Proof of summed form of proper-time expansion for propagator in curved space-time

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We consider the Schwinger-DeWitt proper-time expansion of the kernel of the Feynman propagator in curved space-time. We prove that the proper-time expansion can be written in a new form, conjectured by Parker and Toms, in which all the terms containing the scalar curvature R are generated by a simple overall exponential factor. This sums all terms containing R, including those with nonconstant coefficients, in the proper-time series. This result is valid for an arbitrary space-time and for any spin. It also applies to the heat kernel. This form of the expansion is of importance in connection with nonperturbative effects in quantum field theory.

I. INTRODUCTION

Of crucial importance for the investigation of quantum field theories in curved space-time is the Feynman propagator or Green's function. Exact results are known only in certain special cases, and in general one must have recourse to some approximation scheme. A very powerful and general technique is to make an asymptotic expansion of the kernel of the Green's function in powers of the Riemann curvature tensor. The expansion parameter is related to a fictitious proper time. The terms in this proper-time Schwinger-DeWitt[1] or heat-kernel[2] expansion have a universal form in a general curved space-time. However, so far the terms in the expansion are known explicitly only up to third order in the curvature.

It is possible that there may be infinite subsets of terms in the series which converge and can be summed in closed form. These nonlocal[3] expressions would give information about effects of physical significance which cannot be obtained from the known terms of the series. DeWitt[4] effectively performed such a partial summation in finding an approximation from which he obtained certain nonlocal terms in the effective action. Bekenstein and Parker[5] derived a Gaussian path-integral approximation for the propagator which includes certain nonlocal effects. Motivated by the form of this Gaussian approximation for particular cases, Parker and Toms[6] have recently suggested an exact form for the coincidence limit of the Feynman propagator for a scalar field which, they conjecture, isolates and sums all terms containing the scalar curvature R in the proper-time expansion. This includes terms containing R with nonconstant coefficients. They proceeded to prove this hypothesis to third order in the proper time for general spacetimes. They also have shown that the expression gives significant nonlocal terms in the effective action for quantum fields in curved space-time, and have extended their conjecture to higher spin.[7] In addition to its physical significance, the new form of the proper-time series should make it technically simpler to obtain the terms of the expansion beyond third order in the proper time. A significant fraction of the terms (those containing R) in the original expansion will not be present in the coefficients of the new form of the expansion.

In the present paper, we shall prove their conjecture by induction to all orders for a general space-time, and show that the natural generalization of their result to fields of higher spin also sums all terms involving R, and partially sums extra terms produced by the spin connection and Yang-Mills field as well. In fact, our method of proof demonstrates that the proper-time expansion may be written in a form in which the R dependence is summed before taking the coincidence limit. Exactly analogous results will be true for the heat kernel on a Riemannian manifold.

The organization of the paper is as follows. In Sec. II we introduce the Schwinger-DeWitt proper-time expansion for a scalar field propagator and explain precisely the conjecture of Parker and Toms. In Sec. III we prove some important lemmas concerning the form of the metric in Riemann normal coordinates and the structure of other quantities appearing in the expansion as functions of the curvature. Armed with these preliminaries, in Sec. IV we are then able to prove the conjecture for scalar fields. In Sec. V we present and prove the generalization of the conjecture to fields of higher spin. We offer some remarks on the significance of our results in Sec. VI, and finally in an appendix we display explicitly the coincidence limits of the terms in the proper-time expansion as far as they are known, both for the original version and the new simplified version, and also discuss the inclusion of background gauge-field couplings.

II. THE PROPER-TIME EXPANSION AND THE CONJECTURE OF PARKER AND TOMS

The Feynman propagator $\Delta(x,x')$ of a scalar field coupled to the curvature satisfies

$$[\Box_x+m^2+\xi R(x)]\Delta(x,x')=-\delta(x,x')\ , \qquad (2.1)$$

where $\Box$ is the covariant Laplacian, ξ is an arbitrary dimensionless constant, and $\delta(x,x')$ is defined by $\int dv_x \delta(x,x')f(x)=f(x')$ with dv_x the invariant volume element.[8] If we write

$$\Delta(x,x')=-i\int_0^\infty \langle x,s\,|\,x',0\rangle e^{-im^2s}ds \qquad (2.2)$$

[with $\mathrm{Im}(m^2)<0$ understood] then by virtue of (2.1) the kernel $\langle x,s\,|\,x',0\rangle$ must obey

$$i\frac{\partial}{\partial s}\langle x,s\,|\,x',0\rangle=(\Box+\xi R)\langle x,s\,|\,x',0\rangle \tag{2.3}$$

with the boundary condition $\lim_{s\to 0}\langle x,s\,|\,x',0\rangle=\delta(x,x,')$. Equation (2.3) is the Schrödinger equation for a particle coupled to the curvature, with a fictitious "proper time" s.

The Schwinger-DeWitt proper-time expansion for the kernel is then[1,9]

$$\langle x,s\,|\,x',0\rangle=i(4\pi is)^{-d/2}\exp\left[\frac{i\sigma(x,x')}{2s}\right]\times\Delta_{\rm VM}^{1/2}(x,x')F(x,x';is)\ , \tag{2.4}$$

where $2\sigma(x,x')$ is the square of the proper arc length along the geodesic from x' to x, and $\Delta_{\rm VM}(x,x')$ is the Van Vleck-Morette[10] determinant defined by

$$\Delta_{\rm VM}(x,x')=-\,|\,g(x)\,|^{-1/2}\,|\,g(x')\,|^{-1/2}\det\left[\frac{-\partial^2\sigma(x,x')}{\partial x^{\mu}\partial x^{\nu'}}\right], \tag{2.5}$$

σ and $\Delta_{\rm VM}$ satisfy the important relations[1]

$$\begin{aligned}&\nabla_\mu\sigma\nabla^\mu\sigma=-2\sigma\ ,\\&\Delta_{\rm VM}{}^{1/2}\Box\sigma+2\sigma^{;\mu}\nabla_\mu\Delta_{\rm VM}{}^{1/2}=-d\,\Delta_{\rm VM}{}^{1/2}\ ,\end{aligned} \tag{2.6}$$

where ∇_μ is the covariant derivative. $F(x,x';is)$ is written as a series

$$F(x,x';is)=\sum_{j=0}(is)^jf_j(x,x')\ , \tag{2.7}$$

and then as a consequence of (2.3), the $f_j(x,x')$ satisfy the recurrence relations[1]

$$\begin{aligned}&\sigma^{;\mu}\nabla_\mu f_0=0\ ,\\&jf_j-\sigma^{;\mu}\nabla_\mu f_j=-\Delta_{\rm VM}{}^{-1/2}(\Box+\xi R)(\Delta_{\rm VM}{}^{1/2}f_{j-1})\end{aligned} \tag{2.8}$$

which permit the iterative computation of the coincidence limits $f_j(x,x)$.

Parker and Toms[6] postulated that the coincidence limit of the kernel $\langle x,s\,|\,x',0\rangle$ could be written in the form

$$\langle x,s\,|\,x,0\rangle=i(4\pi is)^{-d/2}\bar F(x,x;is)\times\exp[-is(\xi-\tfrac{1}{6})R(x)]\ , \tag{2.9}$$

where $\bar F(x,x;is)$ has the proper-time series

$$\bar F(x,x;is)=1+\sum_{j=1}^{\infty}(is)^j\bar f_j(x,x) \tag{2.10}$$

such that the $\bar f_j(x,x)$ contain no terms which vanish when R (but not its covariant derivatives) is replaced by zero. Henceforth we shall describe this property by the term "R independent." In other words, all the dependence on R in (2.4) and (2.7) is now comprised in the exponential in (2.9). Our aim in the following two sections is to prove this contention. For our method of proof it is convenient to consider the kernel without taking the coincidence limit, writing it in the form

$$\langle x,s\,|\,x',0\rangle=i(4\pi is)^{-d/2}e^{i\sigma(x,x')/2s}\Delta_{\rm VM}{}^{1/2}(x,x')\bar F(x,x';is)\times\exp[-is(\xi-\tfrac{1}{6})R(x')]\ . \tag{2.11}$$

In the course of proving the original conjecture it will emerge that we can also prove the stronger assertion that $\bar F(x,x';is)$ is R independent before we take the coincidence limit. We should remark here that there are various ways of writing $\langle x,s\,|\,x',0\rangle$ which would reduce to (2.9) in the coincidence limit, e.g.,

$$\langle x,s\,|\,x',0\rangle=i(4\pi is)^{-d/2}e^{i\sigma/2s}\Delta_{\rm VM}{}^{1/2}\exp[-is(\xi-\tfrac{1}{6})R(x)]\bar F'(x,x';is)\ , \tag{2.12a}$$

$$\langle x,s\,|\,x',0\rangle=i(4\pi is)^{-d/2}e^{i\sigma/2s}\Delta_{\rm VM}{}^{1/2}\exp[-\tfrac{1}{2}is(\xi-\tfrac{1}{6})R(x)]\bar F''(x,x';is)\exp[-\tfrac{1}{2}is(\xi-\tfrac{1}{6})R(x')]\ . \tag{2.12b}$$

Although, clearly

$$\bar F(x,x;is)=\bar F'(x,x;is)=\bar F''(x,x;is)\ , \tag{2.13}$$

$\bar F$, $\bar F'$, and $\bar F''$ will differ for noncoincident arguments, while all being R independent, as we shall see in Sec. IV. While (2.12b) is preferable on account of its symmetry, (2.11) is more amenable to computation.

We find, on substituting (2.11) into (2.3), the following recurrence relations for $\bar f_j(x,x')$:

$$\begin{aligned}&\sigma^{;\mu}\nabla_\mu\bar f_0(x,x')=0\ ,\\&j\bar f_j(x,x')-\sigma^{;\mu}\nabla_{\mu x}\bar f_j(x,x')=-\Delta_{\rm VM}{}^{-1/2}\Box_x[\Delta_{\rm VM}{}^{1/2}\bar f_{j-1}(x,x')]+\xi[R(x')-R(x)]\bar f_{j-1}(x,x')-\tfrac{1}{6}R(x')\bar f_{j-1}(x,x')\\&\qquad=\{-M+(\xi-\tfrac{1}{6})[R(x')-R(x)]\}\bar f_{j-1}(x,x')-\Box_x\bar f_{j-1}(x,x')-N_\mu\nabla^\mu\bar f_{j-1}(x,x')\ ,\end{aligned} \tag{2.14}$$

where

$$\begin{aligned}&M=\Delta_{\rm VM}{}^{-1/2}(\Box\Delta_{\rm VM}{}^{1/2})+\tfrac{1}{6}R(x)\ ,\\&N_\mu=\Delta_{\rm VM}{}^{-1/2}\nabla_\mu\Delta_{\rm VM}{}^{1/2}\ .\end{aligned} \tag{2.15}$$

In Sec. IV we shall use these recurrence relations to prove the conjecture of Parker and Toms by induction, but first we need to demonstrate that the coincidence limits of covariant derivatives of $\sigma(x,x')$, $f_0(x,x')$, and M,N_μ defined in (2.15), are R independent. The following section will be devoted to the proof of these subtheorems.

