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Unitarity of conformal supergravity

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The possibility of unitarizing the conformal supergravity of Kaku, Townsend, and van Nieuwenhuizen is investigated by examining the representation content of its linearized equations. Conformal (Weyl) gravitons are seen to correspond to a pair of helicity-conjugate, irreducible, and nonunitary representations of so(4,2). The solutions of the linearized gravitino field equation are seen to carry both unitary and nonunitary representations of so(4,2). The linearized constraint equation of Kaku *et al.*, however, removes the unitary content from the gravitino field equation. The solution space of the linearized constraint and field equations of conformal supergravity is seen to carry a direct sum of two nonunitary and irreducible representations of the superalgebra su(2,2/1). These results imply that current models of conformal supergravity cannot be unitarized without breaking both su(2,2/1) and so(4,2) symmetries. A suggestion as to how one might construct a unitary model of conformal supergravity is made.

I. INTRODUCTION

Since their introduction to the literature a decade ago, models of supergravity have been studied with great intensity, and many wonderful results have followed. The self-consistent coupling of spin-$\frac{3}{2}$ and spin-2 fields, the tremendous improvement in ultraviolet behavior, and the possibility of truly unifying the other $2\frac{1}{2}$ fundamental forces with gravity each constitute a separate miracle. Yet while supergravities have surmounted several important problems in quantum field theory, they have also been seen to possess technical difficulties which block their implementation as complete physical theories.

The situation for conformal supergravities[1,2] is especially acute. Owing to their explicit scale invariance, these models are, from the start, at least power-counting renormalizable. Moreover, these models have local internal symmetries of the type $su(N)\otimes u(1)$ and so allow a particularly natural embedding of the local gauge groups of elementary particle physics. Yet these models are obstructed by a grave lack of unitarity.

It is commonly believed that this lack of unitarity is the inevitable consequence of the higher derivatives which appear in the Lagrange functions of these models. For when one examines the poles of the propagators, it is seen that they are not all positive, and so follows the possibility of negative probabilities in scattering calculations. But these models are not simply Lagrangian field theories; they also involve constraints which are independent of the equations of motion. Since it is known[3] that such constraints can sometimes prevent the negative poles from contributing to scattering calculations, the situation for conformal supergravities invites a closer inspection.

In Ref. 4 it was demonstrated how an appeal to conformal invariance could greatly simplify the analysis of a higher-derivative field theory. Therein it was shown that the solution space of the massless dipole equation carries an irreducible minimal-weight representation of so(4,2). Such representations are endowed with a unique (up to scaling) invariant inner product, the positivity or indefiniteness of which is fixed by the minimal weight of the representation. In turn, the positivity (indefiniteness) of this inner product directly implies both the possibility (impossibility) of constructing an invariant Hilbert space of states and the positivity (indefiniteness) of the poles of the propagators. Thus, by identifying the particular minimal-weight representation that is carried by the solutions of the field equation, the question of unitarity could be resolved in a succinct and canonical manner.

In this letter we present the results of a similar analysis applied to the linearized equations[5] of $N=1$ conformal supergravity. To determine the representations of so(4,2) which are carried by the physical (nongauge) solutions of these equations, we employed the manifestly covariant formalism developed in Refs. 6–8. Rather than repeat that development here, let us instead give a brief synopsis of this technique.

We denote by $\mathscr{C}$ the cone in R^6

$$0=y^2\equiv\delta^{ab}y_a y_b\equiv(y_0)^2-(y_1)^2-(y_2)^2-(y_3)^2-(y_4)^2+(y_5)^2$$

with λy_a identified with y_a for all $\lambda\in R-\{0\}$. The projective space thus defined is locally isomorphic to Minkowski space. The coordinate transformation which connects the two is given by

$$\begin{aligned}x_\mu&=(y_4+y_5)^{-1}y_\mu\\ x_+&=y_4+y_5\ ,\\ x_B&=(y_4+y_5)^{-2}y^2\ .\end{aligned}\tag{1}$$

Note that x_+ and x_B are superfluous parameters; x_B vanishes on the cone and x_+ is identified with λx_+ for all $\lambda\in R-\{0\}$. The singularity at $y_4+y_5=0$ corresponds to infinity in Minkowski space.

