Long- and short-distance contributions to kaon decays

G. Eilam* and M. D. Scadron
Physics Department, University of Arizona, Tucson, Arizona 85721
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We work out in detail both the long- and short-distance graphs for $K_{L\gamma\gamma}$, $K_{L\mu\bar{\mu}}$, $K_{\pi e\bar{e}}$, and $K_{\pi\pi\gamma}$ decays. The long-distance amplitudes, which we find to dominate, are related to the $\Delta I=\frac{1}{2}$ enhancement in $K_{2\pi}$ and $K_{3\pi}$ decays. The short-distance diagrams contribute typically about 20%.
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I. INTRODUCTION

During the past decade we have witnessed increasing enthusiasm for short-distance quark-model calculations of kaon weak decay processes,[1] perhaps at the expense of the long-distance hadronic pole graphs.[2,3] However, we now have a better understanding of (i) a possible $\Delta I=\frac{1}{2}$ $K_{2\pi}$ long-distance scale[3] [derivable[4] in the quark model using the Cabibbo-GIM (Glashow-Iliopoulos-Maiani[5]) left-handed quark current], (ii) the η-η' mixing angle[6] (driven by the quark-annihilation diagram[7]) which enters into many long-distance η and η' weak transitions, (iii) the Kobayashi-Maskawa (KM) weak mixing parameters[8] (due to the recent measurement of the B-meson lifetime[9]) which enter into all the short distance weak transitions, and (iv) short-distance estimates for heavy quark loops.[10,11]

In this paper we survey in a *model-independent* manner the following kaon decays and their relation to $\Delta I=\frac{1}{2}$ enhancement: $K\to\pi\pi$, $K_L\to\gamma\gamma$, $K_L\to\mu\bar{\mu}$, $K\to 3\pi$, $K\to\pi e\bar{e}$, $K\to\pi\pi\gamma$ (abbreviated by $K_{2\pi},\ldots$). We find that the $\Delta I=\frac{1}{2}$ long-distance scale of $\langle\pi|H_w|K\rangle$, which accounts for almost all of $K_{2\pi}$, $K_{3\pi}$, and $K_{\pi\pi\gamma}$ (except for $\sim$5% spectator graphs), also contributes to $\sim$120% of the $K_{L\gamma\gamma}$, $K_{L\mu\bar{\mu}}$, and $K_{\pi e\bar{e}}$ amplitudes. We furthermore show that the short-distance $K_{L\gamma\gamma}$ and $K_{L\mu\bar{\mu}}$ boxes and $s\to d\gamma$ transition in $K_{\pi e\bar{e}}$ are all about 20% of the observed amplitude magnitudes. Since the relative signs between the long- and short-distance $K_{L\gamma\gamma}$ and $K_{\pi e\bar{e}}$ amplitudes can be demonstrated as negative, we suggest that there now exists one universal picture of $\Delta I=\frac{1}{2}$ enhancement which correctly predicts almost all kaon decays. The $K_{S'\pi e\bar{e}}$ and $K_{\pi\nu\bar{\nu}}$ amplitudes are unique in that they receive only short-distance contributions. In a related work,[12] we shall extend this long- and short-distance analysis to the K_L-K_S mass difference.

First, in Sec. II we briefly review the $\Delta I=\frac{1}{2}$ enhancement of $K_{2\pi}$ decays, stressing that any PCAC (partial conservation of axial-vector current) model properly accounting for the momentum variation of the matrix element consistently sets the scale for $\langle\pi|H_w|K\rangle$. The long-distance $K_{3\pi}$ K and π pole graphs also lead to the same magnitude of $\langle\pi|H_w|K\rangle$. Then in Sec. III we investigate in detail the other two-body decays, $K_{L\gamma\gamma}$ and $K_{L\mu\bar{\mu}}$, whose rates are likewise determined by the $\Delta I=\frac{1}{2}$ scale $\langle\pi|H_w|K\rangle$. Recent quark-model analyses of the η,η' mixing mechanism[6,7] have reinforced our approximate belief in the quadratic mass formula, leading to an almost exact cancellation between the long-distance η and η' pole contributions to the $K_{L\gamma\gamma}$ and $K_{L\mu\bar{\mu}}$ amplitudes, leaving the π^0 pole to control both decays. The associated short-distance box graphs as calculated by Refs. 10 and 13 in these two cases are both of order 20%. For $K_{L\gamma\gamma}$, the relative sign between the long- and short-distance amplitudes is then shown to be negative, leading to good agreement between theory and experiment. The observed error on the $\eta_{\mu\bar{\mu}}$ decay rate blurs the conclusion on the $K_{L\mu\bar{\mu}}$ analysis at the 25% level, but still theory and experiment are in rough agreement.

Next in Sec. IV we examine the three-body kaon decays. The four $\Delta I=\frac{1}{2}$ $K_{3\pi}$ decays are linked to $K_{2\pi}$ via current algebra and PCAC[14] and therefore again to the same long-distance scale of $\langle\pi|H_w|K\rangle$. The long-distance $K^+_{\pi e\bar{e}}$ amplitude is also scaled to $\langle\pi|H_w|K\rangle$ by inner-bremsstrahlung diagrams[2] and, although ambiguous up to a factor of 2 due to the uncertainty regarding the difference of K^+ and π^+ charge radii, it is significantly larger than the short-distance $s\to d\gamma$ contribution. The latter we calculate, following Refs. 10, 11, and 15, to be roughly 20% of the observed $K_{\pi e\bar{e}}$ amplitude. Again we show that the relative sign between these contributions is negative, making the net theoretical amplitude roughly compatible with experiment. Lastly, we briefly look at $K_{\pi\pi\gamma}$ decay and appeal to the results of Ref. 16, where it is shown that the inner-bremsstrahlung graphs approximately match the observed $K_S\to\pi^+\pi^-\gamma$ and $K^+\to\pi^+\pi^0\gamma$ branching ratios. We argue that the former is scaled to $K_S\to\pi^+\pi^-$ and therefore to the $\Delta I=\frac{1}{2}$-dominated long-distance scale of $\langle\pi|H_w|K\rangle$. Finally, in Sec. V, we summarize these results, stressing that *all* kaon weak decays (except K_{l2} and $K^+_{\pi\pi}$) stem from the same $\Delta I=\frac{1}{2}$ scale of $\langle\pi|H_w|K\rangle$.