III. R INDEPENDENCE OF DERIVATIVES OF σ, f_0, M, N_μ

All the results in this section can be considered as consequences of the structure of the metric $g_{\alpha\beta}(x)$ when expanded as a Taylor series about x' in Riemann normal coordinates y^μ with origin at x'. In this expansion the coefficient of $y^{\mu_1}\ldots y^{\mu_n}$ is a sum of terms each of which is a product of curvature tensors at x', of which some may be covariantly differentiated. The essential point is that, for $n>2$, each term has the following two properties.

(a) There are no "internal" contractions of indices within any individual curvature tensor.

(b) All the undifferentiated curvature tensors have at least one index which is not one of $\{\alpha,\beta,\mu_1,\ldots,\mu_n\}$ and hence is contracted with an index elsewhere in the product of tensors.

We shall describe an object with these two properties as "curvature connected." Riemann normal coordinates may be set up in the following way:[11] Given a point x, we define the transformation to normal coordinates y with origin x' by

$$x^\nu = x'^\nu + \sum \frac{1}{k!}\left[\left(\frac{\partial^k x^\nu}{\partial y^{\lambda_1}\cdots\partial y^{\lambda_k}}\right)\right]_{y=0} y^{\lambda_1}\cdots y^{\lambda_k}\,, \tag{3.1}$$

where the coefficients

$$\left[\left(\frac{\partial^k x^\nu}{\partial y^{\lambda_1}\cdots\partial y^{\lambda_k}}\right)\right]_{y=0}$$

are chosen so as to ensure that in the y coordinates

$$\Gamma^\nu_{\lambda_1\lambda_2}\Big|_{y=0}=0,\quad \partial_{(\lambda_3}\cdots{}_{\lambda_r}\Gamma^\nu_{\lambda_1\lambda_2)}\Big|_{y=0}=0\,. \tag{3.2}$$

Explicitly,

$$\left[\frac{\partial x^\nu}{\partial y^{\lambda_1}}\right]_{y=0}=\delta^\nu_{\lambda_1}\,,$$

$$\left[\frac{\partial^2 x^\nu}{\partial y^{\lambda_1}\partial y^{\lambda_2}}\right]_{y=0}=-\Gamma^\nu_{\lambda_1\lambda_2}(x')\,, \tag{3.3}$$

$$\left[\frac{\partial^3 x^\nu}{\partial y^{\lambda_1}\partial y^{\lambda_2}\partial y^{\lambda_3}}\right]_{y=0}=[2\Gamma^\nu_{\alpha(\lambda_3}\Gamma^\alpha_{\lambda_1\lambda_2)}-\partial_{(\lambda_3}\Gamma^\nu_{\lambda_1\lambda_2)}]_{y=0}\,,$$

and so on.

When the metric is Taylor expanded in these normal coordinates about x',

$$g_{\alpha\beta}(x)=g_{\alpha\beta}\,|_{y=0}+g_{\alpha\beta,\mu_1}\,|_{y=0}y^{\mu_1}+\cdots+\frac{1}{n!}g_{\alpha\beta,\mu_1\cdots\mu_n}\,|_{y=0}y^{\mu_1}\cdots y^{\mu_n}+\cdots\,, \tag{3.4}$$

then the coefficients of $y^{\mu_1}\ldots y^{\mu_n}$ may be written in terms of Christoffel symbols and their derivatives. For instance,

$$\begin{aligned} g_{\alpha\beta,\mu_1}&=g_{\alpha\beta;\mu_1}+\Gamma^{\lambda_1}_{\mu_1\alpha}g_{\lambda_1\beta}+\Gamma^{\lambda_1}_{\mu_1\beta}g_{\lambda_1\alpha}\\ &=\Gamma^{\lambda_1}_{\mu_1\alpha}g_{\lambda_1\beta}+\Gamma^{\lambda_1}_{\mu_1\beta}g_{\lambda_1\alpha}\,,\end{aligned} \tag{3.5}$$

and furthermore,

$$\begin{aligned} g_{\alpha\beta,\mu_1\mu_2}&=\Gamma^{\lambda_1}_{\mu_1\alpha,\mu_2}g_{\lambda_1\beta}+\Gamma^{\lambda_1}_{\mu_1\alpha}(\Gamma^{\lambda_2}_{\lambda_1\mu_2}g_{\lambda_2\beta}+\Gamma^{\lambda_2}_{\beta\mu_2}g_{\lambda_1\lambda_2})+\alpha\leftrightarrow\beta\\ &=\Gamma^{\lambda_1}_{\alpha(\mu_1,\mu_2)}g_{\lambda_1\beta}+\Gamma^{\lambda_1}_{\alpha(\mu_1}\Gamma^{\lambda_2}_{\mu_2)\lambda_1}g_{\lambda_2\beta}+\Gamma^{\lambda_1}_{\alpha\mu_1}\Gamma^{\lambda_2}_{\mu_2\beta}g_{\lambda_1\lambda_2}+\tfrac{1}{2}R^{\lambda_1}{}_{\alpha\mu_1\mu_2}g_{\lambda_1\beta}+\alpha\leftrightarrow\beta\\ &=(\Gamma^{\lambda_1}_{\alpha(\mu_1,\mu_2)}g_{\lambda_1\beta}+\Gamma^{\lambda_1}_{\alpha(\mu_1}\Gamma^{\lambda_2}_{\mu_2)\lambda_1}g_{\lambda_2\beta}+\Gamma^{\lambda_1}_{\alpha(\mu_1}\Gamma^{\lambda_2}_{\mu_2)\beta}g_{\lambda_1\lambda_2})+\alpha\leftrightarrow\beta\,.\end{aligned} \tag{3.6}$$

By iterating this procedure, $g_{\alpha\beta,\mu_1}\cdots{}_{\mu_n}$ may be expressed as a sum of terms whose general form is

$$\Gamma^{\lambda_1}_{\alpha(\mu_1,\mu_2\cdots\mu_{r_1})}\Gamma^{\lambda_2}_{\lambda_2'(\mu_{r_1+1},\mu_{r_1+2}\cdots\mu_{r_2})}\cdots\Gamma^{\lambda_i}_{\lambda_i'(\mu_{r_{i-1}+1},\mu_{r_{i-1}+2}\cdots\mu_{r_i})}\cdots\Gamma^{\lambda_j}_{\lambda_j'(\mu_{r_{j-1}+1}\cdots\mu_n)}g_{\lambda_{j+1}'\lambda_{j+2}'}\,,$$

where each $\lambda_i'\in\{\lambda_1,\ldots,\lambda_{i-1},\beta\}$ and $r_1<r_2<\ldots<r_{j-1}<n$, together with similar terms with α and β interchanged.

Using the definition of the Riemann tensor

$$\tfrac{1}{2}R^\lambda{}_{\alpha\beta\gamma}=\Gamma^\lambda_{\alpha[\beta,\gamma]}+\Gamma^\mu_{\alpha[\beta}\Gamma^\lambda_{\gamma]\mu} \tag{3.7}$$

in conjunction with (3.2), we may solve for the symmetrized derivatives $\Gamma^\lambda_{\alpha(\mu_1,\mu_2\cdots\mu_r)}\,|_{y=0}$ from which the expansion coefficients in (3.4) are constructed. We obtain immediately from (3.7) and (3.2)[11]

$$\Gamma^\nu_{\mu(\alpha,\beta)}\,|_{y=0}=-\tfrac{1}{3}R^\nu{}_{(\alpha\beta)\mu}\,, \tag{3.8}$$

and after differentiating (3.7),

$$\Gamma^\nu_{\mu(\alpha,\beta\gamma)}\,|_{y=0}=+\tfrac{1}{2}R_{\mu(\gamma}{}^\nu{}_{\beta,\alpha)}\,. \tag{3.9}$$

Differentiating (3.7) again and using (3.8),

$$\Gamma^\nu_{\mu(\alpha,\beta\gamma\delta)}\,|_{y=0}=-\tfrac{3}{5}(\tfrac{2}{9}R^\rho{}_{(\alpha}{}^\nu{}_\beta R_\delta{}^\omega{}_{\gamma)\mu}g_{\rho\omega}-R_{\mu(\delta}{}^\nu{}_{\gamma,\alpha\beta)})\,, \tag{3.10}$$

and so on iteratively. In the process, by differentiating

(3.7) $(n-2)$ times and using (3.2) we can express $\Gamma^{\lambda}_{\alpha(\mu_1,\mu_2\cdots\mu_n)}|_{y=0}$ in terms of $R_{\alpha(\mu_1}{}^{\lambda}{}_{\mu_2,\mu_3\cdots\mu_n)}$ and contractions of pairs of Christoffel symbols with less than $(n-1)$ derivatives, which have already been calculated. The partial derivatives in $R_{\alpha(\mu_1}{}^{\lambda}{}_{\mu_2,\mu_3\cdots\mu_n)}$ may be rewritten in terms of covariant derivatives as was done for the metric tensor following (3.4). It is then clear from this method of construction that each $\Gamma^{\lambda}_{\alpha(\mu_1,\mu_2\cdots\mu_n)}|_{y=0}$ for $n>2$ is curvature connected, and hence, because of the form of $g_{\alpha\beta,\mu_1\cdots\mu_n}$ stated earlier, $g_{\alpha\beta,\mu_1\cdots\mu_n}|_{y=0}$ is curvature connected for $n>2$. In the case of constant curvature where all derivatives of the curvature are zero, it is possible to calculate $g_{\alpha\beta,\mu_1\cdots\mu_n}$ explicitly,[11]

$$g_{\alpha\beta,\mu_1\cdots\mu_{2k}}|_{y=0} = \frac{2^{2k+1}}{(2k+2)(2k+1)} R_{\alpha\mu_1\mu_2}{}^{\sigma_1} R_{\sigma_1\mu_3\mu_4}{}^{\sigma_2}\cdots \times R_{\sigma_{k-1}\mu_{2k-1}\beta\mu_{2k}} \quad (3.11)$$

which is manifestly curvature connected, and $g_{\alpha\beta,\mu_1\cdots\mu_{2k-1}}|_{y=0}=0$.

The curvature connectedness of the $g_{\alpha\beta,\mu_1\cdots\mu_n}|_{y=0}$ in normal coordinates is the key to the proof of the R independence of $\bar{F}$ in (2.9), since it implies that any R dependence may be traced to the quadratic term in (3.4). We have, from (3.2), (3.5), (3.6), and (3.8),

$$g_{\alpha\beta}|_{y=0}=+\eta_{\alpha\beta},\quad g_{\alpha\beta,\mu_1}|_{y=0}=0\ , \qquad g_{\alpha\beta,\mu_1\mu_2}|_{y=0}=-\tfrac{2}{3}R_{\alpha(\mu_1\mu_2)\beta}\ , \quad (3.12)$$

where $\eta_{\alpha\beta}=\mathrm{diag}(+1,-1,-1,-1)$.

