Quantum relativistic oscillator. Modifying the Hamiltonian formalism of the relativistic string

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A new internal position operator $\xi_\mu = -d_\mu$ and a new internal momentum operator π_μ are defined using an analog of constraint Hamiltonian mechanics. The new position and momentum do not fulfill the usual relativistic Heisenberg commutation relations and both have noncommuting components, but in the nonrelativistic limit they contract into the usual three-dimensional position and momentum. The new momentum π_μ is connected with an infinite-dimensional generalization Γ_μ of the Dirac matrices, which together with the intrinsic angular momentum $S_{\mu\nu}$ form an SO(3,2) algebra. A representation of this algebra provides the spectrum of hadron resonances, if hadrons are considered as collective vibrational and rotational excitations. A preliminary comparison between the predictions of this model and the experimental data leads to encouraging results.

I. INTRODUCTION

The primary objective of the quantum-relativistic-oscillator (QRO) model is the same as the original objective of the relativistic string model,[1] to describe the spectrum of hadrons. The general framework in which this is to be achieved for the QRO model is also the same as that for the relativistic string model; it is a relativistic Hamiltonian system with constraint.[2]

Although the primary objective and general theoretical framework are the same, the detailed quantum-mechanical relations of our QRO differ from those obtained by the canonical quantization of the relativistic string, and from the conventional quantum-mechanical relativistic oscillator.[3-6] We were led to the new relativistic Heisenberg commutation relations when we attempted to bring our quantum-relativistic-rotator (QRR) model[7] into congruence with the relativistic string with the idea in mind that "rigid" rotations should be a special motion of the string. To our surprise we found that—in order to obtain a relativistic theory that describes rotational and vibrational motions and goes in the nonrelativistic limit into a picture similar to that of the vibrating rotator of molecular physics or of the collective nuclear models—one should not modify the QRR to agree with the string, but one should modify the conventional relativistic string to accommodate features (Zitterbewegung) of the QRR. The intrinsic position, which in the nonrelativistic limit can be interpreted as the position of the quark relative to the center of mass, should not be described by a commuting four-vector $x^\mu(0,\tau)$, but by the intrinsic position observable

$$d^\mu = S^{\mu\nu}P^\nu/M^2$$

of our QRR model. This intrinsic position observable, and its "canonical conjugate" variable $\pi_\mu \sim \Gamma_\mu$, both have noncommuting components. The use of this new intrinsic position $-d_\mu$ and our unconventional quantization of the vibrating extended relativistic object can also be justified by starting from the usual Poisson-bracket algebra of the relativistic string and then going to the Dirac brackets[2,8] and the primed variables[8] of constrained Hamiltonian mechanics.

In the nonrelativistic limit, our new algebra of observables goes into the algebra for the three-dimensional harmonic oscillator and the new commutation relations of the intrinsic momentum and position go into the three-dimensional Heisenberg commutation relations. Thus our QRO is just a different relativistic generalization of the nonrelativistic oscillator than the conventional generalization.

In Sec. II, we shall conjecture the mathematical structure for the QRO using as a starting point formulas and properties of the relativistic string. As the relativistic string in its numerous variations is much better known than the QRR, this will probably provide the most convincing access to our new model.

In Sec. III, we describe the properties of the QRO, in particular its representation spaces. We also derive some consequences and compare them with the experimental data. A more detailed comparison of the predictions, as well as the derivation of the nonrelativistic limit, will be given in a subsequent paper.

II. CONJECTURING THE ALGEBRA OF OBSERVABLES

The points of the string in $(d-1)$ space dimensions and one time dimension are described by the d-vector

$$x^\mu(\sigma,\tau) = X^\mu(\tau) + \tilde{x}^\mu(\sigma,\tau) , \qquad (2.1)$$

where

$$\tilde{x}^\mu(\sigma,\tau) = i\sqrt{2\alpha'} \sum_{\substack{n=-\infty \\ n\neq 0}}^{+\infty} \frac{\alpha_n^\mu}{n} \cos n\sigma \, e^{-in\tau} \qquad (2.2)$$

and

$$X^\mu(\tau) = X^\mu(0) + 2\alpha' P^\mu \tau . \qquad (2.3)$$

The parameters $\sigma \in [0,\pi]$ and τ are the spacelike and

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timelike coordinates of the string in the orthogonal gauge and α' is a constant with the dimension of cm^2.

The α_m^μ fulfill the Poisson-bracket relation

$$i\{\alpha_m^\mu,\alpha_n^\nu\}\equiv[\alpha_m^\mu,\alpha_n^\nu]^{PB}=-mg^{\mu\nu}\delta_{m,-n} \; . \tag{2.4}$$

One can quantize the string by going from the classical α_m^μ to the operators α_m^μ, fulfilling the relativistic canonical commutation relation (CR)

$$[\alpha_m^\mu,\alpha_n^\nu]=-mg^{\mu\nu}\delta_{m,-n}, \quad \alpha_m^{\mu\dagger}=\alpha_{-m}^\mu \; . \tag{2.5}$$

To simplify writing we will from now on use $(1/i)[\;,\;]^{PB}$ instead of $\{\;,\;\}$ for the Poisson bracket and similarly $(1/i)[\;,\;]^{DB}$ for the Dirac bracket of classical quantities. For the commutator of quantum-mechanical operators we will use [,], so that the correspondence between classical (cl) and quantum mechanics (QM) is

$$[\;,\;]^{DB}\leftrightarrow[\;,\;] \; .$$

The X^μ and P^μ fulfill the (classical) Poisson-bracket or (quantum) commutation relations

$$[X^\mu,X^\nu]=0, \quad [P^\mu,P^\nu]=0, \quad [P^\mu,X^\nu]=ig^{\mu\nu} \; . \tag{2.6a}$$

The α_m^μ commute with X^μ and P^μ, and the $x^\mu(\sigma,\tau)$ along with $p^\mu(\sigma,\tau)$, defined by

$$p^\mu(\sigma,\tau)=\frac{1}{2\alpha'\pi}\dot{\tilde{x}}^\mu(\sigma,\tau) =\frac{1}{\sqrt{2\alpha'}\pi}\sum_{\substack{m=-\infty\\ m\neq 0}}^{+\infty}\alpha_m^\mu \cos m\sigma\, e^{-im\tau} \; , \tag{2.7}$$

fulfill the canonical CR

$$[\tilde{x}^\mu(\sigma,\tau),\tilde{x}^\nu(\sigma'\tau)]=0=[p^\mu(\sigma,\tau),p^\nu(\sigma'\tau)] \; , \tag{2.6b}$$

$$[\tilde{x}^\mu(\sigma,\tau),p^\nu(\sigma'\tau)]=-ig^{\mu\nu}\left[\delta(\sigma'-\sigma)+\delta(\sigma'+\sigma)-\frac{1}{\pi}\right] .$$

We want to focus on one particular position on the string, which may be distinguished by, e.g., localizing the quark or a charge at that position and identifying the string itself with the neutral glue. We choose $\sigma=0$ for this position and define the position operator $Q_\mu(\tau)$ as

$$Q^\mu(\tau)\equiv x^\mu(0,\tau)=X^\mu(\tau)+\tilde{x}^\mu(\sigma=0,\tau) \; . \tag{2.8}$$

As a consequence of the previous relation Q_μ will fulfill

$$[Q_\mu,Q_\nu]=0, \quad [Q^\mu,P^\nu]=-ig^{\mu\nu} \; . \tag{2.6c}$$

With the α_m^μ one can define

$$S^{\mu\nu}\underset{\text{cl}}{=}-i\sum_{\substack{n=-\infty\\ n\neq 0}}^{+\infty}\frac{1}{n}\alpha_{-n}^\mu\alpha_n^\mu \underset{\text{QM}}{=}-i\sum_{n=1,2,\ldots}^{\infty}\frac{1}{n}(\alpha_{-n}^\mu\alpha_{+n}^\nu-\alpha_{-n}^\nu\alpha_{+n}^\mu) \; . \tag{2.9}$$

We will use $S^{\mu\nu}$ only for $d=4$ ($\mu,\nu=0,1,2,3$) when we make the transition to our quantum relativistic oscillator. If one goes to higher dimensions d and uses a noncovariant light-cone gauge,[4,5] one requires (2.9) only for the $d-2$ spacelike dimensions so that d must be at least 5 (cf. the Appendix). The $S^{\mu\nu}$ defined by (2.9) fulfill as a consequence of (2.5) the CR of SO(3,1):

$$[S^{\mu\nu},S^{\rho\sigma}]=-i(g^{\mu\rho}S^{\nu\sigma}+g^{\nu\sigma}S^{\mu\rho}-g^{\mu\sigma}S^{\nu\rho}-g^{\nu\rho}S^{\mu\sigma}) \; . \tag{2.1}$$

The $S^{\mu\nu}$ defined in terms of the α_m^μ describe only integer-angular-momentum representations of $SO(3)_{S_{ij}}$. To obtain half-integer angular momentum one has to adjoin Fermi operators to the α_m^μ, which constitutes no essential complication. On the level of the $S^{\mu\nu}$ nothing changes except that now half-integer representations will also be allowed. We will therefore restrict ourselves here to the integer-spin case.

It is not the light-cone gauge but the center-of-mass gauge[6] which is the departure point from the relativistic string (we restrict ourselves now to $d=3+1$) to our QRO. This c.m. gauge is defined by the constraint

$$\phi_m=\hat{P}_\mu\alpha_m^\mu \underset{C}{=} 0, \quad \hat{P}_\mu=P_\mu(P_\nu P^\nu)^{-1/2} \; . \tag{2.11}$$

The orthogonal gauge does not specify the parameters σ,τ uniquely and (2.11) fixes the parameters further by eliminating the timelike center-of-mass oscillations ("ghost elimination").

