# Strong-coupling quantum gravity. III. Quasiclassical approximation

G. Francisco*
*Imperial College —Theoretical Physics Group, London SW7 2BZ, England*

M. Pilati
*Department of Physics, University of Utah, Salt Lake City, Utah 84112*
(Received 17 July 1984)

The ultralocal representation of gravity is used to obtain a quasiclassical approximation for the evolution of wave functionals towards the initial singularity. The whole dynamical content of this process is encoded in the successive asymptotic scatterings of Kasner wave functions by the gravitational potential term $V=gR$. It is indicated how this can be used to prepare the state (density operator) of the gravitational field in the immediate neighborhood of the singularity.

## I. INTRODUCTION

In previous papers[1,2] the strong-coupling limit of general relativity was studied. In particular, in Ref. 1 a choice of gauge was made while in Ref. 2 the general gauge invariance was maintained.

The strong-coupling limit is obtained by taking the usual constraint Hamiltonian generator

$$\mathscr{H}_\perp=\frac{1}{\sqrt{g}}\left[\kappa G_{ijkl}\pi^{ij}\pi^{kl}-\frac{1}{\kappa}gR\right], \tag{1.1}$$

$$G_{ijkl}=\tfrac{1}{2}(g_{ik}g_{jl}+g_{il}g_{jk}-g_{ij}g_{kl}) \tag{1.2}$$

($g=\det g_{ij}$, $R$ is the scalar curvature of $g_{ij}$, $\pi^{ij}$ is the momentum conjugate to $g_{ij}$, and $\kappa=16\pi G/c^3$) and replacing it by[1-4]

$$\mathscr{H}_0=G_{ijkl}\pi^{ij}\pi^{kl}\ . \tag{1.3}$$

Roughly we have taken $G\to\infty$ in (1.1), whence the name strong-coupling limit.

Since the spatial derivatives of the metric tensor are all contained in the term $gR$ of (1.1) we find that the dynamics generated by (1.3) is characterized by the absence of field correlations between neighboring spatial points, in other words the light cone has collapsed to a line. The concept of ultralocal representations[5,6] offer an ideal framework to study the strong-coupling limit and the quantization of $\mathscr{H}_0$ was already obtained[1,2] (ultralocal means no correlation between spatial points). However, a perturbative technique has to be found so that the term $gR$ can be added to $\mathscr{H}_0$, but as yet it is not known how to represent products of spatial derivatives of the metric tensor.

The absence of physical interpretation of the abstract formalism developed thus far is a compelling reason to study its quasiclassical behavior. The approximation scheme involved here is suggested by the kind of representation in which the strong-coupling limit was expressed in Refs. 1 and 2 (see also Ref. 7 on exponential representations). The way in which this is done in Sec. III allows us to exploit the large amount of information about cosmological models available in the literature.[8-17] It is hoped that such a study will provide us with a better insight into the physics occurring in the full quantum theory.

The physical arguments that led to the discussion of configuration-space $S$-matrix theory in Ref. 2 are again invoked in Sec. IV in connection with classical cosmological solutions. Ordinary $S$-matrix theory deals with asymptotic particle states containing information about some interacting potential. When classical cosmology is understood as a scattering problem in superspace[8-10] then fruitful analogies emerge. The asymptotic behavior of the metric is described by a Kasner universe. The cosmological problem is encoded in the way which Kasner universes are scattered by the gravitational potential $V=gR$.

As a motivation for the quasiclassical approximation described in Sec. III we outline here its quantum-mechanical[18,19] counterpart. Boson and ultralocal scalar fields have already been discussed elsewhere.[18,20]

Suppose that $P$ and $Q$ are operators acting on a Hilbert space $H$ and satisfying the usual canonical commutation relation $[Q,P]=i\hbar$. Assume there is a vector $|0\rangle\in H$ such that $\langle 0|Q|0\rangle=0=\langle 0|P|0\rangle$ and

$$|p,q\rangle=e^{-(i/\hbar)qP}e^{(i/\hbar)pQ}|0\rangle \tag{1.4}$$

span $H$ and denote by $S$ the set of all vectors of the form (1.4) where $p$ and $q$ are $c$ numbers. It can be shown that $S$ is an overcomplete family of states.

The action principle for quantum mechanics is based on arbitrary variations of $\Psi$ in

$$I=\int dt\left[\Psi,\left[i\hbar\frac{d}{dt}-\hat{\mathscr{H}}\right]\Psi\right] \tag{1.5}$$

and gives the Schrödinger equation of motion

$$i\hbar\frac{d}{dt}\Psi=\hat{\mathscr{H}}\Psi\ , \tag{1.6}$$

where $\hat{\mathscr{H}}(P,Q)$ is the Hamiltonian of the system. If, instead of general states $\Psi(t)\in H$ we use states from $S$ the resulting variational principle, called *restricted action principle,*[19] based on arbitrary variations of $p$ and $q$ in

 

$$I=\int dt\left\langle p,q\left|\left[i\hbar\frac{d}{dt}-\hat{\mathscr{H}}\right]\right|p,q\right\rangle , \quad (1.7)$$

gives the equations of motion

$$\dot{q}_{\rm cl}=\frac{\partial\mathscr{H}(p,q)}{\partial p},\ \dot{p}_{\rm cl}=-\frac{\partial\mathscr{H}(p,q)}{\partial q} . \quad (1.8)$$

Notice that

$$\mathscr{H}(p,q)=\langle p,q\,|\,\hat{\mathscr{H}}\,|\,p,q\rangle \quad (1.9)$$

is playing the role of a "classical" Hamiltonian responsible for the evolution of the parameters $p,q$ (for ultralocal scalar fields and quantum gravity these parameters turn out to be smearing functions). This is essentially the content of Klauder's *weak correspondence principle*:[21] given a quantum generator $\hat{\mathscr{G}}(P,Q)$ then its classical counterpart is given weakly by

$$\mathscr{G}(p,q)=\langle p,q\,|\,\hat{\mathscr{G}}\,|\,p,q\rangle . \quad (1.10)$$

Now we relate the "classical" evolution (1.8) to the quantum evolution of the states.

It is important to notice that if $\exp[-(i/\hbar)\hat{\mathscr{H}}t]\,|\,p,q\rangle$ is an element of $S,\forall t$, then the restricted action principle is equivalent to the full action principle. In this case

$$e^{-(i/\hbar)\hat{\mathscr{H}}t}\,|\,p,q\rangle=|\,p_{\rm cl}(t),q_{\rm cl}(t)\rangle , \quad (1.11)$$

where $p_{\rm cl}(t),q_{\rm cl}(t)$ are solutions to (1.8) satisfying the initial conditions $p_{\rm cl}(0)=p,q_{\rm cl}(0)=q$. Systems satisfying the evolution (1.11) are called *exact*. In general, however, (1.11) is not true but a quasiclassical approximation to the full quantum evolution of any system can be found[18] to be

$$e^{-(i/\hbar)\hat{\mathscr{H}}t}\,|\,p,q\rangle\overset{\hbar\to 0}{\approx}|\,p_{\rm cl}(t),q_{\rm cl}(t)\rangle . \quad (1.12)$$

This is essentially the formula we intend to study for quantum gravity and to apply in some concrete circumstances.