II. $K_{2\pi,3\pi}$ PCAC SCALE

We remind the reader that PCAC applied naively to a soft π^0 in $K_{2\pi^0}$ decay or a π^+ or π^- in $K_{\pi^+\pi^-}$ decay leads to different (unphysical) $\Delta I=\frac{1}{2}$ amplitudes even though isospin symmetry dictates that

$$\langle\pi^+\pi^-|H_w|K^0\rangle=\langle\pi^0\pi^0|H_w|K^0\rangle$$

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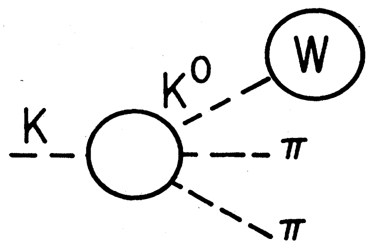

FIG. 1. Long-distance $K_{2\pi}$ kaon tadpole diagram.

for H_w transforming like $\Delta I=\frac{1}{2}$. The problem is in the strong $K\pi$ momentum variation of the model $K_{2\pi}$ "tadpole" amplitude M_P of Fig. 1 (see, e.g., Ref. 3). Accounting for the rapid momentum variation of the Weinberg[17] $K\pi$ amplitude while conserving momentum[3,4] in

$$M=M_{\rm cc}+M_P-M_P(p\to 0)\ ,$$

or alternatively reformulating the entire problem in terms of a nonlinear chiral U(3)×U(3) invariant Lagrangian,[18] one is always led to the *same* PCAC on-shell relation:

$$|\langle\pi\pi\,|\,H_w\,|\,K_S\rangle\,|=\frac{1}{f_\pi}\,|\,\langle\pi^0\,|\,H_w\,|\,K_L\rangle\,|\ , \tag{1}$$

where $f_\pi\simeq 93$ MeV. The right-hand side of (1) is *twice* the inconsistent naive PCAC result as found in Refs. 2. Given the validity of (1), the observed[19] $K^0_{2\pi}$ $\Delta I=\frac{1}{2}$ rate requires the scale

$$|\langle\pi^0\,|\,H_w\,|\,K_L\rangle\,|\simeq 3.9\times 10^{-8}\ {\rm GeV}^2\ . \tag{2}$$

For $K_{3\pi}$ decays, however, the PCAC rapidly varying $K\pi$ and $\pi\pi$ pole structure of the K plus π pole graphs of Fig. 2 directly gives the pure $K_{3\pi}$ pole model on-shell relation between $\Delta I=\frac{1}{2}$ amplitudes:

$$|\langle 3\pi^0\,|\,H_w\,|\,K_L\rangle\,|=\frac{1}{2f_\pi^{\,2}}\,|\,\langle\pi^0\,|\,H_w\,|\,K_L\rangle\,| \tag{3a}$$

for $f_\pi\simeq 93$ MeV, where Figs. 2(a) and 2(b) each contribute equally to (3a). Equation (3a) also follows in the nonlinear-chiral-Lagrangian scheme,[18] and again (3a) is *twice* what one obtains in the naive soft-pion approach of Ref. 2. Fitting (3a) to the observed[19] $K_{3\pi}$ rates with[3]

$$|\langle 3\pi^0\,|\,H_w\,|\,K_L\rangle\,|\simeq 2.5\times 10^{-6}\ ,$$

one finds

$$|\langle\pi^0\,|\,H_w\,|\,K_L\rangle\,|\simeq 4.3\times 10^{-8}\ {\rm GeV}^2\ . \tag{3b}$$

Since (2) and (3b) are practically identical and well within the expected PCAC errors and slight $\Delta I=\frac{3}{2}$ contamination of (3), we shall henceforth assume the scale (2) when considering other kaon decays in this study.

III. $K_L\to\gamma\gamma$ and $K_L\to\mu^+\mu^-$ DECAYS

We now investigate in detail both long- and short-distance contributions to the $K_{L\gamma\gamma}$ decay amplitude, which we write as

$$M_{K_L\gamma\gamma}=M^{\rm LD}_{K_L\gamma\gamma}+M^{\rm SD}_{K_L\gamma\gamma}\ , \tag{4}$$

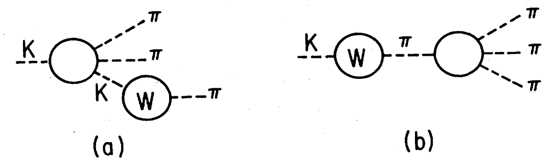

FIG. 2. Long-distance kaon (a) and pion (b) pole graphs for $K\to 3\pi$ decay.