For the QRO we will *not* demand the constraints (2.11), but we will still use the ϕ_m to conjecture a new set of observables. In constraint Hamiltonian mechanics one can for any variable ξ introduce a new variable ξ' by[8]

$$\xi'=\xi-\frac{1}{i}[\xi,\phi_\alpha]^{PB}C^{-1}{}_{\alpha\beta}\phi_\beta \text{ summed over } \alpha \text{ and } \beta \; , \tag{2.12}$$

where ϕ_α is a set of second-class constraints, and the matrix

$$C_{\alpha\beta}\equiv\frac{1}{i}[\phi_\alpha,\phi_\beta]^{PB} \tag{2.13}$$

is nonsingular with $C^{-1}{}_{\alpha\beta}$ in (2.12) as its inverse:

$$C^{-1}{}_{\alpha\gamma}C_{\gamma\beta}=\delta_{\alpha\beta} \; . \tag{2.14}$$

The Dirac brackets are then defined by

$$\frac{1}{i}[\xi,\eta]^{DB}=\frac{1}{i}[\xi,\eta]^{PB}-\frac{1}{i}[\xi,\phi_\alpha]^{PB}C^{-1}{}_{\alpha\beta}[\phi_\beta,\eta]^{PB} \underset{C}{=}\frac{1}{i}[\xi',\eta']^{PB} \; ,$$

where η' is defined analogously to ξ'. The transition to quantum mechanics is taken by

$$[\xi,\eta]^{DB}\rightarrow[\xi,\eta] \tag{2.15}$$

with the quantum-mechanical operators ϕ_α corresponding to the classical constraints

$$\phi_\alpha \underset{C}{=} 0$$

taken to be zero on the Hilbert space of physical states.

We will proceed in a slightly different way: As the second-class constraints (2.11) are gauge-fixing conditions we do not have to impose them. We will, however, still use the ϕ_m to define the new classical and quantum-mechanical observables,

$$Q'^{\mu}=Q^{\mu}-\frac{1}{i}[Q^{\mu},\phi_m]C^{-1}{}_{mn}\phi_n\ , \tag{2.16}$$

$$S'^{\mu\nu}=S^{\mu\nu}-\frac{1}{i}[S^{\mu\nu},\phi_m]C^{-1}{}_{mn}\phi_n\ . \tag{2.17}$$

It is straightforward to calculate

$$C_{mn}=\frac{1}{i}[\alpha_m^{\rho}\hat{P}_{\rho},\alpha_n^{\sigma}\hat{P}_{\sigma}]=im\delta_{m,-n}\ , \tag{2.18}$$

$$C^{-1}{}_{mn}=\frac{i}{m}\delta_{m,-n}\ , \tag{2.19}$$

$$\frac{1}{i}[Q^{\mu},\phi_m]=\alpha_m^{\rho}\frac{1}{i}[Q^{\mu},\hat{P}_{\rho}]=-\alpha_m^{\rho}\breve{g}^{\mu}{}_{\rho}\frac{1}{Mc}\ , \tag{2.20}$$

where

$$\breve{g}^{\mu}{}_{\rho}\equiv g^{\mu}{}_{\rho}-\hat{P}^{\mu}\hat{P}_{\rho},\ \ Mc=(P_{\mu}P^{\mu})^{1/2}\ . \tag{2.21}$$

Thus one obtains

$$\begin{aligned}Q'^{\mu}&=Q^{\mu}-\frac{i}{n}\alpha_{-n}^{\rho}\breve{g}^{\mu}{}_{\rho}\frac{1}{Mc}\hat{P}_{\sigma}\alpha_n^{\sigma}\\ &=Q^{\mu}+S^{\rho\sigma}\breve{g}^{\mu}{}_{\rho}\hat{P}_{\sigma}\frac{1}{Mc}=Q^{\mu}+S^{\mu\sigma}\hat{P}_{\sigma}\frac{1}{Mc}\ ,\end{aligned} \tag{2.22}$$

where the classical form of the definition (2.9) has been used.

If one defines

$$d^{\mu}\equiv S^{\mu\sigma}\hat{P}_{\sigma}\frac{1}{Mc}\ \text{and}\ \hat{d}^{\mu}=d^{\mu}Mc\ , \tag{2.23}$$

one obtains

$$Q'^{\mu}=Q^{\mu}+d^{\mu}\equiv Y^{\mu}\ . \tag{2.24}$$

d^{μ} of (2.23) is the distance operator in the QRR[7(a),9] and the new Q'_{μ} is just the center-of-mass position Y_{μ} of Ref. 7, which is the reason we also denote it here by Y_{μ}.

It is again straightforward to calculate

$$\frac{1}{i}[S^{\mu\nu},\phi_m]=\alpha_m^{\mu}\hat{P}^{\nu}-\alpha_m^{\nu}\hat{P}^{\mu}\ . \tag{2.25}$$

Inserting this and (2.19) into (2.17) and using the definitions (2.9) and (2.23), one obtains

$$\begin{aligned}S'^{\mu\nu}&=S^{\mu\nu}-\hat{d}^{\mu}\hat{P}^{\nu}+\hat{d}^{\nu}\hat{P}^{\mu}\\ &=\breve{g}^{\mu}{}_{\rho}\breve{g}^{\nu}{}_{\sigma}S^{\rho\sigma}\equiv\Sigma^{\mu\nu}\ .\end{aligned} \tag{2.26}$$

This is the spin tensor of Ref. 7, which we have again denoted by $\Sigma^{\mu\nu}$ to contrast it from $S^{\mu\nu}$, which is the intrinsic angular momentum in the rest frame of the position [cf. Ref. 7(a), p. 3024]. The total angular momentum (generator of the Lorentz group) is given by

$$J_{\mu\nu}=Y_{\mu}P_{\nu}-Y_{\nu}P_{\mu}+\Sigma_{\mu\nu}\ . \tag{2.26a}$$

Y^{μ} fulfills the CR

$$[Y^{\mu},Y^{\nu}]=i\frac{1}{(Mc)^2}\Sigma^{\mu\nu} \tag{2.26b}$$

(which is in agreement with the Dirac brackets of Q^{μ}). It is easy to see that

$$P'^{\mu}=P^{\mu} \tag{2.27}$$

and

$$\alpha'^{\mu}_m=\breve{g}^{\mu}{}_{\nu}\alpha_m^{\nu}\equiv a_m^{\mu}\ . \tag{2.28}$$

For further properties of these and other observables, see Appendix.

With the new operator $Y^{\mu}(\tau)$ at hand it is suggestive to use it in place of the ill defined $X^{\mu}(\tau)$ and decompose the vector to the point on the string $x^{\mu}(\sigma,\tau)$ as

$$x^{\mu}(\sigma,\tau)=Y^{\mu}(\tau)+\xi^{\mu}(\sigma,\tau)\ . \tag{2.29}$$

$\xi^{\mu}(\sigma,\tau)$ then describes the vector from the center-of-mass position to the position σ on the string. For $\sigma=0$ we now have

$$x^{\mu}(0,\tau)\equiv Q^{\mu}(\tau)=Y^{\mu}(\tau)+\xi^{\mu}(0,\tau)\ , \tag{2.30}$$

so that by (2.24)

$$\xi^{\mu}(0,\tau)=-d^{\mu}(\tau)\equiv\xi^{\mu}(\tau)\ . \tag{2.31}$$

The new internal position operator $\xi^{\mu}(0,\tau)$ [and $\xi^{\mu}(\sigma,\tau)$] fulfills different CR than the old internal position $x^{\mu}(0,\tau)$ given by (2.6c). As a consequence of its definition (2.23) one has[7]

$$[d^{\mu},d^{\nu}]=-i\frac{1}{(Mc)^2}\Sigma_{\mu\nu}=[\xi^{\mu},\xi^{\nu}]\ , \tag{2.32}$$

which is also consistent with (2.26b). Thus in contrast to the conventional internal position operator $x^{\mu}(0,\tau)$, the new internal position operator $\xi^{\mu}(\tau)$ has noncommuting components.

To find the canonical conjugate of $\xi^{\mu}(\tau)=-d^{\mu}$ one needs to know the Hamiltonian, which in turn may depend upon this canonical conjugate variable. We have to guess: If we want equal spacing for the mass-squared spectrum, as may be suggested by the nonrelativistic oscillator, the conventional relativistic string or linearly rising Regge trajectories, we need an operator whose eigenvalues are

$$\nu=0,1,2,\ldots\ .$$

Such an operator existed in the QRR model[7(a)] and was given by $\hat{P}_{\mu}\Gamma^{\mu}$, where Γ^{μ} is a Lorentz-vector operator which, together with the $S^{\mu\nu}$, form a representation of the algebra of SO(3,2) (Γ^{μ} is an infinite-dimensional generalization of the Dirac γ matrices and thus has something to do with a current or a velocity). We therefore postulate the following operator as the Hamiltonian for the QRO:

$$H=v\left[P_{\mu}P^{\mu}-\frac{1}{\alpha'}\hat{P}_{\mu}\Gamma^{\mu}\right]\ , \tag{2.33}$$

where α' is a constant of dimension $(\text{mass}\,c)^{-2}$ and v is a Lagrange multiplier to be determined later when we fix the meaning of the parameter τ not by imposing the gauge-fixing constraint (2.11), which we do not want to use, but by a different condition.

