# $K_{l3}^{+}$ decays involving finite neutrino mass and mass mixing

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$K_{l3}^{+}$ decays with the inclusion of effects of finite neutrino mass and mass mixing have been investigated. Predictions have been obtained for (i) pion energy spectrum, (ii) decay probability, (iii) $\pi$-$l$ $(=e,\mu)$ angular correlations, (iv) $\pi$-$l$ energy correlations, (v) lepton energy spectrum, (vi) $\pi$-$\nu_i$ $(i=e,\mu,\tau)$ angular correlations, and (vii) $\pi$-$\nu_i$ energy correlations. Finite-mass effects are substantial in all the parameters discussed, whereas the mixing effects are discernible only in $\pi$-$\nu_i$ angular and energy correlations. The mass limits for the $\nu_3$ mass for hierarchical mixing are found to be 174.0±0.9 MeV and 120±2 MeV for electronic and muonic modes, respectively, with nearly identical values for Kobayashi-Maskawa mixing.

## I. INTRODUCTION

At present, major efforts are being made to look for nonvanishing neutrino masses[1] and their mixings.[2–5] A recent measurement by Boris *et al.*[6] (with much improved energy resolution) of the near-end-point shape of the $\beta$ spectrum in tritium decay puts a lower limit $m(\nu_e)>20$ eV, whose upper limit is around[7] 46 eV. The upper mass limits for $\nu_\mu$ and $\nu_\tau$ are substantially high.[8,9] Further, the mass matrix pertaining to finite neutrino mass is not diagonal with respect to its various flavor species. As such, the mass eigenstates are not necessarily the flavor eigenstates, and may show mixing among its various flavor species.[10] Mixings of Kobayashi-Maskawa[11] (KM) and hierarchical[12] type have been suggested and are being made use of in various investigations. Recent experiments have put some limits on some mixing matrices,[2,13] yet their precise estimation shall require many more efforts in this direction.

A finite mass does not appear in $SU_L(2)\times U(1)$ electroweak theory in a natural way. They are introduced[14] by adding (i) an additional Higgs scalar or (ii) an additional lepton or (iii) both. But in all these formulations a massive neutrino turns out to be of Majorana type. A massive neutrino occurs in a number of grand unified theories (GUT's), viz., in SO(10) and $E_6$, etc. In fact, in a recent paper Roy and Shanker[15] have outlined a supersymmetric GUT SO(10) scheme wherein Dirac neutrinos acquire masses. As such, ascertaining finite neutrino masses, mixing matrices, and oscillation phenomena is a necessity and may provide testing ground for GUT's with significant cosmological implications[16] and hint towards the physics that exist beyond $SU(2)_L\times U(1)$ electroweak theory.

In order to detect effects pertaining to finite neutrino mass and mixings a number of processes such as $\beta$ decay,[6,7] muon decay,[12] $\tau^-$ decays,[17,18] two-particle leptonic decays of mesons,[19,20] and inner bremsstrahlung[21] have been investigated with predictions that could possibly detect such contributions. Some phenomena such as the solar neutrino flux[22] and beam-dump experiments[23] do not rule out the possibility of having such effects. At least two experiments, namely, the one involving tritium decay[6] and the Bugey-reactor oscillation experiment,[24] seem to give definite indications about the existence of finite neutrino mass and oscillations phenomena respectively.

In this work, we report our investigations on the three-particle semileptonic decays of $K^+$. At present $K^+$ beam is copiusly available at CERN in the proton-antiproton collider[25] (LEAR), and as such this facility may possibly be made use of for this purpose. The discussion makes use of the phenomenological $V-A$ theory[12,16–19] with the inclusion of finite neutrino mass, treating neutrinos as Dirac particles. The use of the established $SU(2)_L\times U(1)$ theory[14] and GUT's has purposely been avoided in this discussion of preliminary nature, as the latter are primarily expected to lead to refinements towards the predictions of this phenomenological discussion. In addition, Yukawa couplings involved in these models are not yet known.