where SD stands for short distance (or structure dependent in hadron language). The long-distance (LD) amplitude corresponds to the hadronic π^0, η, and η' poles of Fig. 3, which give

$$M^{\rm LD}_{K_L\gamma\gamma}=\frac{\langle\pi^0\,|\,H_w\,|\,K_L\rangle}{m_K^{\,2}-m_\pi^{\,2}}M_{\pi\gamma\gamma}+\frac{\langle\eta\,|\,H_w\,|\,K_L\rangle}{m_K^{\,2}-m_\eta^{\,2}}M_{\eta\gamma\gamma}+\frac{\langle\eta'\,|\,H_w\,|\,K_L\rangle}{m_K^{\,2}-m_{\eta'}^{\,2}}M_{\eta'\gamma\gamma}\ , \tag{5}$$

where the weak transitions are of the dominant $\Delta I=\frac{1}{2}$ tadpole form as given by (2). Note that the first π^0 pole term in (5), of magnitude

$$|\langle\pi^0\,|\,H_w\,|\,K_L\rangle\,|(m_K^{\,2}-m_\pi^{\,2})^{-1}\simeq 1.69\times 10^{-7}\ ,$$

is already close to the experimental value[19]

$$\left|\frac{M_{K_L\gamma\gamma}}{M_{\pi^0\gamma\gamma}}\right|_{\rm expt}=\left[\left(\frac{m_\pi}{m_K}\right)^3\frac{\Gamma_{K_L\gamma\gamma}}{\Gamma_{\pi\gamma\gamma}}\right]^{1/2}=(1.26\pm 0.10)\times 10^{-7}\ . \tag{6}$$

To account for the η and η' contributions in (5), it is most transparent to carry out the mixing relative to the $I=0$ nonstrange-strange quark basis,[7] where the effect of gluon exchanges generates the rediagonalization,

$$|\,\eta\rangle=\cos\phi\,|\,\eta_{\rm NS}\rangle-\sin\phi\,|\,\eta_{\rm S}\rangle\ , \tag{7a}$$

$$|\,\eta'\rangle=\sin\phi\,|\,\eta_{\rm NS}\rangle+\cos\phi\,|\,\eta_{\rm S}\rangle \tag{7b}$$

with the mixing angle found from[6]

$$\tan^2\phi=\frac{(m_{\eta'}^{\,2}-m_S^{\,2})(m_\eta^{\,2}-m_\pi^{\,2})}{(m_{\eta'}^{\,2}-m_\pi^{\,2})(m_S^{\,2}-m_\eta^{\,2})},\quad \phi\simeq 42.0^\circ \tag{8}$$

for $m_S^{\,2}=2m_K^{\,2}-m_\pi^{\,2}$. Then substituting (8) into (7) we obtain

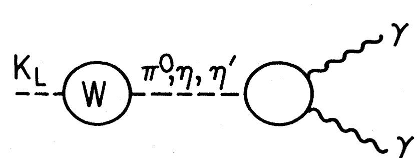

FIG. 3. Long-distance π^0, η, η' meson pole graphs for $K_L\to\gamma\gamma$ decay.

$$\frac{M_{\eta\gamma\gamma}}{M_{\pi\gamma\gamma}}=\tfrac{1}{3}(5\cos\phi-\sqrt{2}\sin\phi)\approx 0.923\ , \tag{9a}$$

$$\frac{M_{\eta'\gamma\gamma}}{M_{\pi\gamma\gamma}}=\tfrac{1}{3}(5\sin\phi+\sqrt{2}\cos\phi)\approx 1.466\ , \tag{9b}$$

and for $\langle P^i | H_w^6 | P^j \rangle$ transforming like d^{i6j} in the U(3) quark model,

$$\frac{\langle \eta | H_w | K_L \rangle}{\langle \pi^0 | H_w | K_L \rangle}=-(\cos\phi-\sqrt{2}\sin\phi)\approx 0.203\ , \tag{10a}$$

$$\frac{\langle \eta' | H_w | K_L \rangle}{\langle \pi^0 | H_w | K_L \rangle}=-(\sin\phi+\sqrt{2}\cos\phi)\approx -1.720\ . \tag{10b}$$

We spell out this mixing analysis in detail in order to emphasize the nearly complete cancellation[3] of the η and η' pole contributions in (5), leading to

$$\left|\frac{M_{K_L\gamma\gamma}}{M_{\pi\gamma\gamma}}\right|_{\mathrm{LD}} \simeq \frac{|\langle \pi^0 | H_w | K_L \rangle|}{m_K{}^2-m_\pi{}^2}[1-0.804+0.864]$$
$$\simeq 1.79\times 10^{-7}\ . \tag{11}$$

If we had ignored the η' pole term, then the magnitude of (11) would drop by a factor of 5. However, the above quark-based mixing picture is further supported by the QCD calculation of the U(1) anomaly[20] and $\phi\simeq 42°$ corresponds to $\theta=\phi-\tan^{-1}\sqrt{2}\simeq -12.7°$, relative to the singlet-octet basis which is close to the conventional quadratic-mass-formula value of $\theta\simeq -11°$. Thus we shall accept (11) as an accurate estimate of the long-distance hadronic component of the $K_{L\gamma\gamma}$ amplitude.