For the simple representations of SO(3,2) that were used in Ref. 7 (Majorana or "remarkable" representations), Γ_μ is completely fixed in terms of the $S_{\mu\nu}$ and therewith by (2.9) also in terms of the α_m^μ. The same is true for the d_μ defined by (2.23). For the more complicated representations of SO(3,2) it is not clear whether Γ_μ is expressible in terms of the α_m^μ (or, if half-integer angular momentum is involved, in terms of $\alpha_{\mp m}^\mu$ and the corresponding anticommuting creation and annihilation operators). With the Hamiltonian (2.33) this is, however, no longer a practical attitude to take and it is much easier to work directly with the more accessible observables $\Sigma^{\mu\nu}$ or $S^{\mu\nu}$, Γ^μ, d^μ. The question of whether there is an underlying structure, e.g., whether these operators can be expressed in terms of creation and annihilation operators, is of as little relevance and is perhaps as difficult to solve for the relativistic rotational and vibrational bands as the question of whether molecular rotation-vibration properties can be expressed in terms of the electron coordinates.

The constraint,

$$P_\mu P^\mu - \frac{1}{\alpha'}\hat{P}_\mu \Gamma^\mu \underset{C}{=} 0 , \tag{2.34}$$

that follows from the Hamiltonian (2.33) looks quite different from the mass constraint for the string, cf. (A16) [$\alpha(0)$ a constant],

$$-\frac{1}{\alpha'}[L_0 - \alpha(0)] \equiv P_\mu P^\mu + \frac{1}{\alpha'}\sum_{k=1}^{\infty} \alpha_{-k}^\mu \alpha_{+k\mu} + \frac{\alpha(0)}{\alpha'} \underset{C}{=} 0 , \tag{2.35}$$

but has similar consequences which become identical in the nonrelativistic limit at least for the lowest mode. This we will see when we derive the spectrum of $\hat{P}_\mu \Gamma^\mu$ and find that it is identical (in certain representations) with the spectrum of the number operator for the three-dimensional harmonic oscillator. This was the first indication that the Hamiltonian (2.33) should have something to do with the quantum-relativistic oscillator; the justification to call the physical system with the Hamiltonian (2.33) a relativistic oscillator came from the nonrelativistic limit $c \to \infty$, in which (2.33) yields the energy operator of the three-dimensional nonrelativistic oscillator.

With the Hamiltonian (2.33) we can now obtain the canonical conjugate of $\xi^\mu = -d^\mu$. We take the proper-time derivative of d^μ defined by

$$\dot{d}_\mu = \frac{1}{i}[d_\mu, H] . \tag{2.36}$$

Using the definition (2.23) and the CR of Γ_ρ with $S_{\mu\nu}$ [Eq. (2.39) of Ref. 7(a)], we obtain by a straightforward calculation

$$\dot{d}_\mu = -\frac{v}{\alpha'(Mc)}[\Gamma_\mu - \hat{P}_\mu(\hat{P}\cdot\Gamma)] = -\frac{v}{\alpha'(Mc)}\breve{g}^{\sigma}{}_{\mu}\Gamma_\sigma . \tag{2.37}$$

$\breve{g}^{\sigma}{}_{\mu}$ was defined by (2.21) and it projects into the plane perpendicular to P_μ. Equation (2.37) shows that the "intrinsic velocity" $\dot{d}_\mu$ is proportional to the component of Γ_μ perpendicular to the momentum direction $\hat{P}_\mu$.

We will now calculate $\dot{Q}_\mu$ and $\dot{Y}_\mu$. To calculate

$$\dot{Q}_\mu = \frac{1}{i}[Q_\mu, H] , \tag{2.38}$$

we need the commutator of Q_μ with $\hat{P}_\nu$ and P_ν as the Q_μ commute with Γ_ν. Equations (2.6c) of this paper and (2.8) of Ref. 7(a) lead to

$$\dot{Q}_\mu = \frac{v}{\alpha'}\breve{g}^{\sigma}{}_{\mu}\Gamma_\sigma \frac{1}{Mc} - v\,2P_\mu . \tag{2.39}$$

From this and (2.34) we obtain for the derivative of

$$Y^\mu = Q^\mu + d^\mu \tag{2.30$'$}$$

the following result:

$$\dot{Y}_\mu = -v\,2P_\mu . \tag{2.40}$$

This relation is identical to (3.21) of Ref. 7(a) for the QRR, for which the Hamiltonian was different. It means that Y_μ follows a straight line parallel to P_μ as τ proceeds, as one would expect of the center of mass. This justifies the name we gave to it in (2.24). Although the components of Y_μ do not commute, their proper-time derivatives do commute among each other.

We now fix the gauge in the same way as we did for the rotator.[7(a)] We choose τ as the proper time of the center-of-mass position Y_μ by demanding

$$\dot{Y}_\mu \dot{Y}^\mu = 1 . \tag{2.41}$$

There are many other equivalent gauge-fixing conditions, e.g., $\dot{Y}_\mu \dot{Y}^\mu = \alpha'$, which makes H dimensionless and gives τ the dimension 1 (or rather $\hbar$), which will all lead to the same result. But if one chooses instead of α' an operator on the right-hand side of (2.41) the result can change. Therefore we must consider (2.41) as a separate assumption, though it is a natural choice. From the gauge-fixing condition (2.41) with (2.40), we find that the unknown velocity v in the Hamiltonian (2.33) is fixed to be

$$v = -\frac{1}{2Mc} . \tag{2.42}$$

Now we calculate the commutator of d_μ with $\dot{d}_\nu$. Using (2.37) and the commutator of $\hat{d}_\mu$ and Γ_ρ given in (2.54) of Ref. 7(a), one can easily find

$$[d_\mu, \dot{d}_\nu] = i\frac{v}{\alpha'(Mc)^2}\breve{g}_{\mu\nu}(\hat{P}_\rho \Gamma^\rho) = -i\breve{g}_{\mu\nu}\frac{1}{2Mc}\frac{\hat{P}_\rho\Gamma^\rho}{\alpha' P_\mu P^\mu} \underset{C}{=} -i\breve{g}_{\mu\nu}\frac{1}{2Mc} . \tag{2.43}$$

The last equality, indicated by C, has been obtained using the constraint (2.34). This suggests the definition of the "canonical-conjugate" variable of $\xi^\mu = -d^\mu$ by

$$\pi_\mu = \frac{1}{v}\dot{d}_\mu = -2Mc\dot{d}_\mu = -2\dot{\hat{d}}_\mu = -\frac{1}{\alpha' Mc}\breve{g}^{\sigma}{}_{\mu}\Gamma_\sigma , \tag{2.44}$$

because then one obtains

$$[\xi_\mu,\pi_\nu]=-i\check{g}_{\mu\nu}\frac{1}{\alpha' P_\mu P^\mu}\hat{P}_\rho\Gamma^\rho \underset{C}{=} -i\check{g}_{\mu\nu} . \tag{2.45}$$

By a straightforward calculation using (2.44), (2.37), and the CR of Γ_σ [Eq. (2.40) of Ref. 7(a)], one obtains

$$[\pi_\mu,\pi_\nu]=-i\frac{1}{\alpha'^2(Mc)^2}\Sigma_{\mu\nu} . \tag{2.46}$$

Equations (2.32), (2.46), and (2.45) are the CR's of the new internal momentum and position that correspond to the CR (2.6b) for the conventional relativistic canonical variables.[9] Though they look quite different we will show in a forthcoming article that in the nonrelativistic limit, $c\to\infty$,

$$\xi_0\to 0,\ \ \xi_i\to\xi_i^{(\infty)} ,$$
$$\pi_0\to 0, \pi_i\to\pi_i^{(\infty)}\ \ (i=1,2,3) ,$$

where $\xi_i^{(\infty)}$ and $\pi_i^{(\infty)}$ are the observables fulfilling the three-dimensional Heisenberg CR. Thus (2.32), (2.46), and (2.45) are relativistic generalizations of the Heisenberg algebra.

Therewith we have completed the conjecturing process of the algebra of observables for the QRO. The basic observables are very much the same as those of the QRR. They consist of the momenta P_μ, total angular momenta $J_{\mu\nu}$, center-of-mass position Y_μ, spin tensor $\Sigma_{\mu\nu}$, intrinsic coordinate ξ_μ, intrinsic momentum π_μ, and $SO(3,2)_{S_{\mu\nu},\Gamma_\mu}$ as the relativistic spectrum generating group. The principal differences are the Hamiltonians and the representations of the relativistic spectrum-generating group. For the QRR the operator $(d_\mu d^\mu)$, representing the (negative square of the) distance between the center of mass Y^μ and the position Q^μ, i.e., the extension of the object, was a constant of the motion. For the vibrating object this cannot be the case. Calculating $(d_\mu d^\mu)^{\cdot}$ using (2.37), (2.21), (2.23), and $d_\mu P^\mu=0$, we obtain

$$(d_\mu d^\mu)^{\cdot}=\{\dot{d}_\mu,d^\mu\}=\frac{1}{2(Mc)^2}\frac{1}{\alpha'}\{\Gamma_\sigma,d^\sigma\}$$
$$=\frac{1}{2(Mc)^2}\frac{1}{\alpha'}\{\Gamma_\sigma,S^{\sigma\nu}\}\hat{P}_\nu\frac{1}{Mc} . \tag{2.47}$$

This is zero in the Majorana representation of the QRR by (2.44) of Ref. 7(a). For the oscillator we need, therefore, a different, more complicated representation.