Briefly the paper is organized as follows. Section II contains a summary of ultralocal quantization of gravity in a fixed gauge. In Sec. III the quasiclassical approximation is presented. This will involve an approximation to the quantum evolution of wave functionals under $\hat{\mathscr{H}}_\perp=\hat{\mathscr{H}}_0+\hat{V}$. Although no representation $\hat{V}$ for the scalar curvature term $V=gR$ has yet been found we conjecture that a similar expression as (1.12) must hold in quantum gravity. Section IV contains more details of how the test functions that smear quantum operators are related to classical cosmologies. Also the concept of density matrix, or state of the system, for the mixmaster universe is introduced. In this model there is no $\vec{x}$ dependence in the smearing functions and the problem is reduced to finite-dimensional quantum mechanics. Finally we make some pertinent remarks about the inhomogeneous case when the smearing functions are $\vec{x}$ dependent.

Throughout the paper we denote by $\vec{x}$ a point in a compact hypersurface $\Sigma$; $g_{ij}(\vec{x})$ is the spatial metric tensor on $\Sigma$ with signature $(+++),i,j,=1,2,3$, in other words $g_{ij}$ is positive definite on $\Sigma$. Sometimes the word "state" means wave function(al) instead of its correct usage as density matrix but the context will make clear which concept is intended.

## II. QUANTIZATION IN A FIXED GAUGE

In the canonical approach to gravity when one tries to quantize the theory in terms of the metric $g_{ij}(\vec{x})$ and its conjugate momenta $\pi^{ij}(\vec{x})$, it is soon found that these operators cannot be Hermitian and at the same time have positive-definite spectrum.[1]

One way of solving this problem is to express the theory in terms of

$$\pi^i_j=\tfrac{1}{2}(g_{jk}\pi^{ik}+\pi^{ik}g_{jk}) . \quad (2.1)$$

with this choice the spectrum of $g_{ij}$ can be taken to have signature $(+++)$.

A convenient rearrangement of the variables $g_{ij}$, $\pi^k_l$ is obtained when we identify an intrinsic time[22,23] among the components of $g_{ij}$,

$$\tau=\tfrac{1}{3}\ln g , \quad (2.2)$$

$$\tilde{g}_{ij}=g^{-1/3}g_{ij} , \quad (2.3)$$

$$\pi=\pi^i_i , \quad (2.4)$$

$$P^i_j=\pi^i_j-\tfrac{1}{3}\pi\delta^i_j , \quad (2.5)$$

where $g=\det g_{ij}$. Note that

$$\det\tilde{g}_{ij}=1 , \quad (2.6)$$

$$P^i_i=0 . \quad (2.7)$$

The only nonvanishing Poisson brackets are

$$[\tau,\pi]=1 , \quad (2.8)$$

$$[\tilde{g}_{ij},P^k_l]=\tfrac{1}{2}(\tilde{g}_{il}\delta^k_j+\tilde{g}_{jl}\delta^k_i) , \quad (2.9)$$

$$[P^i_j,P^k_l]=\tfrac{1}{2}(P^i_l\delta^k_j-P^k_j\delta^i_l) , \quad (2.10)$$

where we omitted factors of $\delta(\vec{x},\vec{x}\,')$; also on the right-hand side one should include $i\hbar$ for quantum-mechanical operators. Notice that $P^i_j$ has the same commutation relations as the generators of the group $SL(3,R)$.

These variables can now be used to reexpress the ultralocal Hamiltonian of the Introduction:

$$\mathscr{H}_0=P^i_jP^j_i-\tfrac{1}{6}\pi^2 . \quad (2.11)$$

Studies of ultralocal cosmological models (or velocity-dominated cosmologies[11,12]) have revealed that only three components of the metric tensor are necessary to specify the dynamical evolution. The fact that three, out of six, components of the metric are relevant in velocity-dominated cosmological models appear to suggest that ultralocality has in some sense reduced the number of degrees of freedom. Strictly speaking, however, no constraint of the theory has really been solved. Rather this reduction is due to the fact that Einstein's equations $R^i_j=0$ plus ultralocality assumption imply that the metric tensor diagonalized on the initial hypersurface $\Sigma$ stays diagonalized when evolved out of $\Sigma$. In other words, $g_{ij}$ can be diagonalized by time-independent triads on $\Sigma$ and, as any $3\times 3$ matrix, only three components need to be considered.

The gauge fixation presented in the first paper of this series seems most adequate to fully explore this peculiarity of velocity-dominated models: for instance, our Hamiltonian (2.16) is precisely the Hamiltonian in Ref. 12, Eq. (5). With this motivation in mind we introduce the gauge-fixed variables.

Among the six components of the metric tensor, three are related to the arbitrariness of the choice of coordinate system on $\Sigma$, two are the true degrees of freedom and one has been identified as an intrinsic time [like $\tau$ in (2.2)] related to the constraint $\mathscr{H}_0$. Let $S$ be the five-dimensional space of all $\tilde{g}_{ij}$ at a given $\vec{x}\in\Sigma$, that is, the space of symmetric positive-definite matrices of unit determinant [see (2.6)]. Then a gauge fixing is obtained by finding a two-dimensional submanifold $E$ of $S$ parametrized by some functions of $\tilde{g}_{ij}$, which we call $t_1$ and $t_2$, and by choosing among the $P^i_j$ two conjugate variables to these functions. Three out of the eight $P^i_j$ generators of SL(3,$R$) [those corresponding to the SO(3) subgroup] can be eliminated by a proper choice of the submanifold $E$. Three more are eliminated by gauge conditions and we are left with two independent combinations $\pi_1,\pi_2$ of the $P^i_j$. It can be shown[1] that

$$[t_1,\pi_1]=1=[t_2,\pi_2]\ , \tag{2.12}$$

$$[t_i,t_i]=0=[\pi_i,\pi_i],\quad i=1 \text{ or } 2\ . \tag{2.13}$$

It is convenient to redefine

$$\pi_+=\sqrt{2}(\pi_1+\pi_2),\quad \pi_-=\sqrt{6}(\pi_1-\pi_2)\ , \tag{2.14}$$

$$t_+=\frac{1}{\sqrt{8}}(t_1+t_2),\quad t_-=\frac{1}{\sqrt{24}}(t_1-t_2)\ , \tag{2.15}$$

and to take the pairs $(t_+,\pi_+),(t_-,\pi_-)$ as our gauge-fixed independent variables.

The Hamiltonian in terms of these variables has a simple form

$$\mathscr{H}_0=\pi_+{}^2+\pi_-{}^2-\pi^2\ , \tag{2.16}$$

where, for convenience, we have dropped numerical factors from the right-hand side of (2.16). Next we quantize this velocity-dominated Hamiltonian using ultralocal techniques[5,6] and in the next section (2.6) will be recovered [see (3.9)].