We identify fields over $\mathscr{C}$ with fields over R^6 which are homogeneous functions of the coordinates $\{y_a\}$, i.e.,

$$y_a\partial^a\phi(y)=N\phi(y),\ \ N\in Z\ .$$

To ensure that the action of so(4,2) on the fiber space is both linear and faithful, we adopt spinorial and tensorial indices which transform according to, respectively, the $4\oplus\bar{4}$- and 6-dimensional representations of so(4,2).

Now according to a theorem by Mack and Salam,[7] there corresponds to each conformally invariant field theory over Minkowski space an equivalent "conformal" field theory over the projective cone $\mathscr{C}$. To write down this equivalent field theory is relatively simple. One first chooses a conformal tensor field with the same type of indices (i.e., tensorial or spinorial or both) and the same symmetry among the indices as the original Minkowski field. One then imposes subsidiary conditions on this field to ensure that the second-order Casimir operator Q_2 of so(4,2) is simply a polynomial in N (the degree of homogeneity) times the identity matrix. Next, the degree N is chosen such that this eigenvalue coincides exactly with the eigenvalue of Q_2 for the original Minkowski field. One then adopts the conformal wave equation appropriate to that value of N. (See Ref. 8.) Finally, one confirms by a coordinate and field transformation that the wave equation and subsidiary conditions on the conformal field are physically equivalent to original equations in Minkowski space (that is, differing at most by a choice of gauge).

The manifest covariance of the conformal field theory thus obtained allows one to view the solution space of its wave equation as a tensor product of a finite-dimensional module and the infinite-dimensional module of a scalar field. This tensor product can be decomposed by identifying the (relative) ground states of the constituent submodules (see Refs. 3 and 8). If we impose the subsidiary conditions on the conformal field, we eliminate from this tensor product all but the subrepresentation carried by the physical and (special) gauge solutions of the wave equation. By identifying and quotienting away the subspace of gauge solutions, we then bare the physical content of the theory.

II. THE LINEARIZED FIELD EQUATIONS

The field equations of Weyl gravity and superconformal gravity can be linearized[5] by first expanding the gauge field $e^a{}_\mu$ associated with local translations about the Minkowski metric $\delta^\mu{}_\nu$,

$$e^a{}_\mu(x)\rightarrow\delta^a{}_\mu+h^a{}_\mu(x)\ ,$$

and then dropping all interaction terms from the field equations. In this manner one obtains

$$0=\Box^2h^{\mu\nu}-\Box\partial^\mu\partial_\lambda h^{\mu\lambda}-\Box\partial^\nu\partial_\lambda h^{\mu\lambda}-\tfrac{1}{3}(\delta^{\mu\nu}\Box-2\partial^\mu\partial^\nu)\partial_\lambda\partial_\rho h^{\lambda\rho} \tag{2}$$

for Weyl gravity and (2) plus

$$0=\Box(\partial\cdot\gamma)\psi^\mu-\tfrac{1}{3}\gamma^\mu(\partial\cdot\psi)-\tfrac{2}{3}\partial^\mu(\partial\cdot\gamma)(\partial\cdot\psi)\ , \tag{3}$$

$$0=\Box A^\mu-\partial^\mu\partial\cdot A \tag{4}$$

for superconformal gravity. These equations are obviously Poincaré invariant. One can verify that if h, ψ, and A have, respectively, conformal degrees 0, $-\frac{1}{2}$, and -1, then each of these equations is invariant under the action of the full conformal group. Moreover, if the field ψ^μ satisfies the additional constraint[1]

$$0=\partial^\mu\psi^\nu-\partial^\nu\psi^\mu-i\gamma_5\epsilon^{\mu\nu\lambda\rho}\partial_\lambda\psi_\rho\ , \tag{5}$$

then the system of Eqs. (2)–(4) is invariant under the action of the superconformal algebra su(2,2/1). Thus this linearization retains the global gauge invariance of the original gauge theory.