Before we study such a representation and derive the consequences for the QRO, we derive the equation of motion for d_μ. It can easily be derived, using the CR of two Γ's, that

$$\dot{\Gamma}_\mu=\frac{v}{\alpha'}\hat{d}_\mu . \tag{2.48}$$

Inserting this into the derivative of (2.37), one obtains

$$\ddot{d}_\mu+\left[\frac{1}{\alpha' 2Mc}\right]^2 d_\mu=0 . \tag{2.49}$$

The solution of this differential equation for the operator $\hat{d}_\mu=d_\mu Mc$ is

$$\hat{d}_\mu(\tau)=A_\mu e^{-i\tau/2\alpha' Mc}+A_\mu^\dagger e^{i\tau/2\alpha' Mc} , \tag{2.50}$$

where the integration constants are

$$\hat{d}_\mu(0)=A_\mu+A_\mu^\dagger ,$$
$$\dot{\hat{d}}_\mu(0)=\frac{-i}{2\alpha' Mc}(A_\mu-A_\mu^\dagger) . \tag{2.51}$$

This, together with (2.30′) and (2.40) shows that the position Q_μ performs a Zitterbewegung about the center of mass which proceeds in the direction of the momentum. Equation (2.50) looks similar to (A43) for the lowest mode of the relativistic string, but is much more complicated since A_μ and $A_\mu^\dagger$ have noncommuting components, which may be complicated functions of the a^μ_{-n} and a^μ_{+n}, if they can even be explicitly given in terms of them. With the Virasoro constraint (A23) imposed, Eq. (A43) describes rotation, whereas (2.50) contains vibrations. Note also that the meaning of τ and of the time development is different in (2.50) and (A43).

In comparison, the equation of motion for the QRR[7(a)] is more complicated and the solution that follows from the QRR Hamiltonian is[10]

$$\hat{d}_\mu=-i\exp\left[-i\frac{\lambda^2}{2Mc}\tau\right]\left[-\exp\left[i\frac{\lambda^2}{Mc}\hat{P}_\mu\Gamma^\mu\tau\right]A_{-1\mu}+\exp\left[-i\frac{\lambda^2}{Mc}\hat{P}_\rho\Gamma^\rho\tau\right]A_{+1\mu}\right] , \tag{2.52}$$

where the constants of integration are given by

$$\hat{d}_\mu(0)=i(A_{-1\mu}-A_{+1\mu}) ,$$
$$\dot{\hat{d}}_\mu(0)=\frac{\lambda^2}{Mc}[(\tfrac{1}{2}-\hat{P}_\rho\Gamma^\rho)A_{-1\mu}-(\tfrac{1}{2}+\hat{P}_\rho\Gamma^\rho)A_{+1\mu}] . \tag{2.53}$$

III. REPRESENTATION SPACES OF THE QRO AND THE VIBRATION-ROTATION SPECTRA OF HADRONS

We will now construct the space of physical states of our QRO which is the representation space of its algebra of observables. Since the basic observables are the same as for the QRR,[7(a)] and only acquire additional properties due to the fact that the extended object now performs vibrations, and not exclusively rigid rotations ($-d_\mu d^\mu$ being a constant for the QRR), we have to go to a larger representation space of $SO(3,2)_{S_{\mu\nu}\Gamma_\mu}$ than the Majorana representation. This is the only new aspect, otherwise the representation is constructed in the same way as described in detail in Ref. 7(b). We will report here only the result.

The Wigner basis system of the irreducible representation space of this algebra of observables is

$$|\hat{p}m\mu jj_3\rangle=U(L^{-1}(\hat{p}))(|\hat{p}_{\text{rest}}m\rangle\otimes|\mu jj_3\rangle) , \tag{3.1}$$

with $\hat{p}_{\text{rest}}=(1,0,0,0)$ and $L^{-1}(\hat{p})$ being the boost. They are generalized eigenvectors of the following complete system of commuting observables (CSCO):[11]

$$\hat{P}_\mu = P_\mu / M, \quad M^2 = P_\mu P^\mu ,$$
$$\hat{P}_\mu \Gamma^\mu, \quad \tfrac{1}{2}\Sigma_{\mu\nu}\Sigma^{\mu\nu}, \quad \Sigma_{12}^{(R)} \tag{3.2}$$

(units are now chosen so that $c=1$), with the eigenvalues[12]

$$\hat{p}_\mu, \quad m^2, \quad \mu, \quad j(j+1), \quad j_3 . \tag{3.3}$$

We use again the four-velocity of the center of mass $\hat{p}_\mu$ (eigenvalues of $\dot{Y}_\mu$), rather than the momenta p_μ, as the set of continuous quantum numbers. If the constraint were trivial ($P_\mu P^\mu = m_0{}^2 = \text{constant}$) or if no constraint were used, these two choices would be equivalent. If a constraint like (2.34) is used, the j- and μ-changing operators will also change the value of p_μ but need not change the value of $\hat{p}_\mu$. Constraint compatibility thus determines the choice to use $\hat{p}_\mu$ eigenvectors.[13]

The Wigner basis vectors at rest, and only those at rest,[14] are the direct product of the basis vectors $|\mu j j_3\rangle$ of our $SO(3,2)_{S_{\mu\nu}\Gamma_\mu}$ representation in which $SO(3)_{S_{ij}} \times SO(2)_{\Gamma_0}$ is diagonal and the basis vectors at rest of the "orbital" Poincaré group $\mathscr{P}_{\hat{P}_\mu, M_{\mu\nu}=J_{\mu\nu}-S_{\mu\nu}}$. Therefore

$$\mu = \text{eigenvalue of } (\hat{P}_\nu \Gamma^\nu \underset{R}{=} \Gamma_0) , \tag{3.4}$$

$$j(j+1) = \text{eigenvalue of } (\tfrac{1}{2}\Sigma_{\mu\nu}\Sigma^{\mu\nu} \underset{R}{=} \tfrac{1}{2} S^{ij} S^{ij}) , \tag{3.5}$$

$$j_3 = \text{eigenvalue of } (\Sigma_{12}^{(R)} \underset{R}{=} S_{12}) \tag{3.6}$$

Here, the R under the equals sign means that this equality holds on the subspace of rest-frame states, i.e., on the subspace which is spanned by the Wigner basis vectors $|\hat{p}_{\text{rest}} m \mu j j_3\rangle$. The spectra of the quantum numbers μ, j, j_3 are, therefore, determined by the choice of the irreducible representation of SO(3,2).

Before we discuss the representations for the QRO, we would like to illustrate our method by considering again the simple, "remarkable," Majorana representation of SO(3,2), which has been used in Ref. 7(a) and for which the representation space has been derived in full detail in Ref. 7(b). However, instead of the rotator constraint (4.6) of Ref. 7(a), we shall use the constraint (2.34).

The reduction of one of the four Majorana representations of SO(3,2) with respect to $SO(3)_{S_{ij}} \times SO(2)_{\Gamma_0}$, i.e., its multiplicity pattern, is depicted in Fig. 1. μ denotes the eigenvalue of Γ_0 which becomes the eigenvalue of $\hat{P}_\mu \Gamma^\mu$ when the SO(3,2) representation is induced to a representation of the whole algebra. j characterizes the irreducible representation of $SO(3)_{S_{ij}}$ and becomes the spin after induction. To each box in the multiplicity pattern there corresponds after induction an irreducible representation space $\mathscr{H}^{\mu=j+1/2}(m,j)$ of the physical Poincaré group, $\mathscr{P}_{P_\mu J_{\mu\nu}}$, representing an "elementary particle" of spin j, mass m, and a new "principle" quantum number μ. For the Majorana representation this new principle quantum number is redundant because $\mu = j + \frac{1}{2}$.

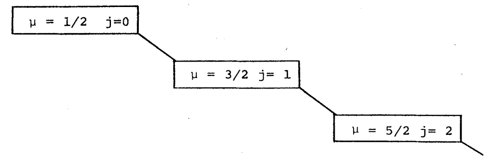

FIG. 1. Multiplicity pattern of the integer-spin Majorana representation of SO(3,2). The eigenvalues of Γ_0 and S_{12} increase in integer steps, as they do along the diagonals of the representation in Fig. 4.

This construction can be done for any value of mass m ($m^2 > 0$), so the representation space will be the continuous direct sum over all $m^2 > 0$. Then one imposes the constraint

$$P_\mu P^\mu - \frac{1}{\alpha'} \hat{P}_\mu \Gamma^\mu \underset{C}{=} 0 \tag{2.34}$$

in order to obtain the subspace of physical states [on which (2.34) holds]

$$\mathscr{H} = \sum_{j=0,1,2,\ldots} \oplus \mathscr{H}^{\mu=j+1/2}(m(j), j) , \tag{3.7}$$

where, as a consequence of (2.34),

$$m^2(j) = \frac{1}{\alpha'}\mu = \frac{1}{2}\frac{1}{\alpha'} + \frac{1}{\alpha'} j, \quad j = 0,1,2,\ldots . \tag{3.8}$$

This result is identical to the results (A39) and (A42) for the lowest mode of the relativistic string, though the mathematical structure used appears to be totally different. Unfortunately, (3.8) is phenomenologically wrong, because in a fit of

$$m^2 = m_0{}^2 + \frac{1}{\alpha'} j \tag{3.9}$$

to the resonances associated with the usual Regge trajectory one obtains phenomenologically $1/\alpha' \approx 1$ GeV^2, $m_0{}^2 \approx -0.5$ GeV^2 [and not $m_0{}^2 = \frac{1}{2}/\alpha' \approx +0.5$ GeV^2 as predicted by (3.8)]. A negative value for $m_0{}^2$ is, in particular, excluded by the construction of the representation. This is a problem which the QRR[7(a)] did not have and the hope is that a combination of rotations with oscillations will bring $m_0{}^2$ up towards zero.