The field operators suitable for the ultralocal limit of gravity are expressed in terms of creation and annihilation operators $A^\dagger,A$ that act on a Fock Hilbert space $H$ possessing a state $|0\rangle$ such that

$$[A(\vec{x};\beta_+,\beta_-,\Omega),A^\dagger(\vec{x}\,';\beta'_+B'_-,\Omega')]=\delta(\vec{x}-\vec{x}\,')\delta(\beta_+-\beta'_+)\delta(\beta_--\beta'_-)\delta(\Omega-\Omega')\ , \tag{2.17}$$

$$[A(\vec{x};\beta_+,\beta_-,\Omega),A(\vec{x}\,';\beta'_+,\beta'_-,\Omega')]=0\ , \tag{2.18}$$

$$[A^\dagger(\vec{x};\beta_+,\beta_-,\Omega),A^\dagger(\vec{x}\,';\beta'_+,\beta'_-,\Omega')]=0\ , \tag{2.19}$$

$$A\,|0\rangle=0\ , \tag{2.20}$$

where $\beta_+,\beta_-,\Omega$ can vary from $-\infty$ to $+\infty$. In the gauge already described the quantized fields of the theory are

$$\hat{t}_\pm(\vec{x})=\int d\beta_+d\beta_-d\Omega\, B^\dagger(\vec{x};\beta_+,\beta_-,\Omega)\beta_\pm B(\vec{x};\beta_+,\beta_-,\Omega)\ , \tag{2.21}$$

$$\hat{\pi}_\pm(\vec{x})=\int d\beta_+d\beta_-d\Omega\, B^\dagger(\vec{x};\beta_+,\beta_-,\Omega)\frac{\hbar}{i}\frac{\partial}{\partial\beta_\pm}B(\vec{x};\beta_+,\beta_-,\Omega)\ , \tag{2.22}$$

$$\hat{\tau}(\vec{x})=\int d\beta_+d\beta_-d\Omega B^\dagger(\vec{x};\beta_+,\beta_-,\Omega)\Omega B(\vec{x};\beta_+,\beta_-,\Omega)\ , \tag{2.23}$$

$$\hat{\pi}(\vec{x})=\int d\beta_+d\beta_-d\Omega\, B^\dagger(\vec{x};\beta_+,\beta_-,\Omega)\frac{\hbar}{i}\frac{\partial}{\partial\Omega}B(\vec{x};\beta_+,\beta_-,\Omega)\ , \tag{2.24}$$

where

$$B(\vec{x};\beta_+,\beta_-,\Omega)=A(\vec{x};\beta_+,\beta_-,\Omega)+C(\beta_+,\beta_-)\ . \tag{2.25}$$

The function $C(\beta_+,\beta_-)$, called *model function,*[1,2,5,6] labels the representations and determines many of their properties; we assume it is not square integrable. This guarantees that $B$ and $A$ are not unitarily equivalent, and that the field operators have continuous spectrum.

An overcomplete family of states analogous to (1.4) is

$$|p_+,p_-,\omega;q_+,q_-,\eta\rangle=\exp\left[-\frac{i}{\hbar}\int d^3x(p_+\hat{t}_++p_-\hat{t}_--\omega\hat{\Omega})\right]\exp\left[\frac{1}{\hbar}\int d^3x(q_+\hat{\pi}_++q_-\hat{\pi}_--\eta\hat{\pi})\right]|0\rangle\ , \tag{2.26}$$

where $p_+,p_-,\omega;\ q_+,q_-,\eta$ are infinitely differentiable real smearing functions on $\Sigma$. We call (2.26) *coherent states* (coherent in the sense that they are eigenstates of the annihilation operator).

The Hamiltonian is now expressed as[1]

$$\hat{\mathscr{H}}_0(\vec{x})=-\hbar^2\int d\beta_+d\beta_-d\Omega\, B^\dagger\left[\frac{\partial^2}{\partial\beta_+{}^2}+\frac{\partial^2}{\partial\beta_-{}^2}-\frac{\partial^2}{\partial\Omega^2}-V(\beta_+,\beta_-)\right]B\ , \tag{2.27}$$

where $V(\beta_+,\beta_-)$ is included to guarantee that

$$\hat{\mathscr{H}}_0(\vec{x})\,|\,0\rangle = 0 \tag{2.28}$$

and thus depends on $C(\beta_+,\beta_-)$. Our choice is the simplest, $C=1$, in which case $V=0$ (other choices are discussed in Ref. 2).

## III. QUASICLASSICAL APPROXIMATION

In this section we intend to elaborate an approximation to the evolution of the quantum states under the full Hamiltonian $\hat{\mathscr{H}}_0+\hat{V}$. Here $\hat{V}$ corresponds to a representation of the classical term $gR$ in (1.1). Since this representation is not known yet we study first the evolution under $\hat{\mathscr{H}}_0$ alone. It is then conjectured how to incorporate $\hat{V}$ in a quasiclassical way.

In superspace quantization of gravity[24] the classical constraints of the theory should become operators annihilating the physical states. Thus the classical ultralocal Hamiltonian constraint (1.3) is implemented as

$$\hat{\mathscr{H}}_0\Psi = 0\ . \tag{3.1}$$

The span of the subset of $H$ satisfying (3.1) is called the *physical subspace*. We would like to restrict the theory so that we only talk about physical states. However, it was recognized that this is not convenient[1] because the physical subspace is not closed under the action of the field operators. A way to circumvent this problem is to introduce an auxiliary time parameter $\theta$, to be identified as a proper time (more on this in Sec. IV), and use the heatlike equation

$$\hat{\mathscr{H}}_0\Psi = i\hbar\frac{\partial}{\partial\theta}\Psi \tag{3.2}$$

with $\hat{\mathscr{H}}_0=\int d^3x\,\hat{\mathscr{H}}_0(\vec{x})$. Notice that this equation can also be obtained from the quantum action functional

$$I=\int\left[\Psi,\left[i\hbar\frac{\partial}{\partial\theta}-\hat{\mathscr{H}}_0\right]\Psi\right], \tag{3.3}$$

where (,) is the inner product of $H$. If an initial condition $\Psi_0\equiv\Psi(0)$ is given, then the unique solution to (3.2) can be expressed as

$$e^{-(i/\hbar)\hat{\mathscr{H}}_0\theta}\Psi_0=\Psi(\theta)\ . \tag{3.4}$$

Thus $\hat{\mathscr{H}}_0$ plays the role of a Hamiltonian generator giving the evolution of the states $\Psi$ in terms of time $\theta$, (3.3) being the associated Schrödinger equation. Next we exhibit the relationship between (3.4) and the evolution of coherent states.