In order to include the short-distance "box" graph of Fig. 4, we follow Ma and Pramudita[13] who estimate that the u-quark graph in Fig. 4 dominates over the c- and t-quark contributions in the KM matrix,[8,21] giving

$$|M_{K_L\gamma\gamma}|_{\mathrm{SD}} \sim \frac{\sqrt{2}}{\pi}\alpha f_K G_F s_1 c_1 c_3\ . \tag{12}$$

Combining (12) with the $\pi^0\gamma\gamma$ anomaly[22] $\alpha/\pi f_\pi$ leads to only a 20% correction to (11). The relative sign between Figs. 3 and 4 is *negative* because, while the relative sign is positive between $M_{K_L\gamma\gamma}^{\mathrm{LD}}$ and $M_{\pi\gamma\gamma}$ in (5) for $\langle \pi^0 | H_w | K_L \rangle$ positive[23] as determined in Ref. 4, and likewise for $M_{K_L\gamma\gamma}^{\mathrm{SD}}$ as found in Ref. 13, the KM-matrix convention for s_1 in Ref. 13 (and in Ref. 21) is opposite to that of the GIM current[5] as used in Ref. 4. Thus we subtract (12) from (11) to obtain the total $K_{L\gamma\gamma}$ amplitude relative to $\pi^0_{\gamma\gamma}$,

$$|M_{K_L\gamma\gamma}/M_{\pi\gamma\gamma}| \simeq (1.79-0.39)\times 10^{-7}$$
$$=1.40\times 10^{-7}\ , \tag{13}$$

very close indeed to the observed ratio (6). In passing, we note that the quark-model choice for the η-η' mixing angle ($\phi=42.0°$ or $\theta=-12.7°$) uniquely leads to (13). If we instead had taken the quadratic mass formula version ($\theta=-10.7°$, $\phi=44.0°$), then the long-distance amplitude (11) decreases by $\sim 20\%$, thus spoiling the role of the short-distance amplitude in (13).

Next we turn to $K_{L\mu\bar{\mu}}$ decay, treating the long-distance hadronic π^0, η, and η', poles of Fig. 5 in a manner similar to $K_{L\gamma\gamma}$ decay. The analog of (5) is then (since the weak transitions can be taken as real),

$$\mathrm{Re}M_{K_L\mu\mu}^{\mathrm{LD}}=\frac{\langle \pi^0 | H_w | K_L \rangle}{m_K{}^2-m_\pi{}^2}\mathrm{Re}M_{\pi\mu\mu}$$
$$+\frac{\langle \eta | H_w | K_L \rangle}{m_K{}^2-m_\eta{}^2}\mathrm{Re}M_{\eta\mu\mu}$$
$$+\frac{\langle \eta' | H_w | K_L \rangle}{m_K{}^2-m_{\eta'}{}^2}\mathrm{Re}M_{\eta'\mu\mu}\ . \tag{14}$$

Just as in the $K_{L\gamma\gamma}$ case, where we set the off-shell scale to the measured $\pi^0_{\gamma\gamma}$ rate and fixed the $\eta_{\gamma\gamma}$, $\eta'_{\gamma\gamma}$ amplitude by SU(3) and mixing, so here we fix the measured $\eta_{\mu\bar{\mu}}$ rate and determine the off-shell $\pi^0_{\mu\bar{\mu}}$, $\eta_{\mu\bar{\mu}}$ amplitudes via the same mixing procedure. Once again the η and η' pole contributions almost cancel for $\phi\approx 42°$, leading to the amplitude ratio determined from (14),

$$\left|\frac{\mathrm{Re}M_{K_L\mu\mu}^{\mathrm{LD}}}{\mathrm{Re}M_{\eta\mu\mu}}\right| \approx (1.83-1.59+1.71)\times 10^{-7}$$
$$\approx 1.95\times 10^{-7}\ . \tag{15}$$

To compare (15) with experiment, we must fold in the observed $K_{L\mu\bar{\mu}}$ and $\eta_{\mu\bar{\mu}}$ decay rates with the absorptive parts of the 2γ intermediate states. In particular, one finds[24]

$$|M_{K_L\mu\mu}|=[4\pi\Gamma/p]^{1/2}=(2.5\pm 0.3)\times 10^{-12}\ , \tag{16a}$$

$$|\mathrm{Im}M_{K_L\mu\mu}|=\frac{\alpha m_\mu}{4}|M_{K_L\gamma\gamma}|\frac{1}{\beta_K}\ln\left[\frac{1+\beta_K}{1-\beta_K}\right]$$
$$\simeq 2.0\times 10^{-12}\ , \tag{16b}$$

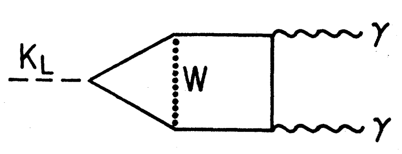

FIG. 4. Short-distance quark box diagram for $K_L\to\gamma\gamma$ decay.

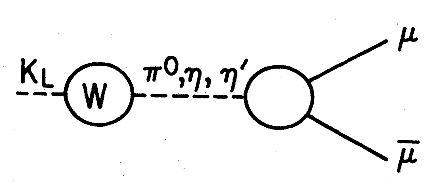

FIG. 5. Long-distance π^0, η, η' meson pole graphs for $K_L\to\mu\bar{\mu}$ decay.

for

$$\beta_K{}^2 = 1 - 4m_\mu{}^2/m_K{}^2$$

and then (16) implies

$$|\,\mathrm{Re}M_{K_L\mu\mu}\,| = (1.5\pm0.2)\times10^{-12}\ .$$

Similarly for $\eta_{\mu\bar{\mu}}$ decay, one obtains

$$|\,\mathrm{Re}M_{\eta\mu\mu}\,| = (1.0\pm0.2)\times10^{-5}\ .$$

Dividing one amplitude by the other, we deduce that

$$\left|\frac{\mathrm{Re}M_{K_L\mu\mu}}{\mathrm{Re}M_{\eta\mu\mu}}\right|_{\mathrm{expt}} = (1.6\pm0.4)\times10^{-7}\ . \tag{17}$$

It is also possible to employ a (quark) model to obtain estimates[25] of $\mathrm{Re}M_{\eta\mu\mu}$, but we shall refer only to experiment here.