Now the solutions of Eqs. (2) and (3) are known to contain nonunitary ghosts.[5,9,10] However, it is not yet clear as to whether or not these ghosts can be removed by a suitable choice of constraint. Certainly, if their solution spaces carry representations of so(4,2) which have no unitary subrepresentations, then no invariant unitarizing constraint will be possible. On the other hand, if unitary subrepresentations do exist, one should be able to discover and impose invariant constraints which project away the remaining nonunitary subrepresentations and thereby "unitarize" the theory. With these two possibilities in mind we shall now determine explicitly the representation content of Eqs. (2) and (3).

III. THE CONTENT OF LINEARIZED WEYL GRAVITY

Let h^{ab} be a symmetric conformal tensor field of degree $N=0$, satisfying the wave equation

$$(\partial_c\partial^c)^2h^{ab}(y)=0\ , \tag{6}$$

and the subsidiary conditions

$$\delta_{ab}h^{ab}(y)=0\ , \tag{7a}$$

$$y_ah^{ab}(y)=0\ , \tag{7b}$$

$$\partial_ah^{ab}(y)=0\ . \tag{7c}$$

In order to demonstrate the physical equivalence of this set of equations with the linearized field equation of Weyl gravity, we simply apply the coordinate transformation (1). Thus setting

$$h^{\alpha\beta}(x)=\frac{\partial y_a}{\partial x_\alpha}\frac{\partial y_b}{\partial x_\beta}h^{ab}(y),\quad \alpha=0,1,2,3,+,B \tag{8}$$

we obtain from (6)

$$0=\Box^2h^{\mu\nu}+8\Box\partial^\mu h^B+8\Box\partial^\nu h^{\mu B}+16\Box\delta^{\mu\nu}h^{BB}+32\partial^\mu\partial^\nu h^{BB}\ , \tag{9a}$$

$$0=\Box^2h^{\mu B}+8\Box\partial^\mu h^{BB}\ , \tag{9b}$$

$$0=\Box^2h^{BB}\ , \tag{9c}$$

$$0=x_+{}^{-1}(\Box h^{\mu+}+8\Box\partial^\mu h^{B+})-4\Box(\partial_\lambda h^{\mu\lambda}+8h^{\mu B})-16\partial^\mu(\partial_\lambda h^{B\lambda}+6h^{BB})\ , \tag{9d}$$

$$0=x_+{}^{-1}\Box^2h^{B+}-4\Box(\partial_\lambda h^{B\lambda}+6h^{BB})\ , \tag{9e}$$

$$0=x_+{}^{-2}\Box^2h^{++}-8x_+{}^{-1}\Box(\partial_\lambda h^{\lambda+}+8h^{B+})+8\partial_\lambda(\partial_\rho h^{\lambda\rho}+8h^{\lambda B})+16(\partial_\lambda h^{\lambda B}+6h^{BB})\ , \tag{9f}$$

and from Eqs. (7)

$$0=h_\mu{}^\mu+4x_+{}^{-1}h^{+B}\ , \tag{10a}$$

$$0=h^{\mu+}=h^{B+}=h^{++}\ , \tag{10b}$$

$$0=\partial_\lambda h^{\lambda\nu}+8h^{B\nu}\,, \tag{10c}$$

$$0=\partial_\lambda h^{\lambda B}+6h^{BB}\,. \tag{10d}$$

Using Eqs. (10) we can eliminate the subsidiary fields $h^{B\nu}$, h^{BB}, h^{+B}, $h^{\mu+}$, $h^{\mu B}$, and h^{++} from the wave equations (9). We then obtain