We will now show that the coincidence limit of more than four covariant derivatives of $\sigma(x,x')$ is R independent. In common with the other results which we will prove later in this section, this can be demonstrated in normal coordinates using the curvature connectedness of the coefficients in (3.4), but in this case it is possible to present a more elegant generally covariant proof founded on the basic relationship for σ in (2.6). We may calculate the coincidence limit of $\sigma_{;\alpha_1\cdots\alpha_n}$ by successive differentiation of (2.6). We immediately have

$$\sigma_{;\alpha_1\mathrm{diag}}=0\ , \quad (3.13)$$

the suffix "diag" indicating the coincidence limit. Differentiating twice,

$$\sigma^{;\mu}{}_{\alpha\beta}\sigma_{;\mu}+\sigma^{;\mu}{}_{\alpha}\sigma_{;\mu\beta}=-\sigma_{;\alpha\beta}$$

which implies, with (3.13)

$$\sigma_{;\alpha\beta\mathrm{diag}}=-g_{\alpha\beta},\quad \sigma^{;\alpha}{}_{\beta\mathrm{diag}}=-\delta^{\alpha}{}_{\beta}\ . \quad (3.14)$$

A further differentiation gives, with use of (3.13) and (3.14),

$$\sigma_{;\alpha\beta\gamma\mathrm{diag}}=0 \quad (3.15)$$

and, differentiating a fourth time,

$$\sigma^{;\mu}{}_{\alpha\beta\gamma\delta}\sigma_{;\mu}+\sigma^{;\mu}{}_{\alpha\beta\gamma}\sigma_{;\mu\delta}+\sigma^{;\mu}{}_{\alpha\beta\delta}\sigma_{;\mu\gamma}+\sigma^{;\mu}{}_{\alpha\gamma\delta}\sigma_{;\mu\beta}+\sigma^{;\mu}{}_{\alpha\beta}\sigma_{;\mu\gamma\delta}+\sigma^{;\mu}{}_{\alpha\gamma}\sigma_{;\mu\beta\delta}+\sigma^{;\mu}{}_{\alpha\delta}\sigma_{;\mu\beta\gamma}+\sigma^{;\mu}{}_{\alpha}\sigma_{;\mu\beta\gamma\delta}=\sigma_{;\alpha\beta\gamma\delta}\ .$$

Taking the coincidence limit and using (3.13), (3.14), and (3.15) yields

$$\sigma_{;\delta\alpha\beta\gamma\mathrm{diag}}+\sigma_{;\gamma\alpha\beta\delta\,\mathrm{diag}}+\sigma_{;\beta\alpha\gamma\delta\,\mathrm{diag}}=0\ , \quad (3.16)$$

which after rearranging indices and using (3.14) again implies

$$\sigma_{;\alpha\beta\gamma\delta\,\mathrm{diag}}=-\tfrac{1}{3}(R_{\alpha\gamma\beta\delta}+R_{\alpha\delta\beta\gamma})\ . \quad (3.17)$$

Clearly this process may be continued indefinitely. After differentiating (2.6) n times, taking the coincidence limit, and using (3.13) and (3.14) we obtain

$$-\sum_{\substack{i\\ i\neq 1}}\sigma_{;\alpha_i\alpha_1\alpha_2\cdots\alpha_{i-1}\alpha_{i+1}\cdots\alpha_n\mathrm{diag}}+\sum_{\substack{i,j\\ i<j}}[\sigma_{;}{}^{\mu}{}_{\alpha_i\alpha_j}\sigma_{;\mu\alpha_1\cdots\alpha_{i-1}\alpha_{i+1}\cdots\alpha_{j-1}\alpha_{j+1}\cdots\alpha_n}]_{\mathrm{diag}}$$
$$+\sum_{\substack{i,j,k\\ i<j<k}}[\sigma_{;}{}^{\mu}{}_{\alpha_i\alpha_j\alpha_k}\sigma_{\mu\alpha_1\cdots\alpha_{i-1}\alpha_{i+1}\cdots\alpha_{j-1}\alpha_{j+1}\cdots\alpha_{k-1}\alpha_{k+1}\cdots\alpha_n}]_{\mathrm{diag}}+\cdots$$
$$+\sum_{\substack{i_1\cdots i_{[n/2]}\\ i_1<\cdots<i_{[n/2]}}}[\sigma^{;\mu}{}_{\alpha_{i_1}\cdots\alpha_{i_{[n/2]}}}\sigma_{;\mu\alpha_1\cdots\alpha_{i-1}\alpha_{i+1}\cdots\alpha_{i_{[n/2]}-1}\alpha_{i_{[n/2]}+1}\cdots\alpha_n}]_{\mathrm{diag}}=0\ , \quad (3.18)$$

where

$$[n/2]=\begin{cases} n/2 & (n\ \text{even})\\ (n-1)/2 & (n\ \text{odd})\ .\end{cases} \quad (3.19)$$

(3.18) relates $\sigma_{;\alpha_1\cdots\alpha_n\mathrm{diag}}$ to coincidence limits of σ with smaller numbers of derivatives, once the indices in the first term are rearranged into the right order. A typical step in this rearrangement would be

$$\sigma_{;\alpha_1\cdots\alpha_{j+1}\alpha_j\cdots\alpha_n}=\sigma_{;\alpha_1\cdots\alpha_j\alpha_{j+1}\cdots\alpha_n}-\sum_{l=1}^{j-1}[R_{\alpha_j\alpha_{j+1}\alpha_l}{}^{\beta}\sigma_{;\alpha_1\cdots\alpha_{l-1}\beta\alpha_{l+1}\cdots\alpha_{j-1}}]_{;\alpha_{j+2}\cdots\alpha_n}\ . \quad (3.20)$$

Let us suppose that $\sigma_{;\alpha_1\cdots\alpha_{k\mathrm{diag}}}$, $4\le k<n$, are R independent. This is true for $k=4$, according to (3.17), provided not all

the indices are internally contracted, which is sufficient to form the basis of the inductive procedure. We then see from (3.20) and (3.18) that any dependence on R (and not its derivatives) could only be generated by letting all the derivatives in the second term on the right-hand side (RHS) of (3.20) act on σ. For $n=4$, as a consequence of (3.14) all the indices in the curvature tensor in (3.20) could be contracted together to give R, but for $n>4$ this is impossible. Hence by induction $\sigma_{;\alpha_1\cdots\alpha_n}$ is R independent for all $n>5$. Using this information about σ, we will now show that in the scalar case

$$\bar{f}_{0;\alpha_1\cdots\alpha_n\mathrm{diag}}=0\ . \tag{3.21}$$

For $\bar{f}_0$ satisfies, from (2.14),

$$\sigma^{;\mu}\bar{f}_{0;\mu}=0\ . \tag{3.22}$$

Differentiating

$$\sigma^{;\mu}{}_{\alpha}\bar{f}_{0;\mu}+\sigma^{;\mu}\bar{f}_{0;\mu\alpha}=0\ , \tag{3.23}$$

and hence, from (3.13) and (3.14),

$$\bar{f}_{0;\alpha}=0\ . \tag{3.24}$$

In general, differentiating n times, and taking the coincidence limit,

$$-\sum_i \bar{f}_{0;\alpha_i\alpha_1\cdots\alpha_{i-1}\alpha_{i+1}\cdots\alpha_n\mathrm{diag}}+\sum_{\substack{i,j\\ i<j}}[\sigma^{;\mu}{}_{\alpha_i\alpha_j}\bar{f}_{0;\mu\alpha_1\cdots\alpha_{i-1}\alpha_{i+1}\cdots\alpha_{j-1}\alpha_{j+1}\cdots\alpha_n}]_{\mathrm{diag}}$$
$$+\sum_{\substack{i,j,k\\ i<j<k}}[\sigma^{;\mu}{}_{\alpha_i\alpha_j\alpha_k}\bar{f}_{0;\mu\alpha_1\cdots\alpha_{i-1}\alpha_{i+1}\cdots\alpha_{j-1}\alpha_{j+1}\cdots\alpha_{k-1}\alpha_{k+1}\cdots\alpha_n}]_{\mathrm{diag}}+\cdots+[\sigma^{;\mu}{}_{\alpha_1\cdots\alpha_n}\bar{f}_{0;\mu}]_{\mathrm{diag}}=0\ . \tag{3.25}$$

After rearranging indices in the first term in (3.25), we obtain an equation for $\bar{f}_{0;\alpha_1\cdots\alpha_n\mathrm{diag}}$. Again, a typical rearrangement of a pair of indices is

$$\bar{f}_{0;\alpha_1\cdots\alpha_{j+1}\alpha_j\cdots\alpha_n}=\bar{f}_{0;\alpha_1\cdots\alpha_j\alpha_{j+1}\cdots\alpha_n}-\sum_{l=1}^{j-1}[R_{\alpha_j\alpha_{j+1}\alpha_l}{}^{\beta}\bar{f}_{0;\alpha_1\cdots\alpha_{l-1}\beta\alpha_{l+1}\cdots\alpha_{j-1}}]_{;\alpha_{j+2}\cdots\alpha_n}\ . \tag{3.26}$$

Hence if $\bar{f}_{0;\alpha_1\cdots\alpha_k\mathrm{diag}}=0, 1\le k\le n$, then from (3.25) and (3.26), $\bar{f}_{0;\alpha_1\cdots\alpha_n\mathrm{diag}}=0$. The statement is true for $n=1$ by (3.24). Hence

$$\bar{f}_{0;\alpha_1\cdots\alpha_n\mathrm{diag}}=0\forall n\ . \tag{3.27}$$

Finally we shall show that $M_{;\alpha_1\cdots\alpha_n\mathrm{diag}}$ is R independent for all n, and $N_{\mu;\alpha_1\cdots\alpha_n\mathrm{diag}}$ is R independent provided there is no α_i, $i=1,\ldots,n$ for which $\alpha_i=\mu$. From (3.12), we may write the normal coordinate expansion of $g_{\alpha\beta}$ in the form

$$g_{\alpha\beta}=\eta_{\alpha\beta}+\tfrac{1}{3}R_{\alpha\lambda\beta\mu}y^{\lambda}y^{\mu}+h_{\alpha\beta},\quad g^{\alpha\beta}=\eta^{\alpha\beta}-\tfrac{1}{3}R^{\alpha}{}_{\lambda}{}^{\beta}{}_{\mu}y^{\lambda}y^{\mu}+h'^{\alpha\beta}\ , \tag{3.28}$$

where $h_{\alpha\beta},h'^{\alpha\beta}$ are curvature connected. If $\bar{g}_{\alpha\beta}$ denotes the analytic continuation of $g_{\alpha\beta}$ to a spacetime of positive (Riemannian) signature, then we have

$$\ln\bar{g}=\mathrm{tr}\ln\bar{g}_{\alpha\beta},\quad \bar{g}=\det\bar{g}_{\alpha\beta}\ . \tag{3.29}$$

