifford algebra $\{\Gamma_A,\Gamma_B\}=-2\eta_{AB}$ are

$$
\begin{aligned}
&\Gamma_0=\gamma_0\times\mathbb{1}_8,\quad \Gamma_i=\gamma_i\times\mathbb{1}_8,\quad i=1,2,3,\\
&\Gamma_5=\gamma_5\times\begin{bmatrix}0 & -\mathbb{1}_4\\ \mathbb{1}_4 & 0\end{bmatrix},\\
&\Gamma_{5+j}=\gamma_5\times\begin{bmatrix}\alpha_j & 0\\ 0 & -\alpha_j\end{bmatrix},\quad j=1,2,3\\
&\Gamma_{8+j}=\gamma_5\times\begin{bmatrix}0 & \beta_j\\ \beta_j & 0\end{bmatrix},\quad j=1,2,3\,.
\end{aligned}
\tag{6}
$$

Here

$$
\gamma_0=\begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix},\quad
\gamma_i=\begin{bmatrix}0 & \sigma_i\\ -\sigma_i & 0\end{bmatrix},\quad
\gamma_5=-i\gamma_0\gamma_1\gamma_2\gamma_3=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix},
$$

and σ_i is the Pauli spin matrix. $\mathbb{1}_8$ and $\mathbb{1}_4$ are 8×8 and 4×4 identity matrices. The α_j and β_j are defined as

$$
\begin{aligned}
&\alpha_1=\begin{bmatrix}0 & \sigma_1\\ -\sigma_1 & 0\end{bmatrix},\quad \alpha_2=\begin{bmatrix}0 & -\sigma_3\\ \sigma_3 & 0\end{bmatrix},\\
&\alpha_3=\begin{bmatrix}i\sigma_2 & 0\\ 0 & i\sigma_2\end{bmatrix},\quad \beta_1=\begin{bmatrix}0 & i\sigma_2\\ i\sigma_2 & 0\end{bmatrix},\\
&\beta_2=\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix},\quad \beta_3=\begin{bmatrix}-i\sigma_2 & 0\\ 0 & i\sigma_2\end{bmatrix}.
\end{aligned}
\tag{7}
$$

Observe that Γ_5, Γ_{5+j}, and Γ_{8+j} are σ_2-, σ_3-, and σ_1-like in 4×4 block matrices. In these representations, Γ_0 and Γ_2 are symmetric, while all other Γ matrices are antisymmetric. Thus the charge conjugation matrix C is

$$
C=\gamma_0\gamma_2\times\mathbb{1}_8\,. \tag{8}
$$

The spinor representation of the tangent space group SO(1,10) is of 32 components, which are represented by

$$
\eta_T=(\Psi_1^T,\ldots,\Psi_8^T)\,, \tag{9}
$$

where $\Psi_j^T=(\lambda_j^T,\chi_j^T)$ with two-component spinor λ_j and χ_j. The Majorana condition $\eta=-C\bar{\eta}^T$ gives the relationship between λ_j and χ_j as

$$
\chi_j=-\sigma_2\lambda_j^*\,. \tag{10}
$$

The χ_j is dependent, and there are only 16 independent components corresponding to eight λ_j.

Since spin connections $\omega_M{}^{AB}$ with $M=t,r,6,\ldots,11$ vanish, their corresponding Killing spinor equations are trivial as

$$
\partial_t\eta=\partial_r\eta=\partial_6\eta=\cdots=\partial_{11}\eta\,. \tag{11}
$$