We will use the restricted action principle and the weak correspondence principle to study the evolution of the states (2.26). Writing the set of smearing functions as

$$\begin{aligned} p(\theta)&=(p_+(\theta),p_-(\theta),\omega(\theta))\ ,\\ q(\theta)&=(q_+(\theta),q_-(\theta),\eta(\theta))\ , \end{aligned} \tag{3.5}$$

the restricted action is

$$\begin{aligned} I&=\int d\theta\left\langle p(\theta),q(\theta)\left|\left[i\hbar\frac{\partial}{\partial\theta}-\hat{\mathscr{H}}_0\right]\right|p(\theta),q(\theta)\right\rangle\\ &=\left\langle 0\left|\int d\theta\,d^3x\,d\beta_+d\beta_-d\Omega\,B^\dagger(p_+\dot{q}_++p_-\dot{q}_-+\omega\dot{\eta})B\right|0\right\rangle-\int\langle p(\theta),q(\theta)\,|\,\hat{\mathscr{H}}_0\,|\,p(\theta),q(\theta)\rangle d\theta\\ &=N\int d\theta\,d^3x(p_+\dot{q}_++p_-\dot{q}_-+\omega\dot{\eta})-N\int d\theta\mathscr{H}_0^{\rm cl}(p(\theta),q(\theta))\ . \end{aligned} \tag{3.6}$$

The overdot means $\theta$-time derivative and

$$N=\left\langle 0\left|\int d\beta_+d\beta_-d\Omega\,B^\dagger B\right|0\right\rangle=\int d\beta_+d\beta_-d\Omega\,C(\beta_+,\beta_-)^2\ , \tag{3.7}$$

$$\langle p(\theta),q(\theta)\,|\,\hat{\mathscr{H}}_0\,|\,p(\theta),q(\theta)\rangle=\mathscr{H}_0^{\rm cl}(p(\theta),q(\theta))=\int d^3x\,\mathscr{H}_0^{\rm cl}(p(\theta,\vec{x}))\ , \tag{3.8}$$

$$\mathscr{H}_0^{\rm cl}(p(\theta,\vec{x}))=\tfrac{1}{2}[p_+{}^2(\theta,\vec{x})+p_-{}^2(\theta,\vec{x})-\omega^2(\theta,\vec{x})]\ . \tag{3.9}$$

From the weak correspondence principle $\mathscr{H}_0^{\rm cl}(p(\theta),q(\theta))$ in (3.8) is taken as a classical Hamiltonian associated to the quantum generator $\hat{\mathscr{H}}_0$. Observe that the model function $C(\beta_+,\beta_-)$ is not square integrable and thus $N$ is divergent (in fact our choice is $C=1$). Since this is only a multiplicative constant it cannot affect the extremal solutions to $\delta I=0$ and we drop it from (3.6). After minimizing the resulting "regularized" action we get

$$\dot{q}_+(\theta,\vec{x})=\frac{\delta\mathscr{H}_0^{\rm cl}(p(\theta),q(\theta))}{\delta p_+(\theta,\vec{x})},\quad \dot{p}_+(\theta,\vec{x})=-\frac{\delta\mathscr{H}_0^{\rm cl}(p(\theta),q(\theta))}{\delta q_+(\theta,\vec{x})}\ , \tag{3.10}$$

$$\dot{q}_-(\theta,\vec{x})=\frac{\delta\mathscr{H}_0^{\rm cl}(p(\theta),q(\theta))}{\delta p_-(\theta,\vec{x})},\quad \dot{p}_-(\theta,\vec{x})=-\frac{\delta\mathscr{H}_0^{\rm cl}(p(\theta),q(\theta))}{\delta q_-(\theta,\vec{x})}\ , \tag{3.11}$$

$$\dot{\eta}(\theta,\vec{x})=\frac{\delta\mathscr{H}_0^{\rm cl}(p(\theta),q(\theta))}{\delta\omega(\theta,\vec{x})},\quad \dot{\omega}(\theta,\vec{x})=-\frac{\delta\mathscr{H}_0^{\rm cl}(p(\theta),q(\theta))}{\delta\eta(\theta,\vec{x})}\ . \tag{3.12}$$

The classical system (3.10)–(3.12) describes the evolution of the smearing functions in terms of the classical Hamiltonian $\mathscr{H}_0^{\rm cl}(p(\theta),q(\theta))$ which we also write as

$$\mathscr{H}_0^{\rm cl}(p(\theta),q(\theta))=\mathscr{H}_0^{\rm cl}(p_+,p_-,\omega) \tag{3.13}$$

because it does not depend on the $q$ variables. When (3.13) is constrained to zero this system has exactly the same structure as the velocity-dominated models of general relativity.[12] Such models describe situations in which spatial derivatives of the metric are negligible as compared with its time derivatives. This means the three-dimensional Ricci tensor can be dropped from the evolution Einstein equations [Ref. 13, Eq. (3.15)]. Equivalently one drops the three-dimensional scalar Ricci curvature term $gR$ from the Hamiltonian [Ref. 12, Eq. (20)] as in the Introduction.

Suppose now that we take the initial condition in (3.4) to be a coherent state $\Psi\equiv\Psi(0)=|p_0,q_0\rangle$. Then the evolved state $\Psi(\theta)$ is not in general a coherent state. However, a property of the ultralocal system under discussion is that $\Psi(\theta)$ *is* a coherent state (i.e., the system is exact) and consequently (3.10)–(3.12) fully specify the quantum evolution and no approximations are necessary. To see this set $\Psi(\theta)=|p(\theta),q(\theta)\rangle$ and compute both sides of (3.2):

$$i\hbar\frac{\partial}{\partial\theta}\exp\left[-\frac{i}{\hbar}\int d^3x(p_+\hat{t}_++p_-\hat{t}_--\omega\hat{\tau})\right]\exp\left[\frac{i}{\hbar}\int d^3x(q_+\hat{\pi}_++q_-\hat{\pi}_--\eta\hat{\pi})\right]|0\rangle$$

$$=\exp\left[-\frac{i}{\hbar}\int d^3x(p_+\hat{t}_++p_-\hat{t}_--\omega\hat{\tau})\right]\int d^3x(\dot{p}_+\hat{t}_++\dot{p}_-\hat{t}_--\dot{\omega}\hat{\tau}-\dot{q}_+\hat{\pi}_+-\dot{q}_-\hat{\pi}_-+\eta\hat{\pi})$$
$$\times\exp\left[\frac{1}{\hbar}\int d^3x(q_+\hat{\pi}_++q_-\hat{\pi}_--\eta\hat{\pi})\right], \tag{3.14a}$$

$$\hat{\mathscr{H}}_0\exp\left[-\frac{i}{\hbar}\int d^3x(p_+\hat{t}_++p_-\hat{t}_--\omega\hat{\tau})\right]\exp\left[\frac{i}{\hbar}\int d^3x(q_+\hat{\pi}_++q_-\hat{\pi}_--\eta\hat{\pi})\right]|0\rangle$$

$$=\exp\left[-\frac{i}{\hbar}\int d^3x(p_+\hat{t}_++p_-\hat{t}_--\omega\hat{\tau})\right]\left\{\hat{\mathscr{H}}_0-\int d^3x\left[(p_+\hat{\pi}_++p_-\hat{\pi}_--\omega\hat{\pi})+\int d\beta_+d\beta d\Omega\,B^\dagger(p_+{}^2+p_-^2-\omega^2)B\right]\right\}$$
$$\times\exp\left[\frac{i}{\hbar}\int d^3x(q_+\hat{\pi}_++q_-\hat{\pi}-\eta\hat{\pi})\right]|0\rangle\,. \tag{3.14b}$$

Now using (3.2), (3.14), (2.28), and the constraint $p_+{}^2+p_--\omega^2=0$, we obtain

$$\begin{aligned}&\dot{q}_+=p_+,\ \dot{p}_+=0\,,\\&\dot{q}_-=p_-,\ \dot{p}_-=0\,,\\&\dot{\eta}=-\omega,\ \dot{\omega}=0\,,\end{aligned} \tag{3.15}$$

which is precisely the system (3.10)–(3.12). Thus the evolution of the smearing functions retains the full content of the quantum evolution and we write

$$e^{-(i/\hbar)\hat{\mathscr{H}}_0\theta}|p_0,q_0\rangle=|p(\theta),q(\theta)\rangle\,, \tag{3.16}$$

where $|p(0),q(0)\rangle=|p_0,q_0\rangle$.