Finally, one must add the short-distance second-order weak box graph[26] of Fig. 6. Following the work of Inami and Lim,[10] Ref. 27 expresses the box-graph contribution (here dominated by the t quark) as

$$\left|M^{\mathrm{SD}}_{K_L\mu\mu}\right| \simeq s_1c_1s_2{}^2\times10^{-9}G(x_t)\ , \tag{18a}$$

$$G(x_t) = \frac{3}{4}\left[\frac{x_t}{x_t-1}\right]^2 \ln x_t + \frac{x_t}{4} + \frac{3}{4}\frac{x_t}{1-x_t}\ , \tag{18b}$$

where $x_t = m_t{}^2/m_W{}^2$. For[9] $s_3\sim0$, $s_2\sim0.1$, and $m_t \lesssim 40$ GeV, (18) is of the order of the error on (17). Thus we conclude that the long-distance amplitude (14) and ratio (15) for second-order photon exchange dominates $K_{L\mu\bar{\mu}}$ decay and the ratio (17) over the short-distance second-order weak box graph. Alternatively, Ref. 27 examines the branching ratio $B(K_L\to\mu\bar{\mu}/\gamma\gamma)$ whose dominant long-distance part is essentially independent of the scale of $\langle\pi\,|\,H_w\,|\,K\rangle$. The dominance of the LD over the SD part again follows, including perhaps a small (LD) K^* pole component.

IV. THREE-BODY KAON DECAYS

A. $K_{3\pi}$

First we return to the long-distance pole-model graphs of Figs. 2 and Eqs. (3). As is well understood,[14] the direct soft-pion limit of the $K_{3\pi}/K_{2\pi}$ ratio is

$$|\,M_{K_L3\pi^0}/M_{K_S2\pi^0}\,| \approx (2f_\pi)^{-1}\ , \tag{19}$$

which also follows from (1) and (3). While this suggests that Fig. 1 is likewise a long-distance $K_{2\pi}$ pole graph, it is also possible to model $K_{2\pi}{}^0$ according to a short-distance "penguin" graph,[1] although the latter scale appears to be too small[28] to be the origin of the $\Delta I=\frac{1}{2}$ rule. Since we wish to present this analysis in as model-independent a fashion as possible, we shall avoid a long- or short-distance interpretation of $K_{2\pi}{}^0$ decays but continue to call $\langle\pi\,|\,H_w\,|\,K\rangle$ and the related meson-pole graphs like Figs. 2, 3, and 5 as "long-distance contributions."

B. $K_{\pi e\bar{e}}$

The general form of this matrix element is

$$M_{K\pi e\bar{e}} = A\,(p_K+p_\pi)^\mu \bar{u}_e\gamma_\mu v_{\bar{e}}\ , \tag{20}$$

where the observed $K^+\to\pi^+e\bar{e}$ rate[19] requires[15]

$$|\,A\,|_{\mathrm{expt}} = (1.8\pm0.2)\times10^{-9}m_K{}^{-2}\ . \tag{21}$$

The associated long-distance "inner-bremsstrahlung" graphs of Fig. 7 correspond to the amplitude

$$|\,A_{\mathrm{LD}}\,| = e^2\frac{|\,\langle\pi^+\,|\,H_w\,|\,K^+\rangle\,|}{m_K{}^2-m_\pi{}^2}\left|\frac{F_{\pi^+}(Q^2)-F_{K^+}(Q^2)}{Q^2}\right|\ , \tag{22}$$

where $Q^2=(p_K-p_\pi)^2$ is the momentum transfer invariant of the photon and the $\Delta I=\frac{1}{2}$ structure of H_w requires[29]

$$\langle\pi^+\,|\,H_w\,|\,K^+\rangle = \langle\pi^0\,|\,H_w\,|\,K_L\rangle\ .$$

The latter is the $\Delta I=\frac{1}{2}$ scale given by (2).

Unfortunately, the charge-radii difference in (22) is somewhat ambiguous at the present time,[30] with the vector-dominance-model (VDM) value[31] and the quark-model value[32] about one-third the observed difference of charge radii[30]

$$\begin{aligned} F'_{\pi^+}(0)-F'_{K^+}(0) &\simeq \tfrac{1}{6}(r_{\pi^+}{}^2 - r_{K^+}{}^2) \\ &\simeq \tfrac{1}{6}(0.48\ \mathrm{fm}^2 - 0.28\ \mathrm{fm}^2) \\ &= (0.21\pm0.08)m_K{}^{-2}\ . \end{aligned} \tag{23}$$

In (23) we have excluded the one observed low value for $r_{\pi^+}{}^2$, the remaining values averaging[30]

$$r_{\pi^+}{}^2 = (0.48\pm0.01)\ \mathrm{fm}^2\ .$$

Combining (23) with (22) and (2), we find

$$|\,A_{\mathrm{LD}}\,| = (3.2\pm1.3)\times10^{-9}m_K{}^{-2}\ . \tag{24}$$

Since (24) is twice the experimental value (21), whether we adopt (24) or the smaller VDM—quark-model version, the

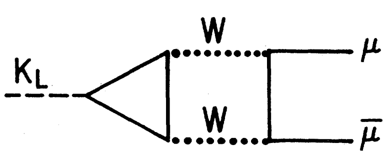

FIG. 6. Short-distance second-order W-exchange box diagram for $K_L\to\mu\bar{\mu}$ decay.