Now $\bar{g}_{\alpha\beta}$ may be written

$$\bar{g}_{\alpha\beta}=(1+Z)_{\alpha\beta}\ , \tag{3.30}$$

where $1_{\alpha\beta}=\delta_{\alpha\beta}$ and $Z_{\alpha\beta}$ is the analytic continuation of $(\frac{1}{3}R_{\alpha\lambda\beta\mu}y^{\lambda}y^{\mu}+h_{\alpha\beta})$. The logarithm of $\bar{g}_{\alpha\beta}$ may then be defined as a formal power series,

$$\ln\bar{g}_{\alpha\beta}=\ln(1+Z)_{\alpha\beta}=\left[\sum_{n=1}(-1)^{n+1}\frac{Z^n}{n}\right]_{\alpha\beta}\ . \tag{3.31}$$

Hence, inserting (3.31) into (3.29) and then continuing back to the original spacetime,

$$\ln(-g)=\tfrac{1}{3}R_{\mu\nu}y^{\mu}y^{\nu}+H,\quad g=\det g_{\alpha\beta}\ , \tag{3.32}$$

where H is curvature connected. Only the term $\frac{1}{3}R_{\mu\nu}y^{\mu}y^{\nu}$ in (3.32) can contribute to R dependence when derivatives with respect to y are taken. In normal coordinates we can express M and N_{μ} exclusively in terms of $g_{\alpha\beta}$ from (3.28) and g from (3.32). We have[9]

$$\Delta_{\mathrm{VM}}{}^{1/2}(x,x')=(-g)^{-1/4}=\exp\left[-\tfrac{1}{12}R_{\mu\nu}y^{\mu}y^{\nu}-\tfrac{1}{4}H\right] \tag{3.33}$$

and

$$\Box=\frac{1}{\sqrt{-g}}\partial_\alpha\sqrt{-g}\; g^{\alpha\beta}\partial_\beta \tag{3.34}$$

or

$$\Box=(\eta^{\alpha\beta}-\tfrac{1}{3}R^{\alpha}{}_{\lambda}{}^{\beta}{}_{\mu}y^\lambda y^\mu+h'^{\alpha\beta})\partial_\alpha\partial_\beta+(\tfrac{2}{3}R^{\beta}{}_{\rho}y^\rho+K^\beta)\partial_\beta\,, \tag{3.35}$$

where

$$K^\beta=\partial_\alpha h'^{\alpha\beta}-\tfrac{1}{9}R_{\alpha\rho}R^{\alpha}{}_{\lambda}{}^{\beta}{}_{\sigma}y^\rho y^\lambda y^\sigma+\tfrac{1}{3}R_{\alpha\rho}y^\rho h'^{\alpha\beta}+\tfrac{1}{2}\eta^{\alpha\beta}\partial_\alpha H-\tfrac{1}{6}\partial_\alpha H R^{\alpha}{}_{\lambda}{}^{\beta}{}_{\mu}y^\lambda y^\mu+\tfrac{1}{2}\partial_\alpha H h'^{\alpha\beta}\,, \tag{3.36}$$

and hence K^β is also curvature connected. Therefore,

$$M=-\tfrac{1}{2}(K^\beta+R^{\beta}{}_{\rho}y^\rho+\partial_\rho h'^{\rho\beta})(\tfrac{1}{6}R_{\beta\sigma}y^\sigma+\tfrac{1}{4}\partial_\beta H)-\tfrac{1}{4}\eta^{\alpha\beta}\partial_\alpha\partial_\beta H+(\tfrac{1}{3}R^{\alpha}{}_{\lambda}{}^{\beta}{}_{\mu}y^\lambda y^\mu-h'^{\alpha\beta})(\tfrac{1}{6}R_{\alpha\beta}+\tfrac{1}{4}\partial_\alpha\partial_\beta H)\,. \tag{3.37}$$

In normal coordinates with origin at x' the coincidence limit corresponds to $y=0$. $M_{\rm diag}$ itself is clearly R independent, and it is apparent from the form of $\Box$ in (3.35) that $(\Box^n M)_{\rm diag}$ is also R independent for any n. *A fortiori* $M_{;\alpha_1\cdots\alpha_n{\rm diag}}$ is R free if the derivatives are not contracted in pairs.

A similar argument may be applied to N_μ. From (3.33) and (2.15), in normal coordinates at x',

$$N_\mu=-(\tfrac{1}{6}R_{\mu\rho}y^\rho+\tfrac{1}{4}\partial_\mu H)\,. \tag{3.38}$$

From (3.28) we may calculate the Christoffel symbol in normal coordinates,

$$\begin{aligned}\Gamma^{\alpha}{}_{\beta\gamma}&=\tfrac{1}{2}g^{\alpha\delta}(g_{\delta\beta,\gamma}+g_{\delta\gamma,\beta}-g_{\beta\gamma,\delta})\\&=-\tfrac{1}{3}(R^{\alpha}{}_{\beta\gamma\rho}+R^{\alpha}{}_{\gamma\beta\rho})y^\rho+G^{\alpha}{}_{\beta\gamma}\,,\end{aligned} \tag{3.39}$$

where $G^{\alpha}{}_{\beta\gamma}$ is curvature connected.

$$N_{\mu;\alpha}=-(\tfrac{1}{6}R_{\mu\alpha}+\tfrac{1}{4}\partial_\alpha\partial_\mu H+\Gamma^{\gamma}{}_{\mu\alpha}N_\gamma)\,, \tag{3.40}$$

so

$$N_{\mu;\alpha{\rm diag}}=-\tfrac{1}{6}R_{\mu\alpha}-\tfrac{1}{4}(\partial_\alpha\partial_\mu H)_{\rm diag}\,, \tag{3.41}$$

$$N_{\mu;\alpha\beta}=\partial_\beta N_{\mu;\alpha}-\Gamma^{\delta}{}_{\mu\beta}N_{\delta;\alpha}-\Gamma^{\delta}{}_{\alpha\beta}N_{\mu;\delta}\,. \tag{3.42}$$

Each successive covariant derivative is constructed from the previous one as in (3.42) so that any curvature tensors arising from the Christoffel symbol in (3.39) will always be curvature connected, and hence $N_{\mu;\alpha_1\cdots\alpha_n{\rm diag}}$ is R independent for all $n>1$ and $N_{\mu;\alpha\,{\rm diag}}$ is R independent for $\alpha\neq\mu$.

We now have all the information about the individual constituents of (2.14) which we need to prove the conjecture, which will be the task of the following section.

IV. PROOF OF THE CONJECTURE OF PARKER AND TOMS

In this section we shall prove the conjecture of Parker and Toms that for a scalar field the kernel $\langle x,s\mid x',0\rangle$ satisfying (2.3) has the expansion

$$\begin{aligned}\langle x,s\mid x',0\rangle&=i(4\pi is)^{-d/2}e^{i\sigma/2s}\Delta_{\rm VM}{}^{1/2}(x,x')\bar F(x,x';is)\\&\quad\times\exp[-is(\xi-\tfrac{1}{6})R(x')]\,,\end{aligned} \tag{4.1}$$

where

$$\bar F(x,x';is)=1+\sum_{j=1}^{\infty}(is)^j\bar f_j(x,x') \tag{4.2}$$

such that for each j, the coincidence limit $\bar f_j(x,x)$ is R independent. In fact, it is necessary to prove the stronger assertion that for all j, $\bar f_{j;\alpha_1\cdots\alpha_n{\rm diag}}$ is R independent for any number n of derivatives. This will then imply that each $\bar f_j(x,x')$ is R independent even for $x\neq x'$. Because of the close relation between the kernel defined by (2.3), and the heat kernel for an operator $-\Box+\xi R$ on a Riemannian manifold, our proof will apply equally to the asymptotic expansion for the heat kernel in powers of s. The proof is by induction on both j and n. Let us assume first that $\bar f_{k;\alpha_1\cdots\alpha_r{\rm diag}}$ is R independent for all r, for all $k<j$. We showed in Sec. III that this statement is true for $k=0$. Let us also assume that $\bar f_{j;\alpha_1\cdots\alpha_m{\rm diag}}$ is R independent for all $m<n$. This is true for $m=0$, since the coincidence limit of (2.14) yields

$$j\bar f_{j\,{\rm diag}}=-[(M\bar f_{j-1})_{\rm diag}+(\Box\bar f_{j-1})_{\rm diag}+N_{\mu\,{\rm diag}}\bar f_{j-1}{}^{;\mu}{}_{\rm diag}]\,, \tag{4.3}$$

and hence by virtue of the first inductive hypothesis and the results for M and N_μ proved in Sec. III, $\bar f_{j\,{\rm diag}}$ is R independent.

Differentiating (2.14) n times, taking the coincidence limit, and using (3.13) and (3.14),

$$
\begin{aligned}
&j\bar f_{j;\alpha_1\cdots\alpha_n\,\mathrm{diag}}+\sum_m \bar f_{j;\alpha_m\alpha_1\cdots\alpha_{m-1}\alpha_{m+1}\cdots\alpha_n\,\mathrm{diag}}\\
&\quad-\sum_{\substack{l,m\\ l<m}}[\sigma^{;\mu}{}_{\alpha_l\alpha_m}\bar f_{j;\mu\alpha_1\cdots\alpha_{l-1}\alpha_{l+1}\cdots\alpha_{m-1}\alpha_{m+1}\cdots\alpha_n}]_{\mathrm{diag}}-\cdots-[\sigma^{;\mu}{}_{\alpha_1\cdots\alpha_n}\bar f_{j;\mu}]_{\mathrm{diag}}\\
&\quad=-[M_{;\alpha_1\cdots\alpha_n\,\mathrm{diag}}+(\xi-\tfrac{1}{6})R_{;\alpha_1\cdots\alpha_n}]\bar f_{j-1\,\mathrm{diag}}\\
&\qquad-\sum_m[M_{;\alpha_1\cdots\alpha_{m-1}\alpha_{m+1}\cdots\alpha_n\,\mathrm{diag}}+(\xi-\tfrac{1}{6})R_{;\alpha_1\cdots\alpha_{m-1}\alpha_{m+1}\cdots\alpha_n}]\bar f_{j-1;\alpha_m\,\mathrm{diag}}\\
&\qquad-\sum_{\substack{l,m\\ l<m}}[M_{;\alpha_1\cdots\alpha_{l-1}\alpha_{l+1}\cdots\alpha_{m-1}\alpha_{m+1}\cdots\alpha_n\,\mathrm{diag}}+(\xi-\tfrac{1}{6})R_{;\alpha_1\cdots\alpha_{l-1}\alpha_{l+1}\cdots\alpha_{m-1}\alpha_{m+1}\cdots\alpha_n}]\bar f_{j-1;\alpha_l\alpha_m\,\mathrm{diag}}\\
&\qquad-\cdots-\sum_m[M_{;\alpha_m\,\mathrm{diag}}+(\xi-\tfrac{1}{6})R_{;\alpha_m}]\bar f_{j-1;\alpha_1\cdots\alpha_{m-1}\alpha_{m+1}\cdots\alpha_n\,\mathrm{diag}}\\
&\qquad-M_{\mathrm{diag}}\bar f_{j-1;\alpha_1\cdots\alpha_n\,\mathrm{diag}}-\bar f_{j-1;\beta}{}^{\beta}{}_{\alpha_1\cdots\alpha_n\,\mathrm{diag}}\\
&\qquad-N_{\mu;\alpha_1\cdots\alpha_n\,\mathrm{diag}}\bar f_{j-1}{}^{;\mu}{}_{\mathrm{diag}}-\sum_m N_{\mu;\alpha_1\cdots\alpha_{m-1}\alpha_{m+1}\cdots\alpha_n\,\mathrm{diag}}\bar f_{j-1}{}^{;\mu}{}_{\alpha_m\,\mathrm{diag}}-\cdots\\
&\qquad-N_{\mu\,\mathrm{diag}}\bar f_{j-1}{}^{;\mu}{}_{\alpha_1\cdots\alpha_n\,\mathrm{diag}}\;.
\end{aligned}
\tag{4.4}
$$