The evolution we want to study however does not take place under $\hat{\mathscr{H}}_0$ as in (3.16) but is given by

$$e^{-(i/\hbar)(\hat{\mathscr{H}}_0+\hat{V})\theta}\Psi=\Psi(\theta)\,, \tag{3.17}$$

where $\Psi(\theta)$ is not a coherent state and a representation $\hat{V}$ for the classical term $V=gR$ is not known. Next we conjecture how to approximate the right-hand side of (3.17) using coherent states.

It has already been shown that boson[18] and ultralocal scalar fields[20] have approximate evolution analogous to the quantum-mechanics case (1.12). From these considerations it is natural to suppose that a similar approximation technique holds for quantum gravity in that the right-hand side of (3.17) is approximately

$$e^{-(i/\hbar)(\hat{\mathscr{H}}_0+\hat{V})\theta}|p_0,q_0\rangle\overset{\hbar\to 0}{\approx}|p(\theta),q(\theta)\rangle\,. \tag{3.18}$$

Here the classical Hamiltonian responsible for the evolution of $p(\theta),q(\theta)$ must contain, according to the weak correspondence principle and the restricted action, two terms:

$$\begin{aligned}\mathscr{H}_{\rm cl}&\equiv\mathscr{H}_{\rm cl}(p(\theta),q(\theta))\\&=\langle p(\theta),q(\theta)|\langle\hat{\mathscr{H}}_0+\hat{V}\rangle|p(\theta),q(\theta)\rangle\,.\end{aligned} \tag{3.19}$$

The first term $\langle p(\theta),q(\theta)|\hat{\mathscr{H}}_0|p(\theta),q(\theta)\rangle$ was shown [(3.9) and (3.13)] to have the same structure [see (2.16)] as the velocity-dominated cosmological Hamiltonian[12] and corresponds to the term $G_{ijkl}\pi^{ij}\pi^{kl}$ of (1.1). As to the second term, whatever representation is found for $V$, it must satisfy, if the weak correspondence principle is to hold,

$$\langle p(\theta),q(\theta)\,|\,\hat{V}\,|\,p(\theta),q(\theta)\rangle \overset{\hbar\to 0}{\approx} gR \ . \tag{3.20}$$

The conclusion then is that $\mathscr{H}_{\rm cl}$ has the same functional form as $\mathscr{H}_{\perp}$ in (1.1) when $\hbar\to 0$, so that $p(\theta),q(\theta)$ are related to the classical solutions to Einstein equations. A more complete discussion of the behavior of these smearing functions is presented next.

## IV. APPLICATIONS

In the previous section an approximation scheme to the evolution of quantum states has been described and the solutions to Einstein dynamical equations played a prominent role. Here we give more details of how these classical solutions are incorporated in the approximation and then close the discussion by giving some applications.

It is a known result that a large class of solutions to Einstein equations are velocity dominated near the initial (cosmological) singularity,[11] i.e., the spatial derivatives of the metric can be dropped since they are much smaller than the time derivatives. It is thus useful to think of the ultralocal quantization of gravity as the quantization one would have got if the gravitational field was quantized sufficiently near the singularity. This fixes the physical asymptotic region in which the ultralocal perturbation theory applies.

A very detailed qualitative description of the classical solutions near a spacelike singularity has been obtained by Belinskii, Khalatnikov, and Lifschitz,[13–15] henceforth referred to as BKL (see also the Hamiltonian treatment of Liang,[12] referred to as L). Notice that in BKL solutions a gauge fixation is involved. They work throughout with a synchronous reference frame where the classical $g_{0\mu}$ components of the metric ($\mu=0,\ldots,3$) satisfy the conditions

$$g_{0i}=0,\ \ g_{00}=-1 \ . \tag{4.1}$$

Thus the line element of spacetime $\Sigma\times\mathbb{R}$ in a covector basis $\{\tau^i\}$ that diagonalizes the metric on $\Sigma$ is

$$ds^2=-dt^2+g_{ij}\tau^i\otimes\tau^j \ . \tag{4.2}$$

From (4.1) it is clear that the proper time parameter $\theta$ is identified (apart from additive constant) with the parameter $t$ in (4.2)

$$\delta\theta=\sqrt{-g_{00}}\,\delta t=\delta t \ . \tag{4.3}$$

Since we are going to use both BKL and L velocity-dominated metrics (see also Barrow[16]) Table I might be useful. The correspondence between L and the smearing functions $q(t,\vec{x})=(q_+(t,\vec{x}),q_-(t,\vec{x}),\eta(t,\vec{x}))$ relies on the fact that $\beta_A,\Omega$ can be written as

$$\beta_1=q_++\sqrt{3}\,q_-,\ \ \beta_2=q_+-\sqrt{3}\,q_- \ , \qquad \beta_3=-2q_+,\ \ \Omega=\eta \ , \tag{4.4}$$

where $\beta_A$, $A=1,2,3$, and $\Omega$ are functions in Table I that describe the spatial tensor $g_{ij}$. In other words (3.15) corresponds to the matter-free version of the dynamical equations in Liang[12] when we rescale our time parameter $dt\to(1/3\omega)d\ln t$, see (4.6a) and Table I [an analogous re-

TABLE I. Solutions around the initial spacelike singularity. Velocity-dominated (Kasner) cosmological models as given by Belinskii, Khalatnikov, and Lifshitz (Refs. 13—15) (BKL) and by Liang (Ref. 12) (L).

| | BKL | L |
|---|---|---|
| Metric | $g_{ij}=\begin{bmatrix} t^{2p_i} & & \\ & t^{2p_2} & \\ & & t^{2p_3}\end{bmatrix}$, $\sum p_i=1=\sum p_i{}^2$ | $g_{ij}=\begin{bmatrix} e^{2(-\Omega+\beta_1)} & & \\ & e^{2(-\Omega+\beta_2)} & \\ & & e^{2(-\Omega+\beta_3)}\end{bmatrix}$, $\sum_{A=1}^{3}\beta_A=0$ <br> $\beta_A=\mathrm{diag}(\beta_++\sqrt{3}\,\beta_-,\beta_+-\sqrt{3}\,\beta_-,-2\beta_+)$ <br> $g\equiv\det g_{ij}=e^{-6\Omega}$ |
| Momentum | | $\pi^i_j\begin{bmatrix} \pi_1+\frac{1}{3}\pi & & \\ & \pi_2+\frac{1}{3}\pi & \\ & & \pi_3+\frac{1}{3}\pi\end{bmatrix}$, $\sum_{A=1}^{3}\pi_A=0$ <br> $12\pi_A=\mathrm{diag}(\pi_++\sqrt{3}\,\pi_-,\pi_+-\sqrt{3}\,\pi_-,-2\pi_+)$ <br> $h\equiv 2\pi^i_i\equiv 2\pi$ |
| Dynamical equations (Kasner) | | $\dot\beta_+=\frac{1}{3}\frac{\pi_+}{h}\frac{1}{t},\ \ \dot\pi_+=0$ <br> $\dot\beta_-=\frac{1}{3}\frac{\pi_-}{h}\frac{1}{t},\ \ \dot\pi_-=0$ <br> $\dot\Omega=-\frac{1}{3}\frac{1}{t},\ \ \dot h_-=0$ |

lationship exists between $\pi_A,\pi$ and $p(t,\vec{x}) =(p_+(t,\vec{x}),p_-(t,\vec{x}),\omega(t,\vec{x}))]$. We show now that the BKL and L metrics are the same.