$$0=\Box^2 h_{\mu\nu}-\Box\partial^\mu\partial_\lambda h^{\lambda\mu}-\Box\partial^\nu\partial_\lambda h^{\mu\lambda}-\tfrac{1}{3}(\Box\delta^{\mu\nu}-2\partial^\mu\partial^\nu)\partial_\rho\partial_\lambda h^{\lambda\rho}\,, \tag{11a}$$

$$0=\Box^2\partial_\lambda h^{\lambda\nu}-\tfrac{4}{3}\Box\partial^\nu\partial_\lambda\partial_\rho h^{\lambda\rho}\,, \tag{11b}$$

$$0=\Box^2\partial_\lambda\partial_\rho h^{\lambda\rho}\,, \tag{11c}$$

which is exactly the linearized equation of Weyl gravity (1) with a conformally invariant gauge fixing.

Now it is evident that the solution space of (6) is just the tensor product of the 20-dimensional representation $D(-2,0,0)$ [to specify the various representations of so(4,2) we have adopted the notation of Mack[11]] of so(4,2) carried by the indices of h and the infinite-dimensional representation carried by the solution space of the scalar dipole equation

$$(\partial_a\partial^a)^2\phi(y)=0\,. \tag{12}$$

It was shown in Ref. 4 that the solution space of (12) carries the indecomposable representation $D(1,\tfrac{1}{2},\tfrac{1}{2})\to D(0,0,0)$ of so(4,2). We are thus interested in the tensor product

$$[D(1,\tfrac{1}{2},\tfrac{1}{2})\to D(0,0,0)]\otimes D(-2,0,0)\,.$$

A calculation of the sort described in Refs. 3 and 8 reveals that this tensor product is equivalent to a direct sum of three indecomposable modules:

$$\left\{\begin{array}{ccccc} & \nearrow & D(2,2,0) & \searrow & \\ & & \oplus & & \\ D(0,1,1) & \to & D(2,0,2) & \to & D(0,1,1) \\ & & \oplus & & \\ & \searrow & D(-1,\tfrac{1}{2},\tfrac{1}{2}) & \nearrow & \end{array}\right\}$$

$$\oplus\left\{\begin{array}{ccccc} & \nearrow & D(2,1,0) & \searrow & \\ & & \oplus & & \\ D(1,\tfrac{1}{2},\tfrac{1}{2}) & \to & D(2,0,1) & \to & D(1,\tfrac{1}{2},\tfrac{1}{2}) \\ & & \oplus & & \\ & \searrow & D(0,0,0) & \nearrow & \end{array}\right\}$$

$$\oplus\{D(-1,\tfrac{3}{2},\tfrac{3}{2})\to D(-2,0,0)\}\,.$$

The subsidiary conditions (7), however, reduce this complicated module to

$$\{D(2,2,0)\oplus D(2,0,2)\}\to D(0,1,1)\,. \tag{13}$$

The invariant subspace which carries $D(0,1,1)$ consists of special gauge fields of the form

$$h_g^{ab}(y)=\partial^a\Lambda^b(y)+\partial^b\Lambda^a(y)$$

with Λ satisfying

$$(\partial_a\partial^a)^3\Lambda^b(y)=0\,,$$

$$y_a\Lambda^a(y)=0\,,$$

$$\partial_a\Lambda^a(y)=0\,.$$

By quotienting away this subspace of gauge solutions we obtain a space of "physical" solutions which carries the representation $D(2,2,0)\oplus D(2,0,2)$. Now in order for a representation $D(E_0,j_1,j_2)$ of so(4,2) to be unitary, we must have[11]

$$\begin{aligned} &E_0>j_1+j_2+2,\quad j_1j_2>0\\ &E_0\ge j_1+j_2+1,\quad j_1j_2=0\,. \end{aligned} \tag{14}$$

The physical solutions thus carry a representation of so(4,2) which has no unitary subrepresentation. We conclude that Weyl gravity cannot be unitarized without breaking conformal invariance. The nonunitary representation $D(2,2,0)$ and its helicity conjugate $D(2,0,2)$ no doubt correspond to the two helicity ± 2 ghosts found by Stelle.[9]