Thus the Killing spinor η is independent of coordinates $t,r,x^6,\ldots,x^{11}$. The Killing spinor equation in the θ direction reads

$$
\begin{aligned}
&\partial_\theta\Psi_j+\frac{i}{2}\omega_\theta{}^{(r)(\theta)}\begin{bmatrix}\sigma_3 & 0\\ 0 & \sigma_3\end{bmatrix}\Psi_j+\frac{1}{2}\omega_\theta{}^{(\phi)(5)}\begin{bmatrix}\sigma_3 & 0\\ 0 & -\sigma_3\end{bmatrix}\Psi_{j+4}=0\,,\\
&\partial_\theta\Psi_{j+4}+\frac{i}{2}\omega_\theta{}^{(r)(\theta)}\begin{bmatrix}\sigma_3 & 0\\ 0 & \sigma_3\end{bmatrix}\Psi_{j+4}-\frac{1}{2}\omega_\theta{}^{(\phi)(5)}\begin{bmatrix}\sigma_3 & 0\\ 0 & -\sigma_3\end{bmatrix}\Psi_j=0,\quad j=1,\ldots,4.
\end{aligned}
\tag{12}
$$

Notice that all eight Ψ_j are not independent, but Ψ_j $(j=1,\ldots,4)$ are related to Ψ_{j+4} $(j=1,\ldots,4)$.

The Killing spinor equation in the ϕ direction reads

$$\partial_\phi \Psi_j - \frac{i}{2}\omega_\phi{}^{(r)(\phi)} \begin{bmatrix} \sigma_2 & 0 \\ 0 & \sigma_2 \end{bmatrix} \Psi_j + \frac{i}{2}\omega_\phi{}^{(\theta)(\phi)} \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_1 \end{bmatrix} \Psi_j + \frac{1}{2}\omega_\phi{}^{(r)(5)} \begin{bmatrix} \sigma_1 & 0 \\ 0 & -\sigma_1 \end{bmatrix} \Psi_{j+4} + \frac{1}{2}\omega_\phi{}^{(\theta)(5)} \begin{bmatrix} \sigma_2 & 0 \\ 0 & -\sigma_2 \end{bmatrix} \Psi_{j+4} = 0 ,$$

$$\partial_\phi \Psi_{j+4} - \frac{i}{2}\omega_\phi{}^{(r)(\phi)} \begin{bmatrix} \sigma_2 & 0 \\ 0 & \sigma_2 \end{bmatrix} \Psi_{j+4} + \frac{i}{2}\omega_\phi{}^{(\theta)(\phi)} \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_1 \end{bmatrix} \Psi_{j+4} - \frac{1}{2}\omega_\phi{}^{(r)(5)} \begin{bmatrix} \sigma_1 & 0 \\ 0 & -\sigma_1 \end{bmatrix} \Psi_j - \frac{1}{2}\omega_\phi{}^{(\theta)(5)} \begin{bmatrix} \sigma_2 & 0 \\ 0 & -\sigma_2 \end{bmatrix} \Psi_j = 0, \quad j=1,\ldots,4. \tag{13}$$

Finally the Killing spinor equation in the fifth direction is

$$\partial_5 \Psi_j + \tfrac{1}{2}\omega_5{}^{(r)(5)} \begin{bmatrix} \sigma_1 & 0 \\ 0 & -\sigma_1 \end{bmatrix} \Psi_{j+4} + \frac{i}{2}\omega_5{}^{(\theta)(\phi)} \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_1 \end{bmatrix} \Psi_j = 0,$$

$$\partial_5 \Psi_{j+4} - \tfrac{1}{2}\omega_5{}^{(r)(5)} \begin{bmatrix} \sigma_1 & 0 \\ 0 & -\sigma_1 \end{bmatrix} \Psi_j + \frac{i}{2}\omega_5{}^{(\theta)(\phi)} \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_1 \end{bmatrix} \Psi_{j+4} = 0, \tag{14}$$