The indices in the second term can be rearranged to the order $\alpha_1\cdots\alpha_n$, introducing further terms consisting of $\bar f_j$ with less than n derivatives contracted into curvature tensors. Such terms are R independent by the second inductive hypothesis. The succeeding terms on the left-hand side (LHS) of (4.4) are R independent by virtue of the result concerning σ in Sec. III, and the second inductive hypothesis yet again. The terms on the RHS of (4.4) are R independent as a consequence of the first inductive hypothesis and the fact that the coincidence limits of derivatives of M and N_μ (provided μ is left as a free index) are R independent. Hence $\bar f_{j;\alpha_1\cdots\alpha_n\mathrm{diag}}$ is R independent. As we remarked earlier, this statement is true

(a) for $j=0$ and all n and

(b) for any j and $n=0$, given it is true for all $k<j$.

Now we have shown that if $\bar f_{k;\alpha_1\cdots\alpha_r\mathrm{diag}}$ is R independent for all r, for $k<j$, and $\bar f_{j;\alpha_1\cdots\alpha_m\mathrm{diag}}$ is R independent for all $m<n$, then $\bar f_{j;\alpha_1\cdots\alpha_n\mathrm{diag}}$ is also R independent. Hence this statement is true for all j and n by induction. The conjecture of Parker and Toms is therefore proved in general, and moreover since we can write

$$
\bar f_j(x,x')=\bar f_{j\,\mathrm{diag}}(x)+\sigma^{;\alpha}\bar f_{j;\alpha\,\mathrm{diag}}+\tfrac{1}{2}\sigma^{;\alpha}\sigma^{;\beta}\bar f_{j;\alpha\beta\,\mathrm{diag}}+\cdots+\frac{1}{n!}\sigma^{;\alpha_1}\cdots\sigma^{;\alpha_n}\bar f_{j;\alpha_1\cdots\alpha_n\,\mathrm{diag}}
\tag{4.5}
$$

the expansion in (4.1) is R independent before taking the coincidence limit.

We should emphasize that the terms in $\bar f_{j;\alpha_1\cdots\alpha_n\mathrm{diag}}$ which do not depend on R are exactly the same as those in the original $f_{j;\alpha_1\cdots\alpha_n\mathrm{diag}}$ computed from (2.8). This can be shown by making the same inductive hypotheses as were used in the proof of the main theorem. Clearly the statement is true with $j=0$, for all n, and since (4.3) has the same form as the coincidence limit of (2.8) apart from terms depending on R, the statement is also true for any j with $n=0$ provided it holds for all $k<j$. Moreover, since when we take derivatives of (2.14) these act only on $R(x)$ and not $R(x')$, the terms in (4.4) have the same form as would be obtained by differentiating (2.8) apart from those involving R and not its derivatives. Hence the statement is proved by induction. We could also see this more simply by expanding $\exp[-is(\xi-\frac{1}{6})R(x')]$ in (4.1) and obtaining $\bar f_j(x,x')$ as a function of the $f_k(x,x')$ by comparing coefficients of $(is)^j$.

On the other hand, if we were to use, for instance, the alternative expansion (2.12a), then the recurrence relations for $\bar f'_j$ would be somewhat more complicated owing to the extra x dependence in the exponential. Nevertheless, we could prove in a similar fashion as for the $\bar f_j$ earlier that the coincidence limits of $\bar f'_j$ and its derivatives are R independent and indeed it is clear from (2.12) and (2.12a) that

$$
\bar f_{j\,\mathrm{diag}}=\bar f'_{j\,\mathrm{diag}}\;.
\tag{4.6}
$$

However, the coincidence limits of derivatives of $\bar f'_j$ will differ from the corresponding quantities involving $\bar f_j$ by terms involving covariant derivatives of R. These terms could be evaluated in principle by expanding $\exp[-is(\xi-\frac{1}{6})R(x')]$ in (4.1) and $\exp[-is(\xi-\frac{1}{6})R(x)]$ in (2.12a) as power series in (is), and then comparing coefficients of powers of is. Similar remarks apply to the other possible expansion written down in (2.12b).

Before leaving the case of the propagator for a scalar field, we must point out that when background gauge fields are contained in the covariant Laplacian, as happens for Faddeev-Popov ghosts, or Higgs fields, then the situation is somewhat more involved, and is considered in detail in the Appendix. The results of this section still hold good even in this case.

V. THE CASE OF HIGHER SPIN

We now turn to consider the kernel $\langle x,s \mid x',0\rangle$ for the propagator of a field with nonzero spin. The field then carries indices according to the particular spin, and the covariant derivative D_μ acting on the field contains the appropriate spin connection. We may write

$$D_\mu = 1_{\mathscr{G}}\nabla_\mu,\ \ \nabla_\mu=\partial_\mu+\Gamma_\mu\ , \tag{5.1}$$

where $1_{\mathscr{G}}$ is the unit matrix for the particular representation of the gauge group $\mathscr{G}$ to which the field belongs and Γ_μ is the relevant spin connection. The case where background gauge fields are included in D_μ is considered in the Appendix. Two covariant derivatives acting on the field successively do not now commute (in contrast to the scalar case) and we have

$$[D_\mu,D_\nu]=W_{\mu\nu}\ , \tag{5.2}$$

where $W_{\mu\nu}$ will be specified later. We shall consider D_μ always to contain the correct spin connection for whatever object it acts on, and denote $D_\mu Y$ by $Y_{;\mu}$.

The kernel is now a bispinor (for half-integral spin) or bitensor (for integral spin), transforming as a direct product of two spinor or tensor representations, at x and x'. Although the calculation will be perfectly general, we shall exemplify it by reference to the spin-$\frac{1}{2}$ and spin-1 (vector) cases. Any higher-spin representation can be constructed as a direct product of two or more of these.[1] Omitting matrix indices, the propagator now satisfies

$$[D^2+X(x)]\Delta(x,x')=-\delta(x,x')1\ , \tag{5.3}$$

where 1 is the unit matrix for the particular spin representation, and $X(x)$ is a matrix whose form will be discussed in detail later, which may include mass terms and in particular any imaginary mass term necessary to ensure convergence.

If background gauge fields are included in the definition of the propagator then the form of X will be modified, as is discussed in the Appendix. However the arguments of this section still apply. Since each of the matrix indices of X transform as a spinor or tensor, the covariant derivative acts according to

$$X_{;\mu}=[D_\mu,X]\ . \tag{5.4}$$

The kernel $\langle x,s\mid x',0\rangle$ which gives $\Delta(x,x')$ according to

$$\Delta(x,x')=-i\int_0^\infty \langle x,s\mid x',0\rangle ds \tag{5.5}$$

has a Schwinger-DeWitt expansion[1] corresponding to (2.4),

$$\langle x,s\mid x',0\rangle = i(4\pi is)^{-d/2}e^{i\sigma(x,x')/2s}\Delta_{\rm VM}{}^{1/2}(x,x')G(x,x';is)\ , \tag{5.6}$$

$$G(x,x';is)=\sum_{j=0}^{\infty}(is)^j g_j(x,x')\ .$$

We now assert that this expansion may be rewritten in the form

$$\langle x,s\mid x',0\rangle = i(4\pi is)^{-d/2}e^{i\sigma(x,x')/2s}\Delta_{\rm VM}{}^{1/2}(x,x')\overline{G}(x,x';is)\times\exp\{-is[X(x')-\tfrac{1}{6}R(x')]\}\ , \tag{5.7}$$

$$\overline{G}(x,x';is)=\sum_{j=0}^{\infty}(is)^j\overline{g}_j(x,x')\ ,$$

where $\exp\{-is[X(x')-\frac{1}{6}R(x')1]\}$ is defined as a formal matrix power series, such that the $\overline{g}_j$ have the following properties:

(1) The coincidence limits of $\overline{g}_j$ and all its derivatives are R independent for all j.

(2) There are no terms in the coincidence limits of $\overline{g}_j$ or any of its derivatives containing factors of X, unless they also contain factors of either W or its derivatives, or derivatives of X.

(3) Any terms in the coincidence limits of $\overline{g}_j$, or its derivatives, which contain X and also contain (a) derivatives of X, or (b) W, or (c) derivatives of W, are such as to sum to zero if X commutes with whichever quantities of type (a), (b), or (c) are present.

The expansion (5.7) was first proposed in Ref. 7, where it was pointed out that in the coincidence limit $\overline{G}$ was R independent to order s^3. In the Appendix, the first three nontrivial terms in $\overline{G}$ are written down explicitly, demonstrating that properties (1)—(3) hold to this order. As a corollary of statement (1) above, $\overline{G}(x,x';is)$ is evidently R independent to all orders even for $x\neq x'$. As for the scalar case, these results apply equally to the heat kernel for an operator $-D^2+X(x)$ defined on a Riemannian manifold.