IV. THE CONTENT OF THE CONFORMAL GRAVITINO EQUATION

Let Ψ^a be a conformal Rarita-Schwinger field[12] of degree $N=-1$ satisfying the wave equation

$$(\partial_a\partial^a)\Psi^a(y)=0 \tag{15}$$

and the subsidiary conditions

$$(y\cdot\beta)(\partial\cdot\beta)\Psi^a(y)=0\,, \tag{16a}$$

$$\beta_a\Psi^a(y)=0\,, \tag{16b}$$

$$y_a\Psi^a(y)=0\,. \tag{16c}$$

We now rewrite these equations using the coordinate transformation (1) and the following field transformation:

$$\begin{aligned} \phi^\alpha(x)&\equiv\tfrac{1}{2}(1+\tau_3)\hat{P}\left[\frac{\partial y_a}{\partial x_\alpha}\right]\Psi^a(y(x))\,,\\ \psi^\alpha(x)&=\tfrac{1}{2}(1-\tau_3)\hat{P}\left[\frac{\partial y_a}{\partial x_\alpha}\right]\Psi^a(y(x))\,, \end{aligned} \tag{17}$$

where

$$\hat{P}\equiv 1+\frac{i}{2}x\cdot\gamma(\tau_1-i\tau_2)\,.$$

From (15) we obtain

$$0=\Box\phi^{\mu}+4\partial^{\mu}\phi^{B}\,, \tag{18a}$$

$$0=\Box\phi^{+}-2\partial_{\mu}\phi^{\mu}-8\phi^{B}\,, \tag{18b}$$

$$0=\Box\phi^{B}\,, \tag{18c}$$

$$0=\Box\psi^{\mu}+4\partial^{\mu}\psi^{B}-2(\partial\cdot\gamma)\psi^{\mu}-4\gamma^{\mu}\psi^{B}\,, \tag{18d}$$

$$0=\Box\psi^{+}-2\partial_{\mu}\psi^{\mu}-8\psi^{B}-2(\partial\cdot\gamma)\psi^{+}+2\gamma_{\mu}\psi^{\mu}\,, \tag{18e}$$

$$0=\Box\psi^{B}-2(\partial\cdot\gamma)\psi^{B}\,, \tag{18f}$$

and from (16a) we obtain

$$\phi^{\mu}=\tfrac{1}{2}[(\partial\cdot\gamma)\psi^{\mu}+2\gamma^{\mu}\psi^{B}]\,, \tag{19a}$$

$$\phi^{+}=\tfrac{1}{2}[(\partial\cdot\gamma)\psi^{+}-\gamma_{\mu}\psi^{\mu}]\,, \tag{19b}$$

$$\phi^{B}=\tfrac{1}{2}(\partial\cdot\gamma)\psi^{B}\,. \tag{19c}$$

Substituting Eqs. (19) back into the wave equations (18) we find Eqs. (18d)–(18f) vanish identically, while Eqs. (18a)–(18c) give third-order equations for ψ^{μ}, ψ^{+}, and ψ^{B}; viz.,

$$0=\Box(\partial\cdot\gamma)\psi^{\mu}+2\Box\gamma^{\mu}\psi^{B}+4\partial^{\mu}(\partial\cdot\gamma)\psi^{B}\,,$$

$$0=\Box(\partial\cdot\gamma)\psi^{+}-\Box_{\mu}\psi^{\mu}-2(\partial\cdot\gamma)\partial_{\mu}\psi^{\mu}-12(\partial\cdot\gamma)\psi^{B}\,,$$

$$0=\Box(\partial\cdot\gamma)\psi^{B}\,.$$

The subsidiary conditions (16b) and (16c) lead to

$$0=\psi^{+}\,,$$

$$0=\gamma_{\mu}\psi^{\mu}\,,$$

$$0=\partial_{\mu}\psi^{\mu}+6\psi^{B}\,.$$

Thus, we arrive at

$$0=\Box(\partial\cdot\gamma)\psi^{\mu}-\tfrac{1}{3}\Box\gamma^{\mu}\partial\cdot\psi-\tfrac{2}{3}\partial^{\mu}(\partial\cdot\gamma)\partial\cdot\psi\,, \tag{20a}$$

$$0=\Box(\partial\cdot\gamma)\partial_{\mu}\psi^{\mu}\,, \tag{20b}$$

which is the linearized field equation for the gravitino of conformal supergravity accompanied by a conformally invariant gauge fixing.