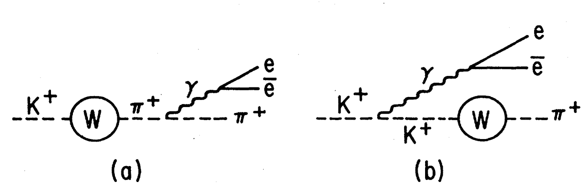

FIG. 7. Long-distance pion (a) and kaon (b) pole inner-bremsstrahlung graphs for $K^+\to\pi^+e\bar{e}$ decay.

long-distance contribution to $K^+_{\pi e\bar{e}}$ certainly cannot be neglected.

As for the short-distance contribution, we follow Ref. 15 and consider only the $\bar{s}\to\bar{d}\gamma$ quark transitions of Fig. 8 for $K^+\to\pi^+ e\bar{e}$ decay. In order to account for the heavy-quark loops in Figs. 8, we again apply the analysis of Refs. 10 and 11, finding from the latter reference the effective $\bar{s}\to\bar{d}\gamma$ quark current ($e>0$)

$$j^{\rm eff}_{\mu,\rm SD}=-\frac{e}{8\pi^2}\frac{G_F}{\sqrt{2}}\left[\sum_{i=u,c,t} V^*_{is}V_{id}F(x_i)\right]Q^2\bar{s}\gamma_\mu d,\quad x_i=m_i^{\,2}/m_W^{\,2}\ , \tag{25a}$$

$$F(x)=\frac{1}{1-x}\left[-\frac{23}{108}-\frac{1}{18}\frac{x}{1-x}-\frac{5}{54}x+\frac{2}{3}\frac{x^2}{(1-x)^2}+\frac{2}{9}\frac{x}{(1-x)^2}-\frac{1}{3}\frac{x}{(1-x)^2}\ln x+\frac{2}{9}\frac{x}{(1-x)^3}\ln x+\frac{2}{3}\frac{x^3}{(1-x)^3}\ln x\right.$$
$$\left.+\frac{1}{2}\frac{x^2}{(1-x)^2}\ln x-\frac{4}{9}\frac{1}{1-x}+\frac{4}{9}\frac{1}{(1-x)^2}+\frac{4}{9}\frac{1}{(1-x)^3}\ln x-\frac{2}{3}\frac{1}{(1-x)^2}\ln x\right]\ . \tag{25b}$$

In (25a) we have retained only the covariant which contributes to the K^+-π^+ transition, i.e.,

$$\langle\pi^+\,|\,\bar{s}\gamma_\mu d\,|\,K^+\rangle=(p_K+p_\pi)_\mu\ .$$

Combining Fig. 8 with the photon propagator and $e\bar{e}$ pair, we obtain the form (20) with

$$|\,A_{\rm SD}\,|\simeq\frac{\alpha G_F}{2\pi\sqrt{2}}(0.714 s_1 c_1 c_2^{\,2} c_3)\simeq 0.36\times10^{-9} m_K^{\,-2} \tag{26}$$

for $s_3 \ll s_2 \sim 0.1$, practically independent of the top-quark mass.

The relative sign between (22) and (26) is *negative* because the *VVF* sum in (25a) is negative, so the sign of the $(p_K+p_\pi)_\mu$ coefficient in (25a) is positive. On the other hand, the effective long-distance current can be expressed as

$$j^{\rm eff}_{\mu,\rm LD}\propto\langle\pi^0\,|\,H_w\,|\,K_L\rangle(p_K+p_\pi)_\mu$$

and $\langle\pi^0\,|\,H_w\,|\,K_L\rangle$ is negative (given the analysis of Ref. 4) for the same KM matrix convention as in Ref. 21. Thus the long- and short-distance contributions to $K_{\pi e\bar{e}}$ interface destructively to

$$|\,A\,|=(2.8\pm1.3)\times10^{-9} m_K^{\,-2}\ . \tag{27}$$

Given the possible reduction in the charge-radii difference in (23) by up to a factor of 2 and the larger error on (27), the agreement with observation (21) appears to be reasonable. We might even say that the short-distance scale (26) pins down the long-distance amplitude (24) and therefore (27) to near the lower error, $|\,A\,|\sim1.8\times10^{-9}m_K^{\,-2}$, which is then close to experiment.

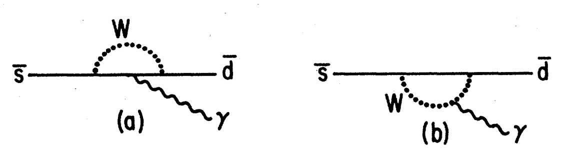

FIG. 8. Short-distance radiative s-d-quark diagram contributing to $K^+\to\pi^+ e\bar{e}$ decay.

Although we do not include QCD corrections here, we note that our short-distance estimate (26) is not incompatible with Ref. 15, whose C_7 coefficient $\ln m_c^{\,2}/\mu^2$ should be evaluated at $\mu\sim1$ GeV for $n_f=3$, where[33] $\Lambda_{\overline{\rm MS}}(3)\approx250$ MeV ($\overline{\rm MS}$ denotes modified minimal-subtraction scheme) for $\alpha_s(\mu^2)\approx0.5$, corresponding to $\Lambda_{\overline{\rm MS}}(5)\approx130$ MeV as is now found from QCD phenomenology. Then one has

$$C_7=(2/9\pi)\ln m_c^{\,2}/\mu^2\sim0.06\ ,$$

which yields 20% of the experimental scale, as does (26). Note, however, that QCD corrections could invalidate our arguments about the relative signs.