The proof of the statements following (5.7) is as follows: from (5.3), (5.5), and (5.7), the coefficients $\overline{g}_j$ satisfy a recurrence relation analogous to (2.14),

$$\sigma^{;\mu}\overline{g}_{0;\mu}=0\ , \tag{5.8a}$$

$$j\overline{g}_j-\sigma^{;\mu}\overline{g}_{j;\mu}=-\overline{g}_{j-1;\mu}{}^{\mu}-N_\mu\overline{g}_{j-1}{}^{;\mu}+\{-M+\tfrac{1}{6}[R(x)-R(x')]\}\overline{g}_{j-1}+[\overline{g}_{j-1}X(x')-X(x)\overline{g}_{j-1}]\ . \tag{5.8b}$$

The proof that the coincidence limits of $\overline{g}_j$ and all its derivatives are R independent follows the same steps as in the scalar case in Secs. III and IV, with some minor differences in detail. The first is that although $\overline{g}_{0;\alpha_1\cdots\alpha_n{\rm diag}}$ remains R independent for all n, it is in general no longer zero. Differentiating (5.8a) twice and taking the coincidence limit,

$$\overline{g}_{0;\alpha\beta\,{\rm diag}}+\overline{g}_{0;\beta\alpha\,{\rm diag}}=0\ . \tag{5.9}$$

We now have, from (5.2),

$$\overline{g}_{0;\alpha\beta\,{\rm diag}}=\tfrac{1}{2}W_{\beta\alpha} \tag{5.10}$$

and in general, in the analogous equation to (3.25) when we rearrange indices in $\sum_i\overline{g}_{0;\alpha_i\alpha_1\ldots\alpha_{i-1}\alpha_{i+1}\ldots\alpha_n{\rm diag}}$ into the order $\alpha_1\ldots\alpha_n$, we obtain terms depending on W, for instance,

$$\bar{g}_{0;\alpha_1\cdots\alpha_{j+1}\alpha_j\ldots\alpha_n}=\bar{g}_{0;\alpha_1\cdots\alpha_j\alpha_{j+1}\cdots\alpha_n}-\sum_{l=1}^{j-1}(R_{\alpha_j\alpha_{j+1}\alpha_l}{}^{\beta}\bar{g}_{0;\alpha_1\cdots\alpha_{l-1}\beta\alpha_{l+1}\cdots\alpha_{j-1}})_{;\alpha_{j+2}\cdots\alpha_n}$$
$$+(W_{\alpha_j\alpha_{j+1}}\bar{g}_{0;\alpha_1\cdots\alpha_{j-1}})_{;\alpha_{j+2}\cdots\alpha_n}\ . \tag{5.11}$$

Since σ and its derivatives are R independent, it is clear from (5.11) and (3.25) with $\bar{g}$ replacing $\bar{f}$, that $\bar{g}_{0;\alpha_1\cdots\alpha_n\text{diag}}$ is built up out of curvature tensors contracted with each other or with W, together with terms involving W alone. There is no explicit R dependence, and in fact there is none concealed in W either. For spin $\frac{1}{2}$, W has spinor indices,[1]

$$W_{\alpha\beta}=-\tfrac{1}{4}R_{\alpha\beta\rho\sigma}\tilde{\gamma}^{\rho}\tilde{\gamma}^{\sigma},\quad \tilde{\gamma}^{\rho}=\gamma^a e_a^{\mu}\ , \tag{5.12}$$

where γ^a are the usual Dirac γ matrices and e_a^{μ} is the vierbein with the properties

$$\delta_{ab}=e_a^{\mu}e_b^{\nu}g_{\mu\nu},\quad g_{\mu\nu}=e_{\mu}^{a}e_{\nu}^{b}\delta_{ab}\ . \tag{5.13}$$

For spin 1, W has Lorentz indices[1]

$$(W_{\alpha\beta})_{\rho\sigma}=-R_{\alpha\beta\rho\sigma}1_{\mathcal{G}}\ . \tag{5.14}$$

The forms of $W_{\alpha\beta}$ in (5.12) and (5.14) must be modified when background gauge fields are included. Details are given in the Appendix. In products of several W's, the $\alpha\beta$ indices in (5.12) and (5.14) are completely independent of the spinor or Lorentz indices ρ,σ and hence they cannot be contracted together to give dependence on R.

Evidently the remainder of Sec. III carries over unaltered. However, Sec. IV requires some modification. Following the procedure of that section we make the inductive hypotheses that

(a) the coincidence limits of $\bar{g}_k$ and all its derivatives are R independent for all $k<j$ and

(b) The coincidence limits of $\bar{g}_{j;\alpha_1\cdots\alpha_m}$ are R independent for all $m<n$.

Then, differentiating (5.8b) n times, taking the coincidence limits and using (3.13) and (3.14), we have

$$j\bar{g}_{j;\alpha_1\cdots\alpha_n\text{diag}}+\sum_i\bar{g}_{j;\alpha_i\alpha_1\cdots\alpha_{i-1}\alpha_{i+1}\cdots\alpha_n\text{diag}}$$
$$-\sum_{\substack{i,j\\i<j}}(\sigma^{;\mu}{}_{\alpha_i\alpha_j}\bar{g}_{j;\mu\alpha_1\cdots\alpha_{i-1}\alpha_{i+1}\cdots\alpha_{j-1}\alpha_{j+1}\cdots\alpha_n})_{\text{diag}}-\cdots-(\sigma^{;\mu}{}_{\alpha_1\cdots\alpha_n}\bar{g}_{j;\mu})_{\text{diag}}$$
$$=-\bar{g}_{j-1;\mu}{}^{\mu}{}_{\alpha_1\cdots\alpha_n\text{diag}}-N_{\mu\text{diag}}\bar{g}_{j-1}{}^{;\mu}{}_{\alpha_1\cdots\alpha_n\text{diag}}-\sum_i(N_{\mu;\alpha_i}\bar{g}_{j-1}{}^{;\mu}{}_{\alpha_1\cdots\alpha_{i-1}\alpha_{i+1}\cdots\alpha_n})_{\text{diag}}$$
$$-\sum_{\substack{i,j\\i<j}}(N_{\mu;\alpha_i\alpha_j}\bar{g}_{j-1}{}^{;\mu}{}_{\alpha_1\cdots\alpha_{i-1}\alpha_{i+1}\cdots\alpha_{j-1}\alpha_{j+1}\cdots\alpha_n})_{\text{diag}}-\cdots-N_{\mu;\alpha_1\cdots\alpha_n\text{diag}}\bar{g}_{j-1}{}^{;\mu}{}_{\text{diag}}$$
$$-(M_{;\alpha_1\cdots\alpha_n\text{diag}}-\tfrac{1}{6}R_{;\alpha_1\cdots\alpha_n}+X_{;\alpha_1\cdots\alpha_n})\bar{g}_{j-1\,\text{diag}}$$
$$-\sum_i(M_{;\alpha_1\cdots\alpha_{i-1}\alpha_{i+1}\cdots\alpha_n\text{diag}}-\tfrac{1}{6}R_{;\alpha_1\cdots\alpha_{i-1}\alpha_{i+1}\cdots\alpha_n}+X_{;\alpha_1\cdots\alpha_{i-1}\alpha_{i+1}\cdots\alpha_n})\bar{g}_{j-1;\alpha_i\text{diag}}$$
$$-\cdots-\sum_i(M_{;\alpha_i\text{diag}}-\tfrac{1}{6}R_{;\alpha_i}+X_{;\alpha_i})\bar{g}_{j-1;\alpha_1\cdots\alpha_{i-1}\alpha_{i+1}\cdots\alpha_n\text{diag}}$$
$$-(M\bar{g}_{j-1;\alpha_1\cdots\alpha_n\text{diag}})+[\bar{g}_{j-1;\alpha_1\cdots\alpha_n\text{diag}},X]\ . \tag{5.15}$$

When we rearrange indices in terms in $\sum_i\bar{g}_{j;\alpha_i\alpha_1\cdots\alpha_{i-1}\alpha_{i+1}\cdots\alpha_n\text{diag}}$, we obtain terms involving W as in (5.11), but according to the remarks following (5.11), they do not introduce any R dependence. Hence, given the inductive hypotheses and by virtue of the results of Sec. III for M, N_{μ}, $\bar{g}_{j;\alpha_1\cdots\alpha_n\text{diag}}$ is R independent. The coincidence limit of (5.8b)

$$j\bar{g}_{j\,\text{diag}}=-\bar{g}_{j-1;\mu}{}^{\mu}{}_{\text{diag}}-M_{\text{diag}}\bar{g}_{j-1\,\text{diag}}+[\bar{g}_{j-1\,\text{diag}},X] \tag{5.16}$$

implies that $\bar{g}_{j\,\text{diag}}$ is R independent given the first inductive hypothesis, that $\bar{g}_{k;\alpha_1\cdots\alpha_r\text{diag}}$ is R independent for $k<j$ and any r, and we have already seen that $\bar{g}_{0;\alpha_1\cdots\alpha_n\text{diag}}$ is R independent. Hence the inductive processes can be initiated and $\bar{g}_{j;\alpha_1\cdots\alpha_n\text{diag}}$ is R independent for every j and all n.

The second statement following (5.7) is readily proved by induction, since it is only derivatives of X, and W and its derivatives with which X may not commute. Given that statement (2) is true for $\bar{g}_{k;\alpha_1\cdots\alpha_r\text{diag}}$ for $k<j$ and arbitrary r, and for $\bar{g}_{j;\alpha_1\cdots\alpha_m\text{diag}}$ for $m<n$, then since terms in $\bar{g}_{j-1;\alpha_1\cdots\alpha_n\text{diag}}$ not containing derivatives of X, or W or its derivatives, will give a vanishing contribution to the commutator in (5.15), the statement is also true for $\bar{g}_{j;\alpha_1\cdots\alpha_n\text{diag}}$. Evidently for the same reason the statement

is true of $\bar{g}_{j\,\text{diag}}$, given it holds for $\bar{g}_{k;\alpha_1\cdots\alpha_r\text{diag}}$ with $k<j$, and obviously it is true for $\bar{g}_{0;\alpha_1\cdots\alpha_n\text{diag}}$ for all n. Hence the second statement is true in general by induction. Conversely, terms involving X and also derivatives of X could only be generated by the commutator in (5.15) and therefore would sum to zero if X commuted with the objects concerned. On the other hand, terms involving X and W or its derivatives can also be generated in a different fashion, since from (5.2) and (5.4)

$$X_{;[\alpha\beta]}=[W_{\beta\alpha},X]\ , \tag{5.17}$$

and so terms containing $W_{\alpha\beta}$ contracted with $X_{;\alpha\beta}$, which are first encountered in evaluating $\bar{g}_{3\text{diag}}$, can yield extra terms involving X and W. It is still the case however that all such terms sum to zero if X commutes with whichever of W or its derivatives appears. This proves the third statement following (5.7). As in the scalar case, since the coincidence limits of all derivatives of each $\bar{g}_j$ are R independent, the expansion (5.7) is R independent in general, without taking the coincidence limit.