After this illustration by an unphysical example we turn to a more complicated representation of SO(3,2), which has a chance to describe a realistic particle spectrum.

There exists a class of representations of SO(3,2) which have the multiplicity pattern depicted in Fig. 2. Like the Majorana representation this is also a class of singleton representations, i.e., irreducible representations in which an irreducible representation of the maximal compact subgroup $SO(3)_{S_{ij}} \times SO(2)_{\Gamma_0}$ appears at most once. The basis vectors are therefore again the $|\mu j j_3\rangle$, only that now μ of (3.4) and j of (3.5) are independent and have the spectrum[16]

$$\mu = \mu_{\min} + \nu, \quad \nu = 0,1,2,3,\ldots, \quad \mu_{\min} > \tfrac{1}{2} ,$$
$$j = 0,2,4,\ldots,\nu \text{ if } \nu = \text{even} , \tag{3.10}$$
$$j = 1,3,5,\ldots,\nu \text{ if } \nu = \text{odd} ,$$

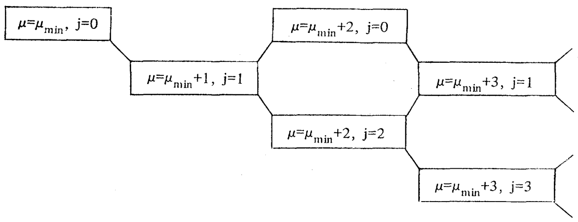

FIG. 2. Multiplicity pattern (Ref. 15) of a class of representations of SO(3,2) with the eigenvalue of Γ_0 given as in (3.10).

which is the scheme of Fig. 2. As in the Majorana case, to each box in Fig. 2 there corresponds an irreducible representation space of $SO(3)\times SO(2)$ labeled by (μ,j) and also an irreducible representation space $\mathscr{H}^{\mu}(m,j)$ of the Poincaré group when induced to the whole algebra. Taking a continuous direct sum for all m ($m^2>0$) and then imposing the constraint (2.34) leads to the physical subspace

$$\mathscr{H}=\sum_{\substack{\nu=0,1,2,3,\ldots\\ j=0,2,\ldots,\nu(\text{even})\\ \text{or } j=1,3,5,\ldots,\nu(\text{odd})}}\oplus\mathscr{H}^{\mu}(m(\nu),j)) \tag{3.11}$$

with

$$m^2(\nu)=\frac{1}{\alpha'}\mu_{\min}+\frac{1}{\alpha'}\nu,\quad \nu=0,1,2,\ldots\,. \tag{3.12}$$

The representation theory of our relativistic spectrum generating group, together with the Hamiltonian (2.33) has, therefore, led to the same spectrum as the three-dimensional oscillator at rest, Eqs. (A38) and (A23) (or the lowest mode of the relativistic string in the center-of-mass gauge before the Virasoro constraint is imposed). What we have gained—in addition to an unambiguous representation theory—is a simpler algebra of operators with essentially only one new basic observable, the Γ_μ, adjoined to the observables P_μ and $J_{\mu\nu}=Q_\mu\wedge P_\nu+S_{\mu\nu}$ of the Poincaré group. Furthermore, these basic observables are the same as for the quantum relativistic rotator,[7(a)] only that for the rotator one has the additional "rigidity" condition

$$(d_\mu d^\mu)^{\cdot}=0\,, \tag{3.13}$$

and a different Hamiltonian. Therefore it is straightforward to suggest a combination of the relativistic rotator with the relativistic oscillator to obtain a relativistic vibrating rotator, in very much the same spirit as is done for molecules. One relaxes (3.13) and takes as the Hamiltonian the combination of the oscillator Hamiltonian (2.33) of this paper and the rotator Hamiltonian (2.55) of Ref. 7(a) to obtain

$$\mathscr{H}=\phi\left[P_\mu P^\mu+\tfrac{9}{4}\lambda^2-\lambda^2\alpha^2-\lambda^2\hat{W}-\frac{1}{\alpha'}\hat{P}_\mu\Gamma^\mu\right]$$
$$(\hat{W}=\tfrac{1}{2}\Sigma_{\mu\nu}\Sigma^{\mu\nu})\,. \tag{3.14}$$

λ^2 is the rotator parameter with dimension of mass^2 and α^2 is a constant. The constraint connected with this Hamiltonian leads to the mass spectrum

$$m^2(\nu,j)=m_0{}^2+\frac{1}{\alpha'}\nu+\lambda^2 j(j+1)\,. \tag{3.15}$$

Thus to each box in Fig. 2 there corresponds a resonance with a different mass which is now determined by its spin j and by the new principal or vibrational quantum number ν. In this way resonances are considered as vibrational and rotational excitations.

We want to emphasize that we have a relativistic quantum mechanics for the rotator which leads to the mass formula (2.57) of Ref. 7(a), and now also relativistic quantum mechanics for the oscillator which leads to the mass formula (3.12) of this paper. These are two consistent theoretical models and the mass formulas are consequences derived within them. Unlike the usual theories with "mass formulas," in which an operator with certain transformation properties is called the mass operator without showing that it *is* the observable $P_\mu P^\mu$ connected with the Poincaré group, our m and j really are the quantities that characterize the representations of the Poincaré group and therewith (according to Wigner) the elementary particle. Thus not only have we derived such simple mass formulas as (2.57) of Ref. 7(a) and (3.12) of this paper, but we have also derived that m *is* the mass.

The mass formula (3.15) has not (yet) been derived because we do not yet have a theory for a quantum-mechanical combination of a relativistic rotator and oscillator. At the present time (3.15) is only a plausible result of such a combination. Here we shall empirically test (3.15) and if it shows promise we will try to develop a relativistic quantum mechanics for the Hamiltonian (3.14) in a subsequent paper.

There are other representations of SO(2,2) which may give more appropriate descriptions of meson (and baryon) resonances. The representations with the multiplicity pattern of Fig. 2 contract in the nonrelativistic limit into the algebra of an oscillator for which the intrinsic angular momentum is entirely orbital

$$S_{ij}=\xi_i^{(\infty)}\wedge\pi_j^{(\infty)}\,, \tag{3.16}$$

i.e., is caused by the intrinsic orbital motion of the constituents. If one wants to obtain the picture of a vibrating and rotating diquark dumbbell as the nonrelativistic limit of our relativistic model one needs a representation of SO(3,2) that gives an angular momentum

$$S_{ij}=\xi_i^{(\infty)}\wedge\pi_j^{(\infty)}+\tilde{S}_{ij}\,, \tag{3.17}$$

where $\tilde{S}_{ij}$ is the total spin of the two quarks.

Such representations of SO(3,2)—with integer as well as with half-integer s [$s(s+1)$=eigenvalue of $\frac{1}{2}\tilde{S}_{ij}\tilde{S}_{ij}$]—exist. Their multiplicity pattern differs from Fig. 2 mainly by starting with $j=s$ in the top left corner instead of with $j=0$. We will discuss these representations, their nonrelativistic contractions, and their detailed comparison with the experimental data in a forthcoming paper.

Here we have fitted[17] formula (3.15) with the spectrum (3.10) for ν,j to the meson resonances which have only nonstrange quarks, the ρ and ω towers, to determine the j mass unit λ^2 ($j$un) and the ν mass unit $1/\alpha'$ (νun) empirically. (We have also included the nucleon resonances assigned to analogous patterns starting with $j=\frac{1}{2}$ and $j=\frac{3}{2}$

in the empirical determination of $j$un and νun.) The empirical values of $j$un and νun will depend upon the assignment of the new vibrational quantum number ν to the resonances. Also, if other representations are used, then other empirical values may be obtained. We found that the empirical values of $j$un and νun for different towers did not differ by much. We therefore performed a single fit with $1/\alpha'$ and λ^2 the same for all the towers and with only $m_0{}^2$ allowed to vary from tower to tower, except that we chose $m_0{}^2(\omega)=m_0{}^2(\rho)$ as required by exchange degeneracy which is experimentally rather accurately fulfilled.

Depending upon the particle assignments to the boxes of Fig. 2 (i.e., the assignment of a value for the quantum number ν to the resonances in the Particle Data Group Table), we found two sets of values for $1/\alpha'$ and λ^2 with acceptable χ^2:

$$\frac{1}{\alpha'}=0.97\pm0.03 \text{ GeV}^2 ,\quad \lambda^2=0.021\pm0.008 \text{ GeV}^2 ,\quad \text{with } \chi^2/n_D=19/31 , \tag{3.18}$$

and

$$\frac{1}{\alpha'}=0.53\pm0.02 \text{ GeV}^2 ,\quad \lambda^2=0.10\pm0.01 \text{ GeV}^2 ,\quad \text{with } \chi^2/n_D=15/31 . \tag{3.19}$$

Equation (3.18) is just a refinement of linearly rising Regge trajectories with universal slope and varying intercept, where the linear trajectory is formed by the particles on the diagonal of Fig. 2 with $\nu=j$. It leads to large negative values of $m_0{}^2$ as one would expect:

$$m_0{}^2(\rho)=m_0{}^2(\omega)=-0.40\pm0.02 \text{ GeV}^2 .$$

Equation (3.19) is slightly better and also has the great advantage of less negative empirical values for $m_0{}^2$:

$$m_0{}^2(\rho)=m_0{}^2(\omega)=-0.11\pm0.02 \text{ GeV}^2 .$$

For the assignment corresponding to (3.19) we could therefore attempt to reduce $m_0{}^2$ to zero by using correction terms:

$$m_{\text{corr}}{}^2=-y_2\nu^2-y_4\nu j(j+1) , \tag{3.20}$$

which describe anharmonicity and correction to the moment of inertia. If we now force $m_0{}^2$ to be non-negative, $m_0{}^2(\rho)=m_0{}^2(\omega)=0$, we obtain a fit which is worse, but still has an acceptable value of χ^2: $\chi^2/n_D=23/30$. It gives the following values for the empirical parameters (in GeV^2):

$$\frac{1}{\alpha'}0.37\pm0.05,\quad \lambda^2=0.12\pm0.02 ,\quad y_2=-0.020\pm0.06,\quad y_4=-0.004\pm0.004 . \tag{3.21}$$

After numerous attempts with many different assignments for the new quantum number ν, but using only the representation space (3.11), we have concluded that the present particle data do not allow a better fit with $m_0{}^2\geq0$ than (3.21) for the ρ and ω towers.