Half of the equations (3.15) says the $p_\pm,\omega$ do not depend on $t$ and we write simply

$$p_\pm \equiv p_\pm(x),\quad \omega\equiv\omega(\vec{x}) , \tag{4.5}$$

where $\vec{x}\in\Sigma$. The other half

$$\frac{dq_\pm}{dt}=\frac{1}{3}\frac{p_\pm}{\omega}\frac{1}{t},\ \frac{d\eta}{dt}=-\frac{1}{3}\frac{1}{t} , \tag{4.6a}$$

gives

$$q_\pm=\frac{1}{3}\frac{p_\pm}{\omega}\ln t+\bar{q}_\pm,\quad \eta=-\tfrac{1}{3}\ln t+\bar{\eta} , \tag{4.6b}$$

where $\bar{q}_\pm$ and $\bar{\eta}$ are independent of $t$ and will be set equal to zero. Then

$$\begin{bmatrix} e^{2(-\Omega+\beta_1)} & & \\ & e^{2(-\Omega+\beta_2)} & \\ & & e^{2(-\Omega+\beta_3)} \end{bmatrix} = \begin{bmatrix} t^{2p_1} & & \\ & t^{2p_2} & \\ & & t^{2p_3} \end{bmatrix} , \tag{4.7}$$

where we defined

$$p_1=\frac{1}{3}\frac{p_+ +\sqrt{3}p_-}{\omega}+\frac{1}{3} ,$$
$$p_2=\frac{1}{3}\frac{p_+ -\sqrt{3}p_-}{\omega}+\frac{1}{3} , \tag{4.8a}$$
$$p_3=-\frac{2}{3}\frac{p_+}{\omega}+\frac{1}{3} .$$

Clearly

$$\sum_{i=1}^{3} p_i=1 \tag{4.8b}$$

and, from ${p_+}^2+{p_-}^2-\omega^2=0$,

$$\sum_{i=1}^{3} {p_i}^2=1 \tag{4.8c}$$

as required for the BKL *generalized Kasner metric* in Table I.

The numbers $p_i(\vec{x})$ satisfying (4.8b) and (4.8c) are called *Kasner indices.* If we assume they are arranged in the order $p_1<p_2<p_3$, they can be uniquely described by a parameter $z(\vec{x})\in(0,1)$ as in Table II.

From Einstein equations of motion one can readily see that[14,17] $\sqrt{g}=\Lambda t$ where $\Lambda$ is some constant. Then $\omega$ can be expressed as[12] $\omega=2g^{-1/2}\partial g/\partial t=4\Lambda$ and (4.8a) gives $p_+,p_-$ as functions of $z$ and $\omega$ (or $\Lambda$) and we write

$$p_\pm=p_\pm(\omega,z) . \tag{4.9}$$

TABLE II. The Kasner indices (4.8). Parametrization of the Kasner indices $p_1,p_2,p_3$ by $z\in(0,1)$.

| | | |
|---|---|---|
| $p_1(z)=\dfrac{-z}{1+z+z^2}$ | $p_2(z)=\dfrac{z(1+z)}{1+z+z^2}$ | $p_3(z)=\dfrac{1+z}{1+z+z^2}$ |

Whenever convenient, as in (4.7), we rescale the BKL metric so that[17] $\Lambda=1$; below when studying transitions between Kasner universes we present the law that gives the corresponding change of $\Lambda$ in terms of $z$.

Suppose now that the smearing functions and the Kasner indices become $\vec{x}$ independent. Then the metric tensor (4.7) is simply called Kasner metric. In homogeneous cosmology language[8] this corresponds to a Bianchi I model. The most interesting of all homogeneous models appears to be the Bianchi IX, or mixmaster universe if it is diagonal. This is an example of chaotic cosmology[16] and it is a paradigm for the general, inhomogeneous solutions to Einstein equations near the cosmological singularity, a remarkable discovery of BKL. The paradigm has recently been criticized,[25] but its fundamental mechanism and assumptions have not convincingly been proved to be incorrect.[26,27]

To review briefly the homogeneous Bianchi IX behavior near the singularity (the inhomogeneous case retains essentially the same features and is discussed later) we recall that, in the BKL point of view, the spacelike singularity is located at $t=0$. So the system evolves "backward" as $t\to 0$ to reach the singularity simultaneously.[27] This fits in with what was said above (in the synchronous frame the line element is time-reversal invariant) and any initial condition has to be given at a later $t_0>0$, rather than earlier time. With this proviso in mind we discuss the evolution of the system toward $t=0$.

To study the approach to the singularity it is useful to divide the interval $[0,t_0]$ into specially chosen consecutive periods $I_k=[t_{k+1},t_k]$, $k=0,1,2,\ldots,$ where $t_k$ converges to $t=0$. They are known as "Kasner epochs" because in each one of these periods the mixmaster metric has the same time dependence as

$$g_{ij}^{(k)}=\begin{bmatrix} t^{2p_1(z_k)} & & \\ & t^{2p_2(z_k)} & \\ & & t^{2p_3(z_k)} \end{bmatrix} ,$$

that is, a Kasner metric characterized by some $z_k\in(0,1)$. The transition from one period to another is rather brief and when $t=t_{k+1}$ the Kasner universe $z_k$ is replaced by $z_{k+1}$. A quantitative description of this mechanism can be obtained after a careful study of Einstein equations.[13–17] It is found that all the dynamical content of the mixmaster universe is summarized in the transitions given by the one-dimensional map $T:(0,1)\to(0,1)$ called *Poincaré map:*

$$Tz_k=\frac{1}{z_k}-\left[\frac{1}{z_k}\right] \tag{4.10}$$

where [ ] means the integer part, e.g., $[\pi]=3$. An analytic expression for $T$ is readily obtained:

$$Tz=\frac{1}{z}-k,\quad \frac{1}{k+1}z\le\frac{1}{k} . \tag{4.11}$$

Observe that this map contains the evolution of the system, i.e., the succession of Kasner universes, without any use of the parameter time $t$. Of course a relationship between a transition time $t_k$ and the number of iterations $k$

can always be found but we do not need to do this. In fact physical results in quantum gravity must not contain any explicit reference to the auxiliary time parameter (proper time) introduced in Sec. III and here identified with $t$. The Poincaré map suits well this purpose and will be used when we deal with the notion of density matrix [see (4.19)].