To determine the representation content of these equations we return to the conformal wave equation (15). Observe that the solution space of this equation is just the tensor product of the 8-dimensional spinor representation

$$D_{\rm sp}\equiv D(-\tfrac{1}{2},\tfrac{1}{2},0)\oplus D(-\tfrac{1}{2},0,\tfrac{1}{2})$$

with the solution space of the wave equation of conformal QED,

$$(\partial_a\partial^a)A^b(y)=0\,. \tag{21}$$

As was seen in Ref. 8, the solutions of (21) carry the indecomposable representation

$$D_{\rm QED}=\begin{array}{ccccc} & \nearrow & D(2,1,0) & \searrow & \\ & & \oplus & & \\ D(1,\tfrac{1}{2},\tfrac{1}{2}) & \rightarrow & D(2,0,1) & \rightarrow & D(1,\tfrac{1}{2},\tfrac{1}{2})\,. \\ & & \oplus & & \\ & \searrow & D(0,0,0) & \nearrow & \end{array}$$

The tensor product $D_{\rm QED}\otimes D_{\rm sp}$ is, of course, rather complicated. There is, however, a theorem by Zuckerman[14] which greatly simplifies the analysis of tensor products between finite- and infinite-dimensional representations of a semisimple Lie group. Applying Zuckerman's theorem, we were able to deduce directly the existence of the following subrepresentation:

$$\begin{array}{ccccc} & \nearrow & D(\tfrac{5}{2},\tfrac{3}{2},0) & \searrow & \\ & & \oplus & & \\ D(\tfrac{1}{2},\tfrac{1}{2},1) & \rightarrow & D(\tfrac{3}{2},0,\tfrac{3}{2}) & \rightarrow & D(\tfrac{1}{2},\tfrac{1}{2},1)\,, \\ & & \oplus & & \\ & \searrow & D(-\tfrac{1}{2},0,\tfrac{1}{2}) & \nearrow & \end{array} \tag{22}$$

as well as its helicity conjugate

$$\begin{array}{ccccc} & \nearrow & D(\tfrac{5}{2},0,\tfrac{3}{2}) & \searrow & \\ & & \oplus & & \\ D(\tfrac{1}{2},1,\tfrac{1}{2}) & \rightarrow & D(\tfrac{3}{2},\tfrac{3}{2},0) & \rightarrow & D(\tfrac{1}{2},1,\tfrac{1}{2})\,, \\ & & \oplus & & \\ & \searrow & D(-\tfrac{1}{2},\tfrac{1}{2},0) & \nearrow & \end{array} \tag{23}$$

within the tensor product $D_{\rm QED}\otimes D_{\rm sp}$. The existence of both these subrepresentations was also confirmed by direct calculation.

By imposing the subsidiary conditions (16), one reduces the entire tensor product to a subrepresentation of (22)⊕(23), viz.,

$$\left\{\begin{array}{ccc} D(\tfrac{5}{2},\tfrac{3}{2},0) & \searrow & \\ \oplus & & D(\tfrac{1}{2},\tfrac{1}{2},1) \\ D(\tfrac{3}{2},0,\tfrac{3}{2}) & \nearrow & \end{array}\right\}\oplus\left\{\begin{array}{ccc} D(\tfrac{5}{2},0,\tfrac{3}{2}) & \searrow & \\ \oplus & & D(\tfrac{1}{2},1,\tfrac{1}{2}) \\ D(\tfrac{3}{2},\tfrac{3}{2},0) & \nearrow & \end{array}\right\}. \tag{24}$$