The best fit, with $m_0{}^2(\rho)=m_0{}^2(\omega)$ as a free parameter and $y_2=0$, is reproduced in Fig. 3. It has been obtained as a fit to 19 baryon and 18 meson resonances; only the meson masses are displayed in Fig. 3. The values of the parameters obtained in this fit are (in GeV^2)

$$\frac{1}{\alpha'}=0.55\pm0.02,\quad \lambda^2=0.134\pm0.019 ,\quad m_0{}^2=-0.19\pm0.04 ,\quad y_4=0.007\pm0.003,\quad \text{with } \chi^2/n_D=11/30 . \tag{3.22}$$

(Excluding the baryons from the fit leads to essentially the same values for the parameters.)

Except for the predictions of a 0^+ around 2.05 GeV and a 3^- around 1.93 GeV there is—ignoring the $I=0$ and $I=1$ degeneracy—a one-to-one correspondence between the experimental data for normal-j^P and positive-C_nP meson resonances and the lower excitation states of the vibrating rotator.

The agreement between the fit and the experimental data is very good. However, since the experimental errors (widths) are large and since other assignments leading to the parameters (3.18) are possible, we should not mistake the goodness of this fit as an empirical proof. The main drawback of the fit (3.18) was that

$$m_0{}^2=m^2(\nu=0,j=0)<0 .$$

In representations that start with $j_{\min}=s=1$, this problem does not arise, and as these representations can be connected in the nonrelativistic limit to the quark model, the νun and $j$un of (3.18) may even be the preferred values.

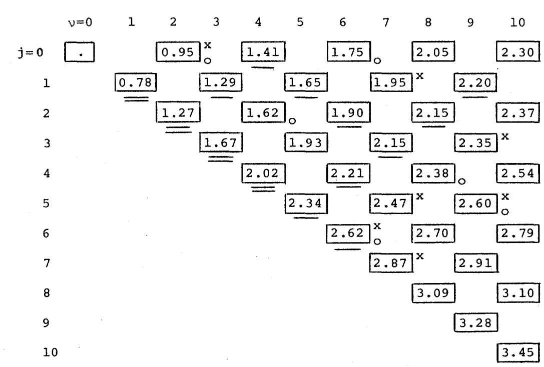

FIG. 3. Predicted masses for the vibrating rotator. The values are the masses in GeV as computed from (3.15) modified by (3.20) and using the parameters (3.22). Underlining the boxes means that the corresponding experimental value has been used in the fit, underlining twice means that the corresponding experimental value for $I=0$ and $I=1$ has been used in the fit. The other masses are predictions of $I=0$ and $I=1$ resonances with normal j^P and positive C_nP. A superscript $\times$ (or subscript $\circ$) means that a resonance with these quantum numbers and with $I=0$ (or $I=1$) has been observed around the predicted mass value [Particle Data Group, Rev. Mod. Phys. **56**, S1 (1984)].

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APPENDIX: COMPARISON WITH THE LOWEST MODE OF THE RELATIVISTIC STRING

In this appendix we collect a few more facts and formulas on the relativistic string and derive the representation spaces for the conventional four-dimensional relativistic oscillator and for the lowest mode of the string. This will provide an easier transition to the representations for the QRR[7(b)] and the QRO, whose nonrelativistic limits have certain similarities with the modes of the string.

The algebraic relations of the new observables a_m^μ and $\Sigma^{\mu\nu}$ follow immediately from their definitions. From (2.28) and (2.4) it follows that the a_m^μ obey the CR

$$[a_m^\mu, a_n^\nu] = -m\breve{g}^{\mu\nu}\delta_{m,-n}, \quad a_m^{\mu\dagger} = a_{-m}^\mu . \tag{A1}$$

As a consequence of the definition (2.28) the $a_{\pm m}^\mu$ fulfill the relation

$$\hat{P}_\mu a_{\mp m}^\mu = 0 . \tag{A2}$$

The $\Sigma^{\mu\nu}$ can be expressed in terms of the a_m^μ by

$$\Sigma^{\mu\nu} = -i \sum_{m=1}^{\infty} \frac{1}{m}(a_{-m}^\mu a_{+m}^\nu - a_{-m}^\nu a_{+m}^\mu) \tag{A3}$$

$$= \breve{g}^\mu{}_\rho \breve{g}^\nu{}_\sigma S^{\rho\sigma} , \tag{A4}$$

and as a consequence of (2.26) and (2.4) they fulfill the CR

$$[\Sigma^{\mu\nu}, \Sigma^{\rho\sigma}] = -i(\breve{g}^{\mu\rho}\Sigma^{\nu\sigma} + \breve{g}^{\nu\sigma}\Sigma^{\mu\rho} - \breve{g}^{\mu\sigma}\Sigma^{\nu\rho} - \breve{g}^{\nu\rho}\Sigma^{\mu\sigma}) \tag{A5}$$

and the relation

$$\hat{P}_\mu \Sigma^{\mu\nu} = \hat{P}_\nu \Sigma^{\mu\nu} = 0 . \tag{A6}$$

Note that (A2) and (A6) are not constraints.

In order to establish the connections we have used here the same notation as for the quantum relativistic rotator.[7] The following relations are identical: (2.10) of this paper and (2.2) of Ref. 7(a), (A5) of this paper and (2.16) of Ref. 7(a), (A6) of this paper, and (2.18) of Ref. 7(a), (2.26) of this paper and (2.13) of Ref. 7(a).

With the α_n^μ one usually defines

$$L_n = -\tfrac{1}{2} \sum_{\substack{k=-\infty \\ k\neq n,0}}^{+\infty} :\alpha_{n-k}^\mu \alpha_{k\mu}: - \sqrt{\alpha'/2} P_\mu \alpha_n^\mu , \tag{A7}$$

where

$$\alpha_0^\mu = \sqrt{2\alpha'} P^\mu . \tag{A8}$$

The L_n fulfill

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{d}{12} m(m^2-1)\delta_{m,-n} \quad (d=4) . \tag{A9}$$

With the a_n^μ one defines

$$\Lambda_n = -\tfrac{1}{2} \sum_{\substack{k=\pm 1, \pm 2, \ldots \\ k\neq n}}^{\pm\infty} :a_{n-k}^\mu a_{k\mu}: , \tag{A10}$$

which, using (A9), one can express in terms of the α_m^ν:

$$\Lambda_n = L_n + \sum_{\substack{k=1,2,\ldots \\ k\neq n}}^{\infty} :\hat{P}_\nu \alpha_{n-k}^\nu \hat{P}_\rho \alpha_k^\rho: + \sqrt{\alpha'/2} P_\mu \alpha_n^\mu . \tag{A11}$$

The Λ_n fulfill relations similar to (A9):

$$[\Lambda_m, \Lambda_n] = (m-n)\Lambda_{m+n} + \frac{d-1}{12} m(m^2-1)\delta_{m,-n} \quad (d=4) . \tag{A12}$$

The L_n and Λ_n are Lorentz invariants. In the rest frame $p_{\text{rest}} = (m,0,0,0)$ (units are now chosen so that $c=1$) one has

$$a_m^0 \underset{R}{=} 0 , \qquad a_m^i \underset{R}{=} \alpha_m^i, \quad i=1,2,3 \text{ or } 1,\ldots,d-1 . \tag{A13}$$

In the rest frame, and only there, one also has

$$\Sigma^{ij} \underset{R}{=} S^{ij} \underset{R}{=} J^{ij}, \quad \Sigma^{0i} \underset{R}{=} S^{0i} - S^{0i} = 0 , \qquad J^{0i} \underset{R}{=} -Y^i M . \tag{A14}$$

Because of (A13) one can write the Lorentz invariant Λ_n also as

$$\Lambda_n \underset{R}{=} \sum_{\substack{k=1,2,\ldots \\ k\neq n,0}}^{\infty} :\alpha_{n-k}^i \alpha_k^i: .$$

One can impose various kinds of constraints. For instance one can demand

$$L_n \underset{C}{=} 0 \text{ for } n\neq 0, \quad L_0 - \alpha(0) \underset{C}{=} 0 \quad (d=3+1) , \tag{A15}$$

where $\alpha(0)$ is a number. The latter can be written

$$P_\mu P^\mu + \frac{\alpha(0)}{\alpha'} \underset{C}{=} -\frac{1}{\alpha'} \sum_{n=1,2,\ldots}^{\infty} :\alpha_{-n}^\mu \alpha_{+n\mu}: . \tag{A16}$$

This constraint follows from the orthogonal gauge condition. On the other hand, one can demand

$$\Lambda_n \underset{C}{=} 0 \text{ for } n\neq 0, \quad \Lambda_0 - \alpha(0) - \alpha' P_\mu P^\mu \underset{C}{=} 0 . \tag{A17}$$

The latter part of (A17) can be written

$$P_\mu P^\mu + \frac{\alpha(0)}{\alpha'} \underset{C}{=} -\frac{1}{\alpha'} \sum_{m=1,2,\ldots}^{\infty} a_{-m}^\mu a_{+m\mu} \underset{C,R}{=} \frac{1}{\alpha'} \sum_{m=1,2,\ldots}^{\infty} \alpha_{-m}^i \alpha_{+m}^i . \tag{A18}$$

If one also uses the constraint

$$\phi_{\pm m}=\hat{P}_\mu \alpha^\mu_{\pm m} \underset{C}{=} 0 \tag{A19}$$

(which we do not use), then one sees immediately from (A11) that the constraints (A17) and (A15) are identical. Otherwise they are two different theories.