Solutions for these equations are obtained in a straightforward way. The Ψ_j $(j=1,\ldots,4)$ are given by

$$\lambda_j = \begin{bmatrix} (A_j e^{i/2\phi} + B_j e^{-i/2\phi}) e^{i/2\theta} \\ (A_j e^{i/2\phi} - B_j e^{-i/2\phi}) e^{-i/2\theta} \end{bmatrix} ,$$

$$\chi_j = -\sigma_2 \lambda_j^* . \tag{15}$$

The Ψ_{j+4} $(j=1,\ldots,4)$ are related to Ψ_j by

$$\lambda_{j+4} = \pm i\lambda_j ,$$

$$\chi_{j+4} = \mp i\chi_j , \tag{16}$$

where the upper (lower) sign corresponds to the positive (negative) monopole charge. Thus there are four independent Killing spinors Ψ_j $(j=1,\ldots,4)$ for Eq. (4), and we obtain $N=4$ remaining supersymmetries in the Kaluza-Klein monopole backgrounds (2) and (3). In the limit of vanishing monopole charge $(n=0)$, the mixing terms between Ψ_j and Ψ_{j+4} $(j=1,\ldots,4)$ due to $\omega_\theta{}^{(\phi)(5)}$ in Eq. (12), $\omega_\phi{}^{(r)(5)}$ and $\omega_\phi{}^{(\theta)(5)}$ in Eq. (13), and $\omega_5^{(r)(5)}$ in Eq. (14) disappear. Then Ψ_{j+4} $(j=1,\ldots,4)$ become independent of Ψ_j, and we recover the well-known $N=8$ solutions of seven-torus compactifications.[8]

It may be worthwhile to comment upon the consistency condition of Eq. (4), which is

$$[D_M, D_N]\eta = -\tfrac{1}{4} R_{MN}{}^{AB}\Gamma_{AB}\eta \propto C_{MN}\eta = 0 . \tag{17}$$

Here Γ_{AB} are the 55 SO(1,10) generators. The nonvanishing C_{MN} for the positive monopole charge are

$$C_{12} = -C_{35} = -\Gamma_{12} + \Gamma_{35} = 4T_1 ,$$

$$C_{13} = C_{25} = -\Gamma_{13} - \Gamma_{25} = 4T_2 , \tag{18}$$

$$C_{23} = -C_{15} = \Gamma_{23} - \Gamma_{15} = 4T_3 ,$$

and form the SO(3) subalgebra $[T_i, T_j] = \epsilon_{ijk} T_k$. Similar results hold also for the negatively charged monopole.

Now let us briefly consider the problem of remaining supersymmetries in D-dimensional supergravity theories other than the 11-dimensional one. The solutions for equations of motion in the torus compactifications are still given by Eq. (3) with the monopole embedded into one U(1) group of the isometry group $U(1)^{D-4}$. Vacuum expectation values of all the bosonic fields vanish, except vielbeins which have the identical vacuum expectation value as in Eq. (3). But if theories contain several fermionic fields other than the gravitino field, the remaining supersymmetry can be drastically altered in this Kaluza-Klein monopole background. For example in $D=6$, $N=2$ supergravity,[15] the supersymmetry transformation law for the gaugino λ_j of Yang-Mills supermultiplets in the Sp(1) direction is [Eq. (22) of Ref. 15 or Eq. (21) of Ref. 16]

$$\delta\lambda^j = -\frac{1}{2\sqrt{2}} e^{\phi/\sqrt{2}} F_{MN}{}^j \Gamma^{MN}\epsilon - \tfrac{1}{2} e^{-\phi/\sqrt{2}} C^{ij} T^i \epsilon , \tag{19}$$

where $C^{ij} = g'[A^i_\alpha (T^j\Phi)^\alpha - \delta^{ij}]$. Due to δ^{ij} in the C^{ij} term, all the supersymmetries are completely broken,

$$\langle \delta\lambda^j \rangle \neq 0 , \tag{20}$$

in the Kaluza-Klein monopole backgrounds. Thus one cannot expect some remaining supersymmetry generally in the Kaluza-Klein monopole background.

It is remarkable that the Kaluza-Klein monopole solution in $D=11$ supergravity is unique in giving $N=4$

remaining supersymmetries with nonvanishing isometry group.

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