The invariant subspace $D(\tfrac{1}{2},\tfrac{1}{2},1)\oplus D(\tfrac{1}{2},1,\tfrac{1}{2})$ is spanned by gauge solutions of the form

$$\Psi^a(y)=\partial^a\zeta(y)$$

with $\zeta(y)$ satisfying

$$(\partial_a\partial^a)^2\zeta(y)=(y\cdot\beta)(\partial\cdot\beta)\zeta(y)=0\ .$$

Quotienting away this gauge subspace, we are left with a physical subspace which carries

$$\{D(\tfrac{5}{2},\tfrac{3}{2},0)\oplus D(\tfrac{5}{2},0,\tfrac{3}{2})\}\oplus\{D(\tfrac{3}{2},\tfrac{3}{2},0)\oplus D(\tfrac{3}{2},0,\tfrac{3}{2})\}\ . \qquad (24')$$

The first two representations are perfectly unitary. In fact, upon restriction to the Poincaré subgroup, these representations become exactly the irreducible unitary representations $D(m=0,s=\pm\frac{3}{2})$ of $\mathscr{P}$. The representations $D(\frac{3}{3},\frac{3}{2},0)$ and $D(\frac{3}{2},0,\frac{3}{2})$, however, do not fulfill the criteria (14). The conformal gravitino field thus contains a pair of unitary, helicity $\pm\frac{3}{2}$ particles along with a pair of nonunitary, helicity $\pm\frac{3}{2}$ ghosts in its physical subspace.

However, just as in the case of linear conformal quantum gravity (Ref. 3), there exists a constraint which removes the nonunitary ghosts from the theory. We found this constraint by noting that there are two types of duality which one can impose on the field Ψ^a:

$$\beta_7\Psi^a(y)=\pm\Psi^a(y)\ , \qquad (25)$$

$$\sum\sigma(a,b,c)y^a\partial^b\Psi^c(y)=\pm i\epsilon^{abcdef}y_d\partial_e\Psi_f(y)\ . \qquad (26)$$

Here $\sum$ denotes a sum over the permutations of the indices a, b, and c and $\sigma(a,b,c)=+1,-1$, respectively, for even and odd permutations. Combining (25) and (26) we obtained a condition of double duality:

$$\sum\sigma(a,b,c)y^a\partial^b\Psi^c(y)=\pm i\beta_7\epsilon^{abcdef}y_d\partial_e\Psi_f(y)\ , \qquad (27)$$

which allows one to split (24) into unitary and nonunitary sectors. If one imposes (27) with the upper sign, then the ghost representations $D(\frac{3}{2},\frac{3}{2},0)$ and $D(\frac{3}{2},0,\frac{3}{2})$ are eliminated from (24); choosing the opposite sign removes instead the unitary massless representations $D(\frac{5}{2},\frac{3}{2},0)$ and $D(\frac{5}{2},0,\frac{3}{2})$.

In terms of the Minkowski field $\psi^\mu(x)$, this double-duality condition is

$$\partial^\mu\psi^\nu(x)-\partial^\nu\psi^\mu(x)=\mp\frac{i}{2}\gamma_5\epsilon^{\mu\nu\lambda\rho}\partial_\lambda\psi_\rho(x)\ .$$

Observe that the lower sign coincides with the constraint (4) of linearized conformal supergravity. This means that the physical gravitinos are absent from conformal supergravity and so one is left with only nonunitary spin-$\frac{3}{2}$ ghosts in the gravitino sector.