The investigation focuses primarily on the following aspects of $K_{l3}^{+}$ decays: (i) Pion energy spectrum, (ii) decay probability and limits of $\nu_3$ mass, (iii) $\pi$-$l$ $(=e,\mu)$ angular correlations as a function of pion energy, (iv) $\pi$-$l$ energy correlations as a function of pion energy, (v) lepton energy spectrum, (vi) $\pi$-$\nu_i$ $(i=e,\mu,\tau)$ angular correlations, and (vii) $\pi$-$\nu_i$ energy correlations as a function of pion energy.

The limits obtained on the $\nu_3$ mass are consistent with presently known experimental upper bound.[9] Finite-$\nu_3$-mass effects have a reasonable influence on all the parameters (i)—(vii). However, mixing contributions could be discernible only in pion-neutrino angular correlations and pion-neutrino energy correlations.

In the next section we give details of our discussion pertaining to all the aspects listed in (i)—(vii) and a summary of conclusions in Sec. III.

## II. $K_{l3}^{+}$ DECAYS

### A. Pion energy spectrum in $K^+\to l^+\nu_i\pi^0$

The matrix element[26] for the processes $K^+\to l^+\nu_i\pi^0$, with the inclusion of the mixing matrix for neutrinos and radiative correction[27] is given by

 

$$M=\frac{G'}{\sqrt{2}}V_{12}\sum_i |U_{li}|\,\bar{u}_{\nu_i}[(p_K+p_\pi)f_+ +(p_K-p_\pi)f_-]\gamma^\lambda(1-\gamma^5)V_l\ , \tag{1}$$

where $G'$ is the coupling constant inclusive of radiative correction, $U_{li}$ are the elements of neutrino-mass-mixing matrix with $l=e,\mu$ and $i=1,2,3$, and $V_{12}$ is the matrix element of the KM mixing matrix in quark sector.[11,28]

The expression for the pion energy spectrum is obtained as

$$\frac{dW}{dx}=\frac{G'^2V_{12}{}^2\sum_i |U_{li}|^2 m_K{}^5(x^2-4\delta_\pi{}^2)^{1/2}\lambda^{1/2}((k-x),\delta_l{}^2,\delta_i{}^2)f_+{}^2}{384\pi^3(k-x)^3}$$

$$\times((x^2-4\delta_\pi{}^2)[2(k-x)^2-(k-x)(\delta_l{}^2+\delta_i{}^2)-(\delta_l{}^2-\delta_i{}^2)^2]$$

$$+3[(k-x)(\delta_l{}^2+\delta_i{}^2)-(\delta_l{}^2-\delta_i{}^2)^2]\{(1-\delta_\pi{}^2)^2+(k-x)[(1-\delta_\pi{}^2)+2\,\mathrm{Re}\xi+|\xi|^2(k-x)]\})\ , \tag{2}$$

where

$$G'^2=1.021G_\mu{}^2\ \text{(Ref. 27)},\quad x=\frac{2E_\pi}{m_K},\quad E_\pi=\text{energy of }\pi^0,\quad \delta_\pi{}^2=\frac{m_{\pi^0}{}^2}{m_K{}^2},\quad \delta_l{}^2=\frac{m_l{}^2}{m_K{}^2},\quad \delta_i{}^2=\frac{m^2(\nu_i)}{m_K{}^2}\ ,$$

$$k=1+\delta_\pi{}^2,\quad \lambda(x,y,z)=x^2+y^2+z^2-2xy-2yz-2zx,\quad \xi=\frac{f_-}{f_+},\quad \xi(0)=\xi\left[1+\frac{\lambda_+q^2}{m_\pi{}^2}\right],$$

$$f_+=f_+(0)\left[1+\frac{\lambda_+q^2}{m_\pi{}^2}\right],\quad q^2=(p_K-p_\pi)^2\ \text{(Refs. 29 and 30)}\ .$$