This complicated pattern of the mixmaster evolution is induced by the action of the potential $V=gR$ because when this term is "switched off" the model is velocity dominated and evolves forever as a Kasner universe. The sudden transitions occur when the potential scatters a given initial Kasner configuration $z_i$ into another $z_f$. Consequently one can study cosmology as a scattering problem in superspace[8–10] and type IX (or VIII) homogeneous cosmologies are examples that require multiple, in fact infinite, scatterings on approach to the singularity.

The mixmaster evolution is then encoded in a sequence of Kasner universes

$$\{z_n\}=\{z_0,Tz_0,\ldots,T^nz_0,\ldots\}\ . \tag{4.12}$$

Asymptotically this sequence acquires a stochastic character.[14] The random nature of this process might seem strange since we are dealing with a completely deterministic system. However, this is the peculiarity of the mixmaster system: the slightest fluctuation of the initial data will tremendously affect the asymptotic behavior rendering the system unpredictable (since an infinite number of scatterings is involved). In the language of dynamical systems this unpredictability is due to the fact that nearby trajectories in phase space diverge asymptotically and the rate at which this occurs has been determined.[14,16] Thus, after a sufficiently large number of iterations the initial conditions will be completely "forgotten." The system then is characterized by a probabilistic distribution[14]

$$\mu(z)=\frac{1}{(1+z)\ln 2}\ , \tag{4.13}$$

where $\int_0^1\mu(z)dz=1$. This gives the probability of a particular Kasner configuration $z$ being visited during the mixmaster evolution. Finally we remark that the measure $\mu$ is preserved by $T$ in the following sense: $\mu(a,b)=\mu(T^{-1}(a,b))$ where $\mu(a,b)=\int_a^b\mu(z)dz$. This is a most important property because, using (4.11) it implies (4.13).[16]

After this digression on classical solutions it is clear that the evolution of the quantum state $|p,q\rangle$ that approximates the behavior of a quantum Bianchi IX model has been completed when $t\to 0$. We use the fact that $\hat{\pi}_\pm|0\rangle=0=\hat{\pi}|0\rangle$ to write

$$|p,q\rangle=e^{-(i/\hbar)(p_+\hat{t}_++p_-\hat{t}_--\omega\hat{\tau})}|0\rangle=|p_+,p_-,\omega\rangle \tag{4.14}$$

because there is no $q$ dependence. Under the action of the potential $V$ the evolution of (4.14) is entirely described by the sequence (4.12). In the asymptotic region the whole dynamics of the system is contained in (4.13). Call (4.14) Kasner states and use (4.9) to rewrite

$$|p_+,p_-,\omega\rangle=|z,\omega\rangle\ . \tag{4.15}$$

The quantity $\omega$ before and after the transition can be shown to be related by[9,14] $\omega_f=(1-2|p_1|)\omega_0$, so $\omega$ is specified by $z$ and some arbitrary constant $\omega_0$.

At this point it is useful to comment on the interesting situation we arrived at. Around the initial singularity as $\hbar\to 0$ the occurrence of a Kasner state $|z,\omega\rangle$ is given by $\mu(z)$ in (4.13). This is essentially the whole dynamical content of a quantum mixmaster universe in that regime. The parameter time $t$ plays no role in this description since the states $|z,\omega\rangle$ are time independent and the transitions in the immediate neighborhood of the singularity are contained in $T$ or $\mu$. However we emphasize that these states do not coexist simultaneously but only one at a time appear during the evolution with probability $\mu(z)$. This situation is quite general and applies not only to homogeneous cosmological models but to inhomogeneous models as well. A property of the mixmaster system is that, in a sense described below, the Kasner states around the singularity are in equilibrium.

The general state of a quantum-mechanical system corresponds to a *density operator*[28] of the form

$$\rho=\sum_{n=0}^{\infty}|\Psi_n\rangle\beta_n\langle\Psi_n|\ , \tag{4.16}$$

where $\beta_n\geq 0$, $\sum_{n=0}^{\infty}\beta_n=1$, and $\{\psi_n\}$ is a complete orthonormal basis. Average values of an Hermitian observable $\hat{O}$ are given by

$$\langle\hat{O}\rangle=\mathrm{Tr}(\rho\hat{O})\ . \tag{4.17}$$

We define the system to be in equilibrium when $\langle\hat{O}\rangle$ is a constant of motion, for any $\hat{O}$. This means the $\rho$ is a stationary state.

In a completely analogous way a density operator for the mixmaster system can be introduced:[29]

$$\rho=\int_0^1 dz\,|z\rangle\mu(z)\langle z|\ , \tag{4.18}$$

where $\mu(z)>0$, $\int_0^1\mu(z)dz=1$, and $\{|z\rangle\}$ can be shown to be orthonormal in the $\hbar\to 0$ limit. The $\omega$ dependence in the Kasner states is not included because we are now only interested in the stationarity of $\rho$ and we know that the change in $\omega$ is contained in $z$.

From (4.18) and (4.11) the state $|z\rangle$ has to evolve to $|Tz\rangle$ and thus the evolution of (4.18) is given by

$$\begin{aligned}\rho^T&=\int_0^1 dz\,|Tz\rangle\mu(z)\langle Tz|\\&=\sum_{k=1}^{\infty}\int_{1/k+1}^{1/k}dz\,|Tz\rangle\mu(z)\langle Tz|\\&=\int_0^1 dz\,|z\rangle\sum_{k=1}^{\infty}\frac{1}{(k+z)(k+z+1)\ln 2}\langle z|\\&=\int_0^1 dz\,|z\rangle\mu(z)\langle z|\end{aligned} \tag{4.19}$$

which is $\rho$, as claimed. This is not, however, a "thermal" equilibrium since no maximization of entropy (see below) is involved.

A very interesting concept will now be introduced into the formalism. The *entropy* $S(\rho)$ of a general state $\rho$ of a system is defined by[28]

$$S(\rho) = -\operatorname{Tr}(\rho \ln \rho) . \tag{4.20}$$

In the canonical diagonal form (4.16) the above equation reads

$$S(\rho) = -\sum_{n=0}^{\infty} \beta_n \ln \beta_n . \tag{4.21}$$

Analogously from (4.18) the entropy (4.20) for the mixmaster universe in the asymptotic region can be shown to be

$$S(\rho) = -\int_0^1 dz\, \mu(z) \ln \mu(z) \tag{4.22}$$

in the limit $\hbar \to 0$. Substituting (4.13) in (4.22) we get

$$S(\rho) = \frac{1}{2\ln 2}[\ln^2(2\ln 2) - \ln^2(\ln 2)] . \tag{4.23}$$

The numerical result (4.23) gives a measure of how mixed, or chaotic, the state $\rho$ is.[30,31] Significant results could now be obtained if we propagated the state (4.18) away from the singularity and then compared the resulting entropy with (4.23). This is not an easy task since the only way available to evolve $\rho$ is through the Poincaré map as given by the BKL approximation. However, we have just shown that $\rho$ is an equilibrium state with respect to $T$. [An attempt to prepare a state "far" from the singularity in order to compare its entropy with (4.23) was made in Ref. 29.] Thus a satisfactory solution to this problem appears to require that the state $\rho$ be propagated out of the region where the BKL approximation holds. Then the variation of entropy $\Delta S$ would become a physically useful quantity. For instance, if $\Delta S > 0$ then the final state is more chaotic than the initial one and so it contains more information that is inaccessible to the observer.[31] Results of this kind could help to throw some light into the nature of the singularity and the present state of the universe.