The spectrum of the mass squared that follows from (A17) is the same as the energy spectrum of a three-dimensional [or $(d-1)$-dimensional] oscillator at rest for every mode m. In contrast, Eq. (A15) [(A16) for $d=3+1$] still has ghosts and other problems. For $d=4+1$, $5+1$, or $25+1$ and the case that one demands (A15), one fixes the remaining freedom of conformal transformations by the choice of light-cone variables (light-cone gauge). The constraints (A15) can then be used to eliminate the nontransverse components

$$\alpha_n^\pm=(\alpha_n^0\pm\alpha_n^{d-1})\frac{1}{\sqrt{2}}, \quad n\neq 0 \tag{A20}$$

so that one remains with $d-2$ independent transverse components α_n^i, $i=1,2,\ldots,d-2$. The S^{ij} are again defined in terms of these independent components by a formula like (2.9) for $\mu,\nu=i,j$ only and the J^{ij} by something like (2.26a), but the S^{i0} and J^{i0} are problematic. The constraint (A15) leads then to

$$P_\mu P^\mu+\frac{\alpha(0)}{\alpha'} \underset{C}{=} \frac{1}{\alpha'}\sum_{n=1,2,\ldots}^{\infty} \alpha^i_{-n}\alpha^i_{+n}\,, \quad i \text{ summed over } 1,2,\ldots,d-2 \tag{A21}$$

so that the mass spectrum is again that of a $(d-2)$-dimensional oscillator for every mode n.

A representation theory for the quantum relativistic string is very complicated and has been developed only in a very rudimentary form by applying the creation operators α^μ_{-n} successively to a vacuum state and projecting out a physical subspace with the use of the constraint. We will describe it here only for the lowest mode:

$$\alpha^\mu_{\pm m}=0 \text{ (or } a^\mu_{\pm m}=0) \text{ except for } m=1\,. \tag{A22}$$

Then the string spectrum in the form (A21) for $d=5$ or the form (A18) becomes the spectrum of a three-dimensional oscillator with the additional constraints

$$\Lambda_{-2}=a^\mu_{-}a_{-\mu} \underset{C}{=} 0, \quad \Lambda_{+2}=a^\mu_{+}a_{+\mu} \underset{C}{=} 0\,. \tag{A23}$$

For all the other Λ_n we have identically

$$\Lambda_n=0 \quad \text{for } n\neq 0,\pm 2 \tag{A24}$$

due to (A22). The representation theory for the lowest mode of the relativistic string is given completely by the three-dimensional oscillator and therefore it is fully developed. For the sake of definiteness we use the version (A18) rather than (A21). Then we have the relativistic quantum system which at rest is a three-dimensional oscillator with constraints (A23). We review this briefly here because it will make the representation spaces for the QRR and the QRO—which use SO(3,2) representations[7(b)]—more easily acceptable.

The representation space must be a representation space of the physical Poincaré group $\mathscr{P}_{P_\mu J_{\mu\nu}}$ and of the three-dimensional oscillator algebra at rest. As basis vectors of the representation space we use the Wigner basis vectors

$$|\hat{p}mjj_3\nu\rangle\,, \tag{A25}$$

which are generalized eigenvectors of the CSCO

$$\hat{P}_\mu=P_\mu/M,\quad M^2=P_\mu P^\mu,\quad \hat{W}=\tfrac{1}{2}\Sigma_{\mu\nu}\Sigma^{\mu\nu},\quad \Sigma_{12}^{(R)}\,,\quad \Lambda_0=-a_{-1\mu}a^\mu_{+1} \tag{A26}$$

with eigenvalues

$$\hat{p}_\mu,\quad m^2,\quad j(j+1),\quad j_3,\quad \nu\,, \tag{A27}$$

respectively. (R) indicates that it is the operator at rest[11] and the choice of $\hat{P}_\mu$ rather than P_μ for relativistic systems with constraints, such as (A18), has been explained before[18] and is based on the assumption that

$$[\hat{P}_\mu,a^\nu]=0 \quad\text{or}\quad [\hat{P}_\mu,\alpha^\nu]=0\,. \tag{A28}$$

Under this assumption the $\alpha^i_\pm$ do not transform away from rest, though they can change the value of m, which by the constraint (A18) is connected with the value of ν:

$$m^2={m_0}^2+\frac{1}{\alpha'}\nu \quad \left[{m_0}^2=-\frac{\alpha(0)}{\alpha'}\right]. \tag{A29}$$

The basis vectors (A25) at rest,

$$|\hat{p}_{\text{rest}}mjj_3\nu\rangle=U(L(\hat{p}))\,|\hat{p}mjj_3\nu\rangle\,, \tag{A30}$$

[with $L(\hat{p})$ being the inverse boost] therefore span the representation space of the three-dimensional oscillator. For the three-dimensional oscillator the familiar basis vectors are

$$|n_1n_2n_3\rangle \tag{A31}$$

with

$$n_1=0,1,2,\ldots,\quad n_2=0,1,2,\ldots,\quad n_3=0,1,2,\ldots, \tag{A32}$$

which are eigenvectors of

$$N^{(1)}=\alpha^1_-\alpha^1_+,\quad N^{(2)}=\alpha^2_-\alpha^2_+,\quad N^{(3)}=\alpha^3_-\alpha^3_+\,. \tag{A33}$$

It is well known[19] how to transform from the familiar basis into the angular momentum basis

$$|\nu jj_3\rangle\,, \tag{A34}$$

which consists of eigenvectors of

$$\alpha^i_-\alpha^i_+,\quad \tfrac{1}{2}S^{ij}S^{ij},\quad S^{12}\,, \tag{A35}$$

where

$$S^{ij}=-i(\alpha^i_-\alpha^j_+-\alpha^j_-\alpha^i_+) \tag{A36}$$

in accordance with (2.9). The spectrum of νjj_3 is also well known and is given by[19]

$$\nu=0,1,2,3,\ldots,\quad j=0,2,4,\ldots,\nu \quad \text{for } \nu=\text{even}$$
$$j=1,3,5,\ldots,\nu \quad \text{for } \nu=\text{odd}\ ,$$
$$-j \le j_3 \le +j\ . \tag{A37}$$

We depict this spectrum by the scheme shown in Fig. 4. Each box represents the space spanned by

$$|\,\nu=\text{fixed}\ j=\text{fixed}\ j_3\,\rangle$$

with $-j \le j_3 \le +j$. If these vectors are used as the factor in the Wigner basis vectors at rest,[20]

$$|\,\hat{p}_{\text{rest}} m j j_3 \nu\rangle = |\,\hat{p}_{\text{rest}} m\,\rangle \otimes |\,\nu j j_3\,\rangle\ ,$$

which are then boosted to any arbitrary momentum [inverse of Eq. (A30)] to give

$$|\,\hat{p} m j j_3 \nu\rangle\ ,$$

then each box in Fig. 4 represents the space spanned by $|\,\hat{p}m\ j=\text{fixed}\ j_3\nu=\text{fixed}\rangle$. This is an irreducible representation space of the Poincaré group $\mathcal{H}^{\nu}(m,j)$, which is in addition labeled by the number ν. The whole space depicted by the scheme in Fig. 4 is thus the direct sum

$$\mathcal{H}^{(\text{3-dim osc})} = \sum_{\substack{\nu=0,1,2,\ldots \\ j=0,2,4,\ldots,\nu \\ \text{or } j=1,3,5,\ldots,\nu}} \oplus \mathcal{H}^{\nu}(m(\nu),j)\ . \tag{A38}$$

Due to the constraint (A18), the mass m is related to ν by (A29). Each box, therefore, represents an elementary particle as an oscillator excitation with $\nu=0$, $j=0$ being the vacuum state of the oscillator.[21] The particle spectrum predicted by (A38) shows therefore that each spin j is infinitely degenerate with the mass depending only upon the vibrational quantum number ν, which is—in general—also degenerate. However, this is the spectrum of the three-dimensional spacelike oscillator (or three-dimensional oscillator at rest). In order to obtain the spectrum for the lowest mode of the string one has to apply the constraint (A23). We have indicated by the dashed line the actions of Λ_2 and Λ_{-2} in the scheme of Fig. 4. They transform along the horizontal changing the eigenvalue ν by two units without changing the spin. Thus if (A23) is imposed the diagram breaks up and, starting out with $\nu=0$, $j=0$, one can only reach the states

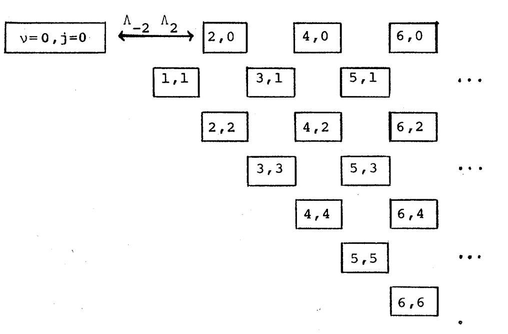

FIG. 4. Spectrum of the three-dimensional oscillator at rest. The numbers in the boxes are values of ν, j.