There are no long-distance contributions to either $K^0_S\to\pi^0 e^+e^-$ by virtue of $\langle\pi^0\,|\,H_w\,|\,K_S\rangle=0$ up to *CP* violation, or to $K^+\to\pi^+\nu\bar{\nu}$ since the Z^0 mass is much larger than the momentum transfer involved. These decays are then pure short-distance processes. Note also that $K^0_L\not\to\pi^0e^+e^-$ if *CP* is conserved as long as a single photon is exchanged.[15]

C. $K\to\pi\pi\gamma$

Since both $K_S\to\pi^+\pi^-\gamma$ and $K^+\to\pi^+\pi^0\gamma$ have now been measured,[19] we can form the decay rate ratio,

$$\left.\frac{\Gamma_{K_S\to\pi^+\pi^-\gamma}}{\Gamma_{K^+\to\pi^+\pi^0\gamma}}\right|_{\rm expt}\approx930\ , \tag{28a}$$

which displays the striking $\Delta I=\frac{1}{2}$ enhancement analogous to

$$\left.\frac{\Gamma_{K_S\to\pi^+\pi^-}}{\Gamma_{K^+\to\pi^+\pi^0}}\right|_{\rm expt}\approx450\ . \tag{28b}$$

Alternatively we may examine the branching ratios

$$\begin{aligned}&B(K_S\to\pi^+\pi^-\gamma/\pi^+\pi^-)_{\rm expt}\approx0.0027\ ,\\ &B(K^+\to\pi^+\pi^0\gamma/\pi^+\pi^0)_{\rm expt}\approx0.0013\ ,\end{aligned} \tag{28c}$$

both of which are suppressed by the typical bremsstrahlung scale factor of $\alpha/\pi\sim0.0023$. These results argue convincingly for the dominance of the long-distance hadronic-bremsstrahlung graphs and for the continuation of the $\Delta I=\frac{1}{2}$ rule via the amplitudes

$$M^{\rm LD}_{K_S\to\pi^+\pi^-\gamma}=eM_{K_S\pi^+\pi^-}\left[\frac{p_{\pi^+}}{p_{\pi^+}\cdot k}+\frac{p_{\pi^-}}{p_{\pi^-}\cdot k}\right]\cdot\epsilon^*(k)\ , \tag{29a}$$

$$M^{\rm LD}_{K^+\to\pi^+\pi^0\gamma}=eM_{K^+\pi^+\pi^0}\left[\frac{p_{\pi^+}}{p_{\pi^+}\cdot k}-\frac{p_{K^+}}{p_{K^+}\cdot k}\right]\cdot\epsilon^*(k)\ . \tag{29b}$$

The factor-of-2 difference between (28a) and (28b) or in (28c) could then be due to the kinematical variation in (29a) and (29b), or to smaller $s\to d\gamma$ contributions ($\sim 20\%$ or less) as in $K\to\pi e\bar{e}$ or due to both effects.

We follow Ref. 16, which argues that the $s\to d\gamma$ short-distance contribution is small and integrates over (29a) for $\omega_\gamma^*>50$ MeV to find [calling (29a) the IB contribution]

$$\Gamma^{\rm IB}_{K_S\to\pi^+\pi^-\gamma}=0.002\,55\Gamma_{K_S\to\pi^+\pi^-}\ . \tag{30}$$

Since (30) is so close to (28c), we may presume that (29a) is essentially the entire $K_S\to\pi^+\pi^-\gamma$ amplitude. We only differ from Ref. 16 in interpreting (29a) as a long-distance contribution driven directly by $\langle\pi\,|\,H_w\,|\,K\rangle$ in (1)–(3).

V. SUMMARY

We have first shown that $K_{2\pi}$ and $K_{3\pi}$ decays self-consistently require the same $\Delta I=\frac{1}{2}$ scale of

$$|\langle\pi^0\,|\,H_w\,|\,K_L\rangle|\simeq 3.9\times10^{-8}\ {\rm GeV}^2\ ,$$

which is one-half the value found in the original references on this work based on the naive "chain rule"[2,26]

$$|\langle n\pi^0\,|\,H_w\,|\,K^0\rangle|\simeq(2f_\pi)^{-1}|\langle(n-1)\pi^0\,|\,H_w\,|\,K^0\rangle|\ .$$

This $\Delta I=\frac{1}{2}$ scale then uniformly predicts dominant long-distance contributions to the other observed kaon decays: $K_{L\gamma\gamma}$, $K_{L\mu\bar{\mu}}$, $K^+_{\pi^+e\bar{e}}$, $K^0_{2\pi\gamma}$. We have also estimated the short-distance quark contributions to the latter four decays and find them typically $\sim 20\%$ of the associated long-distance contributions and of the opposite sign.

Note added in proof. While the theoretical estimate in Ref. 4 of $\langle 0\,|\,H_w\,|\,K^0\rangle$ is somewhat unclear, a more recent treatment of $\langle\pi^0\,|\,H_w\,|\,K_L\rangle$ for the *s*-*d* quark self-energy graph with W exchange based on light-plane wave functions obtains a $\Delta I=\frac{1}{2}$ scale of -3.4×10^{-8} GeV^2, consistent with the phenomenological findings of this study [N. Fuchs and M. Scadron, University of Arizona report, 1985 (unpublished)].

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*Permanent address: Physics Department, Technion, Haifa, Israel.

[1]M. A. Shifman, A. I. Vainshtein, and V. I. Zakharov, Zh. Eksp. Teor. Fiz. **22**, 123 (1975) [JETP Lett. **22**, 55 (1975)]; Nucl. Phys. **B120**, 316 (1977).

[2]For a review of the original literature, see R. E. Marshak, Riazuddin, and C. P. Ryan, *Theory of Weak Interactions in Particle Physics* (Academic, New York, 1969); D. Bailin, *Weak Interactions* (Sussex University Press, U.K., 1977).