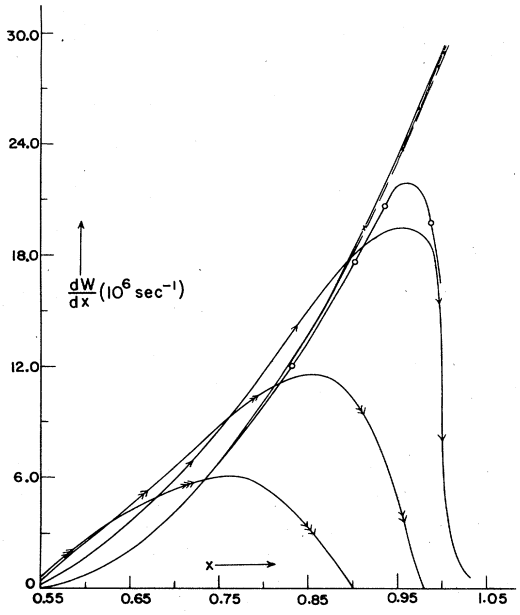


FIG. 1. Pion energy spectrum in the decay $K^+\to e^+\nu_i\pi^0$. Solid curve, single-arrow curve, double-arrow curve, and triple-arrow curve are for $m(\nu_3)=0$, 100, 150, and 200 MeV, respectively, without mixing. Dashed curve and single-circle curve are with KM mixing for $m(\nu_3)=100$ and 200 MeV, respectively. Cross curve is for $m(\nu_3)=100$ MeV with hierarchical mixing. Curves with hierarchical mixing (not shown) for $m(\nu_3)=150$ and 200 MeV almost coincide with the curve of $m(\nu_3)=0$.

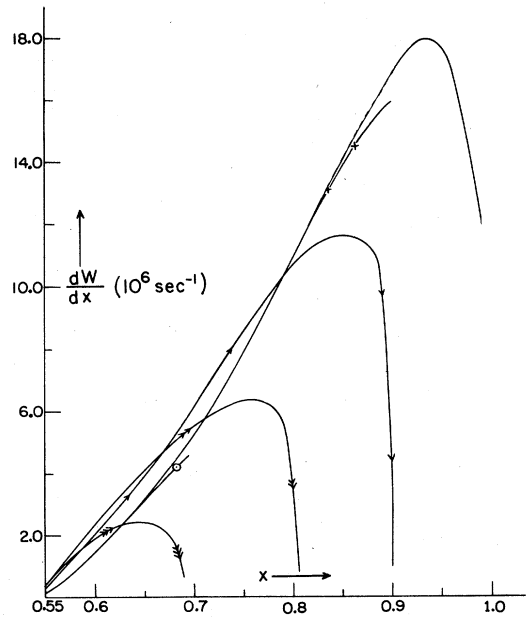


FIG. 2. Pion energy spectrum in the decay $K^+\to\mu^+\nu_i\pi^0$. Description of the curves is identical to that given in Fig. 1.

We take

$$f_+(0)V_{12}=0.2161 \text{ (Refs.27 and 28)},$$

$$\xi(0)=0,\quad \lambda_+=0.0285\pm0.0043$$

for $K_{e3}^{+}$ decays[29] and

$$\lambda_+=0.032\pm0.008,$$

$$\xi(0)=-0.35\pm0.15$$

for $K_{\mu3}^{+}$ decays[29] for the purpose of numerical calculations.

Expression (2) shows that in general the pion energy spectrum for $l=e$ or $\mu$, will be the sum of three different spectra for the cases $i=1,2,3$. Confining to the three-neutrino world, and, in order to have an order-of-magnitude estimate, we adopt the convention that the neutrino masses $m(\nu_i)$ are in ascending order of values, i.e., $m_1<m_2<m_3$. Further, for the case of nondegenerate neutrinos, we take $\nu_1$, $\nu_2$, and $\nu_3$ to be, respectively, $\nu_e$, $\nu_\mu$, and $\nu_\tau$. Using the present experimental bounds on the masses of various neutrino species,[6-9] 20 eV $<m(\nu_e)<$ 46 eV, $m(\nu_\mu)<0.50$ MeV, and $m(\nu_\tau)<164$ MeV, we obtain $\delta_1<9.32\times10^{-8}$, $\delta_2<1.01\times10^{-3}$, and $\delta_3<0.33$, where $\delta_i=m(\nu_i)/m_K$. Thus the dominant contribution comes from $\delta_3$ only. Retaining $\delta_3$, we obtain for the pion energy spectrum, from Eq. (2), the expression