(irreducible representation spaces of the Poincaré group) on the lower diagonal. The constraint (A23) therefore establishes a connection between ν and j similar to the one for the Majorana representation of SO(3,2).[22] The space of physical states is thus the following direct sum of irreducible representation spaces of the Poincaré group:

$$\mathcal{H}^{(\text{one mode})} = \sum_{j=0,1,2,\ldots} \oplus \mathcal{H}^{\nu=j}(m(j),j)\ . \tag{A39}$$

To see this more clearly one can express the operator $\frac{1}{2}\Sigma_{\mu\nu}\Sigma^{\mu\nu}$ in terms of a_-^{μ}, a_+^{μ} using (A3) for $m=1$. The result is

$$\tfrac{1}{2}\Sigma_{\mu\nu}\Sigma^{\mu\nu} = (a_{-\mu}a_+^{\mu})^2 - (a_{-\mu}a_+^{\mu}) - (a_-^{\mu}a_{-\mu})(a_+^{\mu}a_{+\nu})\ , \tag{A40}$$

or when applied to the basis vectors

$$j(j+1) = \nu^2 + \nu - \Lambda_{-2}\Lambda_{+2}\ . \tag{A40$'$}$$

When the constraint (A23) holds, one obtains

$$j=\nu\ , \tag{A41}$$

so that the degeneracy in j is removed and the mass formula (A29) becomes

$$m^2 = m_0{}^2 + \frac{1}{\alpha'}j,\quad j=0,1,2,\ldots\ . \tag{A42}$$

The lowest mode of the string, i.e., the three-dimensional spacelike oscillator with the constraint (A23) imposed, is a rotator rather than a vibrator because it is rigid in the sense that the length of the intrinsic coordinate perpendicular to the momentum, namely,

$$-\tilde{x}^2 = \tilde{x}_{\perp}^{\mu}(\tau)\tilde{x}_{\perp_{\mu}}(\tau)$$

with

$$\tilde{x}_{\perp}^{\mu}(\tau) = \check{g}^{\mu}{}_{\nu}\tilde{x}^{\nu}(\tau)\ ,$$

is a constant of the motion. $\tilde{x}_{\perp}^{\mu}(\tau)$ for the string is the analog of $-d^{\mu}(\tau)$ for the rotator and $\tilde{x}_{\perp}^{\mu} = \tilde{x}^{\mu}$ if the constraint (A19) is imposed. $d_{\mu}d^{\mu}$ is a constant of the motion for the QRR, but not for the QRO. To show that $\tilde{x}^2$ is a constant of the motion, we use (2.2) for the lowest mode $n=\pm 1$ and (2.28) to obtain

$$\tilde{x}_{\mu^{\perp}} = -\sqrt{2\alpha'}\,i(a_-^{\mu}e^{i\tau} - a_+^{\mu}e^{-i\tau})\ . \tag{A43}$$

Then one obtains

$$\tilde{x}_{\mu^{\perp}}\tilde{x}_{\perp}^{\mu} = -2\alpha'(a_-^{\mu}a_{-\mu}e^{2i\tau} + a_+^{\mu}a_{+\mu}e^{-2i\tau} - \{a_-^{\mu}, a_{+\mu}\})\ , \tag{A44}$$

which does not depend upon τ when the constraint (A23) is used.

If one has n (or an infinite number of) modes instead of a single mode one can repeat the construction of the oscillator spaces as long as one does not impose the constraints (A17) or (A15). One thus arrives at the n-fold direct product of representation space like (A38) for all masses and then has to apply the constraints (A17) to project away the space of "unphysical states," i.e., the subspace on which (A17) is not fulfilled. Only for a single mode is (A17) as simple as (A23); in general the creation and annihilation operators of different modes intermingle and it is probably impossible to find a new set of independent oscillator variables in terms of which the constraints (A17) become simple identities. For the QRO we have, therefore, followed a different path using new variables d_μ and $\hat{d}_\mu \sim \Gamma_\mu$ in terms of which the representation theory can be solved using results from group theory.

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[7](a) R. R. Aldinger *et al.*, Phys. Rev. D **28**, 3020 (1983); (b) A. Bohm, M. Loewe, L. C. Biedenharn, and H. van Dam, *ibid.* **28**, 3032 (1983); (c) R. R Aldinger *et al., ibid.* **29**, 2828 (1984).

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[9]This vector d_μ is as old as the Pauli-Lubanski vector. M. Mathisson, Acta Phys. Pol. **6**, 163 (1937); **6**, 218 (1937). Following M. M. L. Pryce [Proc. R. Soc. London **A195**, 62 (1948)], it is usually constrained to zero (Ref. 8), abolishing the extension of the relativistic object. For an extensive review of its use in the classical theory of relativistic rotations, see H. C. Corben, *Classical and Quantum Theories of Spinning Particles* (Holden Day, San Francisco, 1968). See also K. Rafanelli, Phys. Rev. D **30**, 1707 (1984), and references therein.

[10]R. R. Aldinger, Ph.D. thesis, University of Texas at Austin, 1984.

[11]$\Sigma_{12}^{(R)}$ is defined as $\Sigma_{12}^{(R)} \equiv U(L^{-1}(\hat{p}))\hat{w}_3 U(L(\hat{p}))$, where $\hat{w}_3$ is the third component of the Pauli-Lubanski vector: $\hat{w}^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\hat{P}_\nu J_{\rho\sigma}$.

[12]We have changed here the notation as compared to Ref. 7(b): The spin quantum numbers called s,s_3 there we call j,j_3 here to conform with common usage. The label of the spinor basis called $\overset{j}{j}_3$ in Ref. 7(b), and the spinor basis itself, we shall not use here.

[13]D. J. Almond, Z. Phys. C **15**, 71 (1982); H. van Dam and L. C. Biedenharn, Phys. Rev. D **14**, 405 (1976); A. Bohm, Phys. Rev. **175**, 1767 (1968). (See also Ref. 18.)

[14]The spinor basis is the direct-product basis of an $SO(3,2)_{S_{\mu\nu}\Gamma_\mu}$ representation in which $SO(3,1)_{S_{\mu\nu}}$ is diagonal and the basis vectors of the representation of $\mathscr{P}_{\hat{P}_\mu, M_{\mu\nu}=J_{\nu\mu}-S_{\mu\nu}}$ for any value of $\hat{p}_\mu$. The transformation between the spinor basis and the Wigner basis is given by a generalized Foldy-Wouthuysen transformation which is $\hat{p}$ dependent.

[15]J. B. Ehrman, thesis, Princeton University, 1954, Sec. VII.i.3; L. Jaffe, J. Math. Phys. **12**, 882 (1971); E. Angelopoulos, in *Quantum Theory, Groups, Fields and Particles,* edited by A. O. Barut (Reidel, Dordrecht, 1983), p. 101.

[16]$\mu_{\min} \geq 0$ characterizes, together with the eigenvalues of the second- and fourth-order Casimir operators, the irreducible representations of SO(3,2). For every irreducible representation characterized by a $\mu_{\min} \geq 0$ there exists also an irreducible representation characterized by $\mu_{\max} = -\mu_{\min}$ in which the spectrum of μ is given by $\mu = \mu_{\max} - \nu$, $\nu = 0,1,2,\ldots$.

[17]The values of the parameters have been obtained by minimizing the values of χ^2, with m^2 fitted and with the error taken as $\Delta m^2 = m\Gamma$, where m is the mass of a resonance and Γ is its width.

[18]Reference 7(b) and references therein. The reason is that $\alpha^i_{\pm m}$ cannot commute with P_μ after the constraint (A18) has been imposed, whereas it can still be assumed that $[\hat{P}_\mu, \alpha^i_{\mp m}] = 0$ (Ref. 13).

[19]See, e.g., V. Bargmann and M. Moshinsky, Nucl. Phys. **18**, 697 (1960); **23**, 127 (1961); E. Chacon and M. de Llano, Rev. Mex. Fis. **XII**, 2 (1963); James D. Louck, J. Math. Phys. **6**, 1786 (1965). The latter reference contains a generalization which may be used when higher modes are included.

[20]At rest, and only at rest, the Wigner basis vectors $|\hat{p} m j j_3 \nu\rangle$ split into a direct product $|\hat{p}_{\text{rest}} m j j_3 \nu\rangle = |\hat{p}_{\text{rest}} m\rangle \otimes |\nu j j_3\rangle$ with $\Sigma_{ij}^{(R)} \underset{R}{=} S_{ij}$ acting only on the second factor.

[21]If, instead of (A18), one would impose the constraint $P^\mu P_\mu \underset{C}{=} m_0{}^2$, then each box in (A38) will represent an $\mathscr{H}^\nu(m_0, j)$, where m_0 is independent of ν and j; the boxes describe oscillator excitation but their mass does not depend upon the oscillator quantum numbers.

[22]$\nu = j + \frac{1}{2}$, where ν is the eigenvalue of Γ_0, see, e.g., A. Bohm, in *Lectures in Theoretical Physics,* edited by A. O. Barut and W. E. Brittin (Gordon and Breach, New York, 1968), Vol. X-B.