We could also postulate the alternative expansions

$$\langle x,s\,|\,x',0\rangle=i(4\pi is)^{-d/2}e^{i\sigma/2s}\Delta_{\text{VM}}{}^{1/2}\times\exp\{-is[X(x)-\tfrac{1}{6}R(x)1]\}\bar{G}'(x,x';is) \tag{5.18}$$

or the symmetrical

$$\langle x,s\,|\,x',0\rangle=i(4\pi is)^{-d/2}e^{i\sigma/2s}\Delta_{\text{VM}}{}^{1/2}\times\exp\{-\tfrac{1}{2}is[X(x)-\tfrac{1}{6}R(x)1]\}\bar{G}''(x,x';is)\times\exp\{-\tfrac{1}{2}is[X(x')-\tfrac{1}{6}R(x')1]\}\ , \tag{5.19}$$

where $\bar{G}'$ and $\bar{G}''$ have a series expression similar to $\bar{G}$ in (5.7).

These expansions also share the properties of (5.7). The immediate difficulty in proving this is that there is no convenient expression for the derivative of $\exp\{-is[X(x)-\frac{1}{6}R(x)1]\}$, and hence it is not clear how to obtain the recurrence relations for $\bar{g}'_j$ and $\bar{g}''_j$ from (5.3), (5.5), and (5.18) or (5.19). This is easily overcome in the case of (5.18) since, neglecting any complications due to possible zero modes, the propagator also satisfies

$$\Delta(x,x')[\overleftarrow{D}^2_{x'}+X(x')]=-\delta(x,x')1 \tag{5.20}$$

and hence recurrence relations can be written down which are similar in form to (5.8) except that derivatives are now with respect to x' rather than x. The proof then carries through as previously. In the case of (5.19), however, another approach is evidently required. Comparing (5.7) with (5.19), we see

$$\bar{G}(x,x')\exp\{-\tfrac{1}{2}is[X(x')-\tfrac{1}{6}R(x')1]\}=\exp\{-\tfrac{1}{2}is[X(x)-\tfrac{1}{6}R(x)]\}\bar{G}''(x,x')\ . \tag{5.21}$$

Expanding the exponentials and equating coefficients of $(is)^j$ we obtain

$$\bar{g}''_j(x,x')+\sum_{l=0}^{j-1}\frac{[Y(x)]^{j-l}}{(j-l)!}\bar{g}''_l(x,x')=\sum_{l=0}^{j}\bar{g}_l(x,x')\frac{[Y(x')]^{j-l}}{(j-l)!}\ , \tag{5.22}$$

where

$$Y(x)=\tfrac{1}{2}[\tfrac{1}{6}R(x)-X(x)]\ . \tag{5.23}$$

It is straightforward to show by induction that the solution for $\bar{g}''_j$ in terms of $\bar{g}_k$, $k=1,\ldots,j$ is

$$\bar{g}''_j(x,x')=\bar{g}_j(x,x')+[\bar{g}_{j-1}(x,x')Y(x')-Y(x)\bar{g}_{j-1}(x,x')]+\tfrac{1}{2}[\bar{g}_{j-2}Y(x')^2-2Y(x)\bar{g}_{j-2}Y(x')+Y(x)^2\bar{g}_{j-2}]+\cdots+\frac{1}{k!}\sum_{l=0}^{k}\begin{bmatrix}k\\ l\end{bmatrix}(-1)^l[Y(x)]^l\bar{g}_{j-k}(x,x')[Y(x')]^{k-l}+\cdots\ , \tag{5.24}$$

where

$$\begin{bmatrix}k\\ l\end{bmatrix}=\frac{k!}{l!(k-l)!}\ .$$

It is then a matter of tedious combinatorics, using in particular the relation

$$\sum_{l=0}^{k}\begin{bmatrix}k\\ l\end{bmatrix}(-1)^l=0 \tag{5.25}$$

to show that the properties of $\bar{g}_j$ imply the same properties for $\bar{g}''_j$.

Finally let us display explicitly the forms of X for the particular cases of spin $\frac{1}{2}$ and spin 1. In the case of spin $\frac{1}{2}$, the propagator Δ_F satisfies a first-order equation

$$\tilde{\gamma}^{\mu}D_{\mu}\Delta_F(x,x')=-\delta(x,x')\ , \tag{5.26}$$

with $\tilde{\gamma}^{\mu}$ as in (5.12). In order to be able to work with a second-order operator as required for the Schwinger-DeWitt expansion, we write

$$\Delta_F(x,x')=\tilde{\gamma}\cdot D\Delta(x,x') \tag{5.27}$$

so that, from (5.26),

$$(\tilde{\gamma}\cdot D)^2\Delta(x,x')=-\delta(x,x')\ . \tag{5.28}$$

Using (5.2) and (5.12) together with the usual γ-matrix properties, (5.28) may be written in the form (5.3) with

$$X=\tfrac{1}{4}R\,1 \tag{5.29}$$

and hence X does commute with all quantities involved. Thus for spin $\frac{1}{2}$, there are no terms in the coincidence limits of $\bar{g}_j$ or any of its derivatives which vanish when X (but not its derivatives) is set to zero. In other words,

$\overline{G}(x,x';is)$ remains R independent even when the R dependence in X is taken into account.

For the spin-1 case, on the other hand, assuming that (5.3) applies to the propagator for a quantum gauge theory with a Feynman-type gauge, then

$$X_{\mu\nu}=R_{\mu\nu}1_{\mathscr{G}}\ . \tag{5.30}$$

Thus in this instance X does not commute with its derivatives, nor with W and its derivatives as given by (5.14). However, evidently the terms containing X can introduce no R dependence into the coincidence limits of $\overline{g}_j$ and its derivatives. The reader is referred to the Appendix for a discussion of the inclusion of background gauge fields.

VI. CONCLUSION

We have now proved the theorem stated in Eq. (5.7) and the paragraph following it. The quantity X of Eq. (5.7) is given for spin $\frac{1}{2}$ in Eq. (5.29) and for a spin-1 Yang-Mills field in Eq. (5.30). In the scalar case, X is given by ξR and the theorem reduces to Eqs. (4.1) and (4.2) with the $\overline{f}_j(x,x')$ being R independent, even when x and x' are not equal.

An immediate application of the theorem is in simplifying the calculation of the higher-order coefficients of the heat kernel or Schwinger-DeWitt expansion. We have proved that, when written in the new form, the set of terms containing the scalar curvature will not appear in the expansion coefficients for any spin.

The new form of the expansion should also be useful in probing nonlocal effects which could not be obtained from a finite set of terms. In quantum field theory, for example, the exponential term in Eq. (5.7) or Eq. (4.1) gives rise to nonlocal curvature dependence of the effective coupling constants, and introduces significant modifications into the gravitational field equations.[7] Finally, by retaining a finite known set of terms in the expansion of $\overline{F}$ in Eq. (4.2) or $\overline{G}$ in Eq. (5.7), one obtains useful approximations for the heat kernel and Feynman propagator.

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APPENDIX

In this Appendix we list the explicit forms of the coincidence limits of the first three nontrivial terms in the expansions of $G,\overline{G}$ in (5.6) and (5.7) so that the properties stated after (5.7) may be checked in these cases. For an operator of the form D^2+X where the covariant derivative D_μ satisfies the commutation relation (5.2), we have[2]

$$g_{1\mathrm{diag}}=\tfrac{1}{6}R-X\ , \tag{A1}$$

$$\overline{g}_{1\mathrm{diag}}=0\ , \tag{A2}$$

$$g_{2\mathrm{diag}}=\tfrac{1}{2}(\tfrac{1}{6}R-X)^2-\tfrac{1}{180}R_{\mu\nu}R^{\mu\nu}+\tfrac{1}{180}R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}-\tfrac{1}{30}\Box R+\tfrac{1}{6}\Box X+\tfrac{1}{12}W_{\alpha\beta}W^{\alpha\beta}\ , \tag{A3}$$

$$\overline{g}_{2\mathrm{diag}}=-\tfrac{1}{180}R_{\mu\nu}R^{\mu\nu}+\tfrac{1}{180}R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}-\tfrac{1}{30}\Box R+\tfrac{1}{6}\Box X+\tfrac{1}{12}W_{\alpha\beta}W^{\alpha\beta}\ , \tag{A4}$$

$$\begin{aligned}g_{3\mathrm{diag}}=\frac{1}{7!}[&18\Box^2R-17R_{;\mu}R^{;\mu}+2R_{\mu\nu;\rho}R^{\mu\nu;\rho}+4R_{\mu\nu;\rho}R^{\mu\rho;\nu}-9R_{\mu\nu\rho\sigma;\tau}R^{\mu\nu\rho\sigma;\tau}-28R\Box R\\&+8R_{\mu\nu}\Box R^{\mu\nu}-24R_{\mu\nu}R^{\mu\rho;\nu}{}_{\rho}-12R_{\mu\nu\rho\sigma}\Box R^{\mu\nu\rho\sigma}+\tfrac{35}{9}R^3-\tfrac{14}{3}RR_{\mu\nu}R^{\mu\nu}\\&+\tfrac{14}{3}RR_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}-\tfrac{208}{9}R_{\mu\nu}R^{\mu}{}_{\rho}R^{\nu\rho}+\tfrac{64}{3}R_{\mu\nu}R_{\rho\sigma}R^{\mu\rho\nu\sigma}-\tfrac{16}{3}R_{\mu\nu}R^{\mu}{}_{\rho\sigma\tau}R^{\nu\rho\sigma\tau}\\&+\tfrac{44}{9}R_{\mu\nu\rho\sigma}R^{\mu\nu\alpha\beta}R^{\rho\sigma}{}_{\alpha\beta}+\tfrac{80}{9}R_{\mu\nu\rho\sigma}R^{\mu\alpha\rho\beta}R^{\nu}{}_{\alpha}{}^{\sigma}{}_{\beta}]\\&-\tfrac{1}{45}W_{\alpha\beta;\gamma}W^{\alpha\beta;\gamma}-\tfrac{1}{180}W^{\alpha\beta}{}_{;\beta}W_{\alpha\gamma}{}^{;\gamma}-\tfrac{1}{60}(\Box W_{\alpha\beta})W^{\alpha\beta}-\tfrac{1}{60}W_{\alpha\beta}\Box W^{\alpha\beta}+\tfrac{1}{30}W_{\alpha\beta}W^{\beta\gamma}W_{\gamma}{}^{\alpha}\\&+\tfrac{1}{60}R_{\alpha\beta\gamma\delta}W^{\alpha\beta}W^{\gamma\delta}-\tfrac{1}{90}R_{\mu\nu}W^{\mu\alpha}W^{\nu}{}_{\alpha}+\tfrac{1}{72}RW_{\alpha\beta}W^{\alpha\beta}-\tfrac{1}{60}\Box^2X\\&-\tfrac{1}{12}X(\Box X)-\tfrac{1}{12}(\Box X)X-\tfrac{1}{12}X_{;\mu}X^{;\mu}-\tfrac{1}{6}X^3-\tfrac{1}{30}XW_{\alpha\beta}W^{\alpha\beta}\\&-\tfrac{1}{60}W_{\alpha\beta}XW^{\alpha\beta}-\tfrac{1}{30}W_{\alpha\beta}W^{\alpha\beta}X+\tfrac{1}{36}R\Box X+\tfrac{1}{90}R^{\alpha\beta}X_{;\alpha\beta}+\tfrac{1}{30}R_{;\mu}X^{;\mu}+\tfrac{1}{60}X_{;\beta}W^{\alpha\beta}{}_{;\alpha}-\tfrac{1}{60}W^{\alpha\beta}{}_{;\alpha}X_{;\beta}+\tfrac{1}{12}X^2R\\&+\tfrac{1}{30}X\Box R-\tfrac{1}{72}XR^2+\tfrac{1}{180}XR_{\mu\nu}R^{\mu\nu}-\tfrac{1}{180}XR_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\end{aligned} \tag{A5}$$