[3]For a recent review, see M. D. Scadron, Rep. Prog. Phys. **44**, 213 (1981).

[4]B. H. J. McKeller and M. D. Scadron, Phys. Rev. D **27**, 157 (1983). See note added in proof.

[5]N. Cabibbo, Phys. Rev. Lett. **10**, 531 (1963); S. Glashow, J. Iliopoulos, and L. Maiani, Phys. Rev. D **2**, 1285 (1970).

[6]H. F. Jones and M. D. Scadron, Nucl. Phys. **B155**, 409 (1979); H. Genz, Acta Phys. Austriaca Suppl. **21**, 559 (1979); M. D. Scadron, Phys. Rev. D **29**, 2076 (1984).

[7]A. DeRújula, H. Georgi, and S. Glashow, Phys. Rev. D **12**, 147 (1975); N. Isgur, *ibid.* **12**, 3770 (1975).

[8]M. Kobayashi and T. Maskawa, Prog. Theor. Phys. **49**, 652 (1973).

[9]See, e.g., M. W. Reay, invited talk at *International Symposium on Lepton and Photon Interactions at High Energies, Ithaca, New York,* edited by D. G. Cassel and D. L. Kreinick (Newman Laboratory of Nuclear Studies, Cornell University, Ithaca, 1983), and references therein.

[10]F. Inami and C. S. Lim, Prog. Theor. Phys. **65**, 297 (1981).

[11]N. G. Deshpande and G. Eilam, Phys. Rev. D **26**, 2463 (1982).

[12]G. Eilam and M. D. Scadron, University of Arizona report (in preparation).

[13]E. Ma and A. Pramudita, Phys. Rev. D **24**, 2476 (1981).

[14]C. G. Callan and S. B. Treiman, Phys. Rev. Lett. **16**, 197 (1966); Y. Hara and Y. Nambu, *ibid.* **16**, 875 (1966).

[15]F. J. Gilman and M. B. Wise, Phys. Rev. D **21**, 3150 (1980).

[16]J. L. Lucio, Phys. Rev. D **24**, 2457 (1981).

[17]S. Weinberg, Phys. Rev. Lett. **17**, 616 (1966); H. Osborn, Nucl. Phys. **B15**, 501 (1970).

[18]J. Cronin, Phys. Rev. **161**, 1483 (1967).

[19]Particle Data Group, M. Roos *et al.*, Phys. Lett. **111B**, 1 (1982).

[20]R. Delbourgo and M. D. Scadron, Phys. Rev. D **28**, 2345 (1983); M. D. Scadron, Z. Phys. C **23**, 237 (1984).

[21]We follow the KM phase conventions as in F. J. Gilman and M. B. Wise, Phys. Lett. **83B**, 83 (1979).

[22]S. L. Adler, Phys. Rev. **177**, 2426 (1979); J. S. Bell and R. Jackiw, Nuovo Cimento **60**, 47 (1969).

[23]This relative sign follows independently by directly computing the quark tadpole graph or by comparing the $\Delta I=\frac{1}{2}$ $\langle\pi\,|\,H_w\,|\,K\rangle$ in $K^0_{2\pi}$ decays with the $\Delta I=\frac{3}{2}$ vacuum-saturated transition which occurs at the 5% level in $K^0_{2\pi}$ decays, but dominates $K^+_{2\pi}$ decay.

[24]L. M. Sehgal, Nuovo Cimento **45**, 785 (1966).

[25]Ll. Ametller, L. Bergström, A. Bramon, and E. Massó, Nucl. Phys. **B228**, 301 (1983); M. D. Scadron and M. Visinescu, Phys. Rev. D **29**, 911 (1984).

[26]M. K. Gaillard and B. W. Lee, Phys. Rev. D **10**, 897 (1974).

[27]L. Bergström, E. Massó, P. Singer, and D. Wyler, CERN report, 1983 (unpublished).

[28]C. Hill and G. C. Ross, Phys. Lett. **94B**, 234 (1980); M. B.

Gavela, A. Le Yaouanc, L. Oliver, O. Pène, and J. C. Raynal, Phys. Lett. **148B**, 248 (1984); J. F. Donoghue, Phys. Rev. D **30**, 1499 (1984); T. N. Pham, Ecole Polytechnique Report No. A606.0584, 1984 (unpublished).

[29]We choose Condon and Shortley phase conventions here. Such sign phases have been a problem in $D_{K\pi}$ decays; see G. Eilam, M. D. Scadron, and B. H. J. McKellar, University of Arizona report, 1983 (unpublished); M. D. Scadron, Phys. Rev. D **29**, 1375 (1984).

[30]Ll. Ametller, C. Ayala, and A. Bramon, Phys. Rev. D **24**, 233 (1981).

[31]A. I. Vainshtein, V. I. Zakharov, L. B. Okun, and M. A. Shifman, Yad. Fiz. **24**, 820 (1976) [Sov. J. Nucl. Phys. **24**, 427 (1976)]. We disagree with their (non-$\Delta I=\frac{1}{2}$) vacuum-saturated estimate of $\langle \pi^+ | H_w | K^+ \rangle$ in Eq. (21).

[32]R. Tarrach, Z. Phys. C **2**, 221 (1979); S. B. Gerasimov, Yad. Fiz. **29**, 513 (1979) [Sov. J. Nucl. Phys. **29**, 259 (1979)].

[33]R. Miller and B. McKellar, J. Phys. G **7**, L247 (1981); Phys. Rep. **106**, 170 (1984).