V. THE su(2,2/1) CONTENT OF CONFORMAL SUPERGRAVITY

To uncover the representation of su(2,2/1) that is carried by the physical solutions of Eqs. (2)–(5), we simply adopted the analysis that Flato and Fronsdal[15] used to discover the irreducible "massless" representations of su(2,2/1). In their notation this representation is

$$\tilde{D}=D_s(\tfrac{3}{2},\tfrac{3}{2},0;\tfrac{3}{2})\oplus D_s(\tfrac{3}{2},0,\tfrac{3}{2};-\tfrac{3}{2})\ .$$

Upon restriction to the even part of the superalgebra, su(2,2)⊗u(1)∼so(4,2)⊗u(1) the representation D reduces to

$$\tilde{D}\,|_{\mathrm{so}(4,2)\otimes\mathrm{u}(1)}\simeq\{D(\tfrac{3}{2},\tfrac{3}{2},0)\otimes D(\tfrac{3}{2})\}\oplus\{D(\tfrac{3}{2},0,\tfrac{3}{2})\otimes D(-\tfrac{3}{2})\}$$
$$\oplus\{D(2,1,0)\otimes D(0)\}\oplus\{D(2,0,1)\otimes D(0)\}$$
$$\oplus\{D(2,2,0)\otimes D(0)\}\oplus\{D(2,0,2)\otimes D(0)\}\ .$$

It is evident that this reduction corresponds precisely to the two axially charged conformal gravitinos, the two neutral axial photons, and the two neutral conformal gravitons of conformal supergravity. Since both $D_s(\frac{3}{2},\frac{3}{2},0;\frac{3}{2})$ and $D_s(\frac{3}{2},0,\frac{3}{2};-\frac{3}{2})$ are nonunitary and irreducible,[15] we conclude that the conformal supergravity of Kaku *et al.* is inherently nonunitarizable.

VI. DISCUSSION

It may seem a surprise that the constraint that permits the superalgebra to close is identical to the constraint that removes the physical particles from the gravitino field equation. We stress that this should not be interpreted as supersymmetry actually worsening the situation with respect to unitarity; for the existence of a symmetry group should not be confused with the unitarity or nonunitarity of its representations. The real impact of the gravitino constraint is that it projects onto a particular, in fact the maximal, su(2,2/1) module lying within the solution space of the field equations (2)–(4). From a purely mathematical point of view it is rather coincidental that this module is nonunitary.

Yet from another perspective this lack of unitarity is not so accidental. The $N=1$ conformal supergravity of Kaku *et al.*,[1] and the models of extended conformal supergravity[2] as well, are supersymmetric extensions of Weyl gravity. We saw in Sec. III that in terms of representations of so(4,2), the physical content of Weyl gravity is a direct sum of two nonunitary irreducible representations. Since a unitary representation of a supergroup must also be unitary with respect to each subgroup of the supergroup, there was never any real chance of obtaining a unitary model of conformal supergravity by extending Weyl gravity.

In a recent paper Fronsdal[16] has proposed a unitary alternative to Weyl gravity. Like Kaku *et al.*[1] Fronsdal relies on so(4,2) gauge invariance as a constructive principle. However, he imposes a different constraint on the field strengths; instead of setting the curvature associated with local translations equal to zero, Fronsdal generalized the unitarizing constraint of Ref. 3. By this choice of constraint, he avoided the nonunitary theory of Weyl and obtained instead a unitary model of conformal gravity.

We saw in Sec. IV that the conformal gravitino can be unitarized by imposing the double-duality condition

$$0=\partial^{\mu}\psi^{\nu}-\partial^{\nu}\psi^{\mu}+i\gamma_5\epsilon^{\mu\nu\lambda\rho}\partial_{\lambda}\psi_{\rho}\ .$$

It is remarkable that the linearized constraint of Fronsdal's model can also be formulated as a condition of double duality. We strongly suspect that these two unitarizing constraints are, in fact, supersymmetric partners in a linear su(2,2/1)-invariant theory of massless spin-$\frac{3}{2}$ and spin-2 particles. Indeed, by generalizing these two constraints to a form compatible with local su(2,2/1) gauge invariance, we expect to find a unitary version of conformal supergravity.

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$$\{\beta_a,\beta_b\}=\delta_{ab},\quad a,b=0,\ldots,5\ .$$

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