and

$$\bar{g}_{3\text{diag}}=\frac{1}{7!}[18\Box^2R-17R_{;\mu}R^{;\mu}+2R_{\mu\nu;\rho}R^{\mu\nu;\rho}+4R_{\mu\nu;\rho}R^{\mu\rho;\nu}-9R_{\mu\nu\rho\sigma;\tau}R^{\mu\nu\rho\sigma;\tau}$$
$$+8R_{\mu\nu}\Box R^{\mu\nu}-24R_{\mu\nu}R^{\mu\rho;\nu}{}_{\rho}-12R_{\mu\nu\rho\sigma}\Box R^{\mu\nu\rho\sigma}-\tfrac{208}{9}R_{\mu\nu}R^{\mu}{}_{\rho}R^{\nu\rho}+\tfrac{64}{3}R_{\mu\nu}R_{\rho\sigma}R^{\mu\rho\nu\sigma}$$
$$-\tfrac{16}{3}R_{\mu\nu}R^{\mu}{}_{\rho\sigma\tau}R^{\nu\rho\sigma\tau}+\tfrac{44}{9}R_{\mu\nu\rho\sigma}R^{\mu\nu\alpha\beta}R^{\rho\sigma}{}_{\alpha\beta}+\tfrac{80}{9}R_{\mu\nu\rho\sigma}R^{\mu\alpha\rho\beta}R^{\nu}{}_{\alpha}{}^{\sigma}{}_{\beta}]-\tfrac{1}{45}W_{\alpha\beta;\gamma}W^{\alpha\beta;\gamma}$$
$$-\tfrac{1}{180}W^{\alpha\beta}{}_{;\beta}W_{\alpha\gamma}{}^{;\gamma}-\tfrac{1}{60}(\Box W_{\alpha\beta})W^{\alpha\beta}-\tfrac{1}{60}W_{\alpha\beta}\Box W^{\alpha\beta}+\tfrac{1}{30}W_{\alpha\beta}W^{\beta\gamma}W_{\gamma}{}^{\alpha}+\tfrac{1}{60}R_{\alpha\beta\gamma\delta}W^{\alpha\beta}W^{\gamma\delta}$$
$$-\tfrac{1}{90}R_{\mu\nu}W^{\mu\alpha}W^{\nu}{}_{\alpha}-\tfrac{1}{60}\Box^2X+\tfrac{1}{12}X\Box X-\tfrac{1}{12}(\Box X)X-\tfrac{1}{12}X_{;\mu}X^{;\mu}+\tfrac{1}{90}R^{\mu\nu}X_{;\mu\nu}+\tfrac{1}{30}R_{;\mu}X^{;\mu}$$
$$+\tfrac{1}{60}X_{;\beta}W^{\alpha\beta}{}_{;\alpha}-\tfrac{1}{60}W^{\alpha\beta}{}_{;\alpha}X_{;\beta}+\tfrac{1}{20}XW_{\alpha\beta}W^{\alpha\beta}-\tfrac{1}{60}W_{\alpha\beta}XW^{\alpha\beta}-\tfrac{1}{30}W_{\alpha\beta}W^{\alpha\beta}X\ . \quad \text{(A6)}$$

In the main text we considered explicitly the usual propagators for fields Q of spin 0, $\frac{1}{2}$, or 1 satisfying equations like (5.3) where the operator $\Box+X$ is derived simply from the kinetic part of the action $I[Q]$, in other words, the part quadratic in Q. However, the Schwinger-DeWitt or heat-kernel expansion is also a very powerful tool when used in conjunction with the background-field method.[1,4,12] In this case the field Q is written

$$Q=\hat{q}+q\ , \quad \text{(A7)}$$

where now q is the quantum field and $\hat{q}$ a classical background field. While some authors who work with the background-field method find it convenient to continue to use the usual propagator as before,[13,14] so that all the terms in the expansion of $I[\hat{q}+q]$ containing $\hat{q}$ are considered as interactions, some calculations are simplified by using the full propagator in the presence of the background fields,[1,4,15,16] so that the operator $\Box+X$ is that which is associated with the full set of terms quadratic in q in the expansion of $I[\hat{q}+q]$, and then X and D_μ may acquire contributions from the background fields $\hat{q}$. In this appendix we will consider in detail the modifications induced by including a background gauge field A_μ, but no other background fields. The covariant derivative in (5.1) will now become in general,

$$D_\mu=\nabla_\mu 1_{\mathscr{G}}-ie_gA_\mu\ , \quad \text{(A8)}$$

where A_μ is the matrix-valued vector-gauge connection with coupling constant e_g. The forms of $W_{\alpha\beta}$ and X for each spin will now be modified and are displayed below. (Scalar and spinor background fields may also be introduced without further alteration in $W_{\alpha\beta}$, but the forms of X will then be more complicated and are listed elsewhere in the literature.[16]) We shall consider each spin in turn.

A. Spin 0

Let us suppose we have an n-dimensional multiplet of scalar fields transforming under a representation of the gauge group with generators t_a^ϕ. These generators are chosen to be Hermitian and satisfy

$$[t_a^\phi,t_b^\phi]=if_{abc}t_c^\phi\ . \quad \text{(A9)}$$

The propagator then satisfies an equation of the form (5.3) with the covariant derivative given by (A8) with now

$$A_\mu=A_{\mu,a}t_a^\phi\ , \quad \text{(A10)}$$

where $A_{\mu,a}$ are the components of the gauge field. Consequently, $W_{\alpha\beta}$, as defined in (5.2), is given by

$$W_{\alpha\beta}=-ie_gF_{\alpha\beta}^\phi,\quad F_{\alpha\beta}^\phi=F_{\alpha\beta,a}t_a^\phi\ , \quad \text{(A11)}$$

where

$$F_{\alpha\beta,a}=\partial_\alpha A_{\beta,a}-\partial_\beta A_{\alpha,a}+e_gf_{abc}A_{\alpha,b}A_{\beta,c}\ . \quad \text{(A12)}$$

The form of X depends on the type of scalar fields we are considering. For instance, when they are anticommuting Faddeev-Popov ghost fields, then $X=0$ and t_a^ϕ are the generators of the adjoint representation

$$t_a^\phi=t_a^{\text{adj}},\quad (t_a^{\text{adj}})_{bc}=-if_{abc}\ . \quad \text{(A13)}$$

On the other hand, for a multiplet of Higgs fields, then

$$X=-M^2+\xi R 1\ , \quad \text{(A14)}$$

where M^2 is the mass-squared matrix. The properties of the expansion (5.7) then imply that all the M^2 dependence is contained in the exponential, which could be seen directly as for the scalar field propagator without background gauge field in (2.2).

B. Spin $\frac{1}{2}$.

For a multiplet of spinor fields transforming under a representation of $\mathscr{G}$ with Hermitian generators t_a^ψ with commutation relations as for t_a^ϕ in (A9), the covariant derivative has the form (A8) with

$$A_\mu=A_{\mu,a}t_a^\psi \quad \text{(A15)}$$

and we find

$$\begin{aligned}&W_{\alpha\beta}=W_{\alpha\beta}^{\text{curv}}+W_{\alpha\beta}^{\text{gauge}}\ ,\\ &W_{\alpha\beta}^{\text{curv}}=-\tfrac{1}{4}R_{\alpha\beta\rho\sigma}\tilde{\gamma}^{\rho}\tilde{\gamma}^{\sigma},\quad W_{\alpha\beta}^{\text{gauge}}=-ie_gF_{\alpha\beta}^\psi 1\ ,\\ &X=X^{\text{curv}}+X^{\text{gauge}}\ ,\\ &X^{\text{curv}}=\tfrac{1}{4}R\,1\otimes 1_{\mathscr{G}},\quad X^{\text{gauge}}=-ie_g\sigma^{\mu\nu}F_{\mu\nu}^\psi\ ,\end{aligned} \quad \text{(A16)}$$

where

$$F_{\mu\nu}^\psi=F_{\mu\nu,a}t_a^\psi,\quad \sigma^{\mu\nu}=\tfrac{1}{4}[\tilde{\gamma}^{\mu},\tilde{\gamma}^{\nu}]\ , \quad \text{(A17)}$$

where $\tilde{\gamma}^{\rho}$ is defined in (5.12) and $F_{\mu\nu,a}$ in (A12).

Clearly the terms involving X will still contribute no R dependence to $\bar{g}_j$ in (5.7) and its derivatives. However X^{gauge} does not commute with $W^{\text{gauge}}_{\alpha\beta}$ or its derivatives, or with derivatives of X^{gauge}, and hence there will in general be sets of terms in $\bar{g}_j$ and its derivatives containing products of X^{gauge} together with W^{gauge} or its derivatives, or derivatives of X^{gauge}.

C. Spin 1

For a spin-1 field, in order to obtain an equation of the form (5.3), when we include the background gauge field, we need to use the so-called background field gauge.[14] This leads in general to a propagator Δ_ζ satisfying

$$\left[\delta^\mu{}_\lambda\Box+\left[\frac{1}{\zeta}-1\right]D^\mu D_\lambda+X^\mu{}_\lambda\right]\Delta_\zeta{}^\lambda{}_\nu=-\delta^\lambda{}_\nu\delta(x,x'), \quad (\text{A18})$$

where ζ is a variable gauge-fixing parameter, and the covariant derivative D_μ is given by (A8) with

$$A_\mu=A_{\mu,a}t_a^{\text{adj}} , \quad (\text{A19})$$

t_a^{adj} being defined in (A13). We have in this case

$$(W_{\alpha\beta})_{\mu\nu}=-R_{\alpha\beta\mu\nu}1_{\mathscr{G}}-ie_gF^{\text{adj}}_{\alpha\beta}\delta_{\mu\nu} ,$$
$$X_{\mu\nu}=R_{\mu\nu}1_{\mathscr{G}}-2ie_gF^{\text{adj}}_{\mu\nu} , \quad (\text{A20})$$

where

$$F^{\text{adj}}_{\mu\nu}=F_{\mu\nu,a}t_a^{\text{adj}} . \quad (\text{A21})$$

Evidently to obtain the form (5.3) we must take $\zeta=1$.

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