The main obstacle to pursue further the considerations above belongs to classical cosmology. It is not known at present how the mixmaster universe behaves when the BKL approximation breaks down. All we know is that sufficiently far from the singularity such approximation necessarily fails since the potential term $gR$ cannot be ignored even for small periods of time. As a starting point to the solution of this difficult problem one must try to obtain an estimate of the size of the region where the BKL mechanism is valid. This and related issues are under investigation.

To conclude we discuss briefly the case where the smearing functions and the Kasner indices are $\vec{x}$ dependent. The BKL construction of a general solution around the singularity contains a mechanism in many ways analogous to the homogeneous case already studied. The $\vec{x}$-dependent potential $V = gR$ is now responsible for the scattering of generalized Kasner metrics. Recall that in this case the Kasner indices are labeled by an arbitrary function $z(\vec{x})$. At each $\vec{x} \in \Sigma$ the law that describes the scattering of the metric from a given initial generalized Kasner configuration to another can be shown to be similar as that for the Bianchi IX model. Because of this fact it is often useful to think of the inhomogeneous metric around the singularity as if a Bianchi IX metric had been attached at every point $\vec{x} \epsilon \Sigma$. This is the essence of the BKL paradigm.[15]

Suppose that the metric scatters simultaneously over $\Sigma$. Then it is reasonable to infer from the BKL paradigm that the $\hbar \to 0$ behavior of quantum gravity-wave functionals around the initial singularity is contained in the probability

$$\mu(z) = \frac{1}{(1+z)\ln 2} \tag{4.24}$$

to find the system in a generalized Kasner state

$$|z,\omega\rangle = \exp\left[-\frac{i}{\hbar}\int d^3x (p_+\hat{t}_+ + p_-\hat{t}_- - \omega\hat{\tau})\right]|0\rangle . \tag{4.25}$$

The function $z = z(\vec{x})$ in this case is arbitrary because we are not dealing with any specific model. The next step would be to find the Poincaré map $T$. However this is not as straightforward as it seems and our discussion must come to a halt. In any case $T$ will not be one dimensional and in fact it may correspond to higher-dimensional extensions called $F$ expansions (see Barrow[14] and references therein). Also the final decision as to whether (4.24) is the appropriate measure for this system will depend on it being preserved by the Poincaré map.

## V. DISCUSSION

A quantum theory of gravity does not exist at the moment. However, we believe that in first-order approximation any prospective canonical quantization of the gravitational field around the initial singularity should exhibit the behavior just described.

The quasiclassical approximation we conjectured to hold in quantum gravity constitutes a generalization of a similar situation in homogeneous cosmology.[30] For a homogeneous model one is able to prove formula (3.18) since the potential term $V = gR$ does not involve any spatial derivatives but is constructed solely from the metric and the structure constants of the homogeneity group [SO(3) for the mixmaster universe].

The concept of entropy promises to be potentially very useful in discussing cosmological problems. Several kinds of entropies can be defined in this context (but the relationship between them is not clear): Penrose's gravitational entropy,[32] Kolmogorov entropy,[16] topological entropy (these two coincide for the mixmaster universe[16]), and recently Hu[33] has proposed still another one. Our formula (4.22) appears to be able, among other things, to say something about the quantum properties of the initial singularity but a full appreciation of these possibilities has to await further developments.

## ACKNOWLEDGMENTS

The authors would like to thank J. Barrow, C. Isham, T. Kakas, and M. MacCallum for helpful discussions. One of us (G. F.) acknowledges the hospitality extended to him by K. Kuchar, University of Utah, where part of this paper was written, and support by Coordenação de Aperfeiçoamento de Pessoal de Nivel Superior (Brazil).

*Permanent address: Instituto de Física Teórica, Rua Pamplona, 145 São Paulo, Brazil.

[1]M. Pilati, Phys Rev. D **26**, 2645 (1982).
[2]M. Pilati, Phys. Rev. D **28**, 729 (1983).
[3]C. J. Isham, Proc. R. Soc. London **A351**, 209 (1976).
[4]C. Teitelboim, Phys. Rev. D **25**, 3159 (1982).
[5]J. R. Klauder, Acta Phys. Austriaca Suppl. **VIII**, 227 (1971).
[6]J. R. Klauder in *Functional Techniques,* edited by W. E. Brittin (Colorado University Press, Boulder, 1973).
[7]J. R. Klauder, J. Math. Phys. **11**, 609 (1970).
[8]M. P. Ryan and L. C. Shepley, *Homogeneous Relativistic Cosmology* (Princeton University Press, Princeton, New Jersey, 1975).
[9]C. Misner, Phys. Rev. **186**, 1319 (1969).
[10]C. Misner, in *Magic Without Magic,* edited by J. R. Klauder (Freeman, San Francisco, 1972).
[11]D. Eardley, E. Liang, and R. Sachs, J. Math. Phys. **13**, 99 (1972).
[12]E. Liang, J. Math. Phys. **13**, 386 (1972).
[13]E. M. Lifshitz and I. M. Khalatnikov, Adv. Phys. **12**, 185 (1963).
[14]V. A. Belinskii, I. M. Khalatnikov, and E. M. Lifshitz, Adv. Phys. **19**, 525 (1970).
[15]V. A. Belinskii, I. M. Khalatnikov, and E. M. Lifshitz, Adv. Phys. **31**, 639 (1982).
[16]J. D. Barrow, Phys. Rep. **85**, 1 (1982).
[17]L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields* (Pergamon, Oxford, 1975).
[18]K. Hepp, Commun. Math. Phys. **35**, 265 (1974).
[19]J. R. Klauder, in *Path Integrals,* edited by C. J. Papadopoulos and J. T. Devresee (Plenum, New York, 1978).
[20]G. Francisco and M. Pilati (unpublished).
[21]J. R. Klauder, J. Math Phys. **8**, 2392 (1967).
[22]J. W. York, Phys. Rev. Lett. **28**, 1082 (1972).
[23]B. S. DeWitt, Phys. Rev. **160**, 1113 (1967).
[24]C. J. Isham, in *Quantum Gravity: An Oxford Symposium,* edited by C. J. Isham, R. Penrose and D. Sciama (Oxford University Press, Oxford, 1975).
[25]J. D. Barrow and F. J. Tipler, Phys. Rep. **56**, 372 (1979).
[26]M. MacCallum, (private communication).
[27]R. M. Wald and P. Yip, J. Math. Phys. **22**, 2659 (1981).
[28]J. R. Klauder and E. Sudarshan, *Quantum Optics* (Benjamin, New York, 1968).
[29]G. Francisco, Imperial College report, 1983 (unpublished).
[30]A. Katz, *Principles of Statistical Mechanics* (Freeman, San Francisco, 1967).
[31]W. Thirring, *A Course in Mathematical Physics,* Vol. 4 (Springer, Berlin, 1983).
[32]R. Penrose, in *General Relativity: An Einstein Centenary Survey,* edited by S. W. Hawking and W. Israel (Cambridge University Press, Cambridge, 1979).
[33]B. L. Hu, Phys. Lett. **97A**, 368 (1983).