$$\begin{aligned}\frac{dW}{dx}=&\frac{G'^2V_{12}^2m_K^5(x^2-4\delta_\pi^2)^{1/2}f_+^2}{384\pi^3(k-x)^3}\\
&\times[(1-U_{l3}^2)(k-x-\delta_l^2)((x^2-4\delta_\pi^2)[2(k-x)^2-\delta_l^2(k-x)-\delta_l^4]\\
&\quad+3[(k-x)\delta_l^2-\delta_l^4]\{(1-\delta_\pi^2)^2+(k-x)[(1-\delta_\pi^2)2\,\mathrm{Re}\xi+|\xi|^2(k-x)]\})\\
&+U_{l3}^2\lambda^{1/2}((k-x),\delta_l^2,\delta_3^2)((x^2-4\delta_\pi^2)[2(k-x)^2-(k-x)(\delta_l^2+\delta_3^2)-(\delta_l^2-\delta_3^2)^2]\\
&\quad+3[(k-x)(\delta_l^2+\delta_3^2)-(\delta_l^2-\delta_3^2)^2]\\
&\quad\times\{(1-\delta_\pi^2)^2+(k-x)[(1-\delta_\pi^2)2\,\mathrm{Re}\xi+|\xi|^2(k-x)]\})].\end{aligned}\tag{3}$$

The pion energy spectrum in the decay $K^+\to e^+\pi^0\nu_i$ is shown in Fig. 1 for $m(\nu_3)=0$, 100, 150, and 200 MeV. Matrices used for hierarchical[12] and KM (Ref. 5) mixings, respectively, are

$$|U_{li}|^2=\begin{bmatrix}0.9947 & 0.005 & 0.0003\\ 0.0043 & 0.9367 & 0.059\\ 0.001 & 0.01 & 0.94\end{bmatrix},\tag{4}$$

$$|U_{li}|=\begin{bmatrix}0.95 & 0.22 & 0.21\\ -0.23 & 0.97 & 0.04\\ -0.21 & -0.08 & 0.98\end{bmatrix}.\tag{5}$$

The effect of finite $m(\nu_3)$ on the pion energy spectrum throughout the pion energy range is substantial and becomes pronounced towards the high-energy end, i.e., for $x>0.85$. Mixing effects are, however, negligible, the spectra involving hierarchical mixing are almost coincident with those for $m(\nu_3)=0$. KM mixing effects for $m(\nu_3)=200$ MeV, for $x>0.9$ may, however, be observable. The pion energy spectrum for the decay $K^+\to\pi^0\mu^+\nu_i$, shown in Fig. 2, has qualitatively the same features as that for $K^+\to e^+\nu_i\pi^0$ except for numerical differences.

### B. Decay probability and limits on $\nu_3$ mass

The expression for the decay probability, obtained by integrating Eq. (3), is given by

$$\begin{aligned}W=&\frac{G'^2V_{12}^2m_K^5}{384\pi^3}\int_{x_{\min}}^{x_{\max}}\frac{f_+^2}{(k-x)^3}[(1-U_{l3}^2)(k-x-\delta_l^2)((x^2-4\delta_\pi^2)[2(K-x)^2-\delta_l^2(k-x)-\delta_l^4]+3[(k-x)\delta_l^2-\delta_l^4]\\
&\quad\times\{(1-\delta_\pi^2)^2+(k-x)[(1-\delta_\pi^2)2\,\mathrm{Re}\xi+|\xi|^2(k-x)]\})\\
&+U_{l3}^2\lambda^{1/2}((k-x),\delta_l^2,\delta_3^2)\end{aligned}$$

$$\times((x^2-4\delta_\pi{}^2)[2(k-x)^2-(k-x)(\delta_l{}^2+\delta_3{}^2)-(\delta_l{}^2-\delta_3{}^2)^2]$$

$$+3[(k-x)(\delta_l{}^2+\delta_3{}^2)-(\delta_l{}^2-\delta_3{}^2)^2]$$

$$\times\{(1-\delta_\pi{}^2)^2+(k-x)[(1-\delta_\pi{}^2)2\,\mathrm{Re}\xi+|\xi|^2(k-x)]\})]dx\ , \tag{6}$$

where

$$x_{\max}=\frac{m_K{}^2+m_{\pi^0}{}^2-[m_l+m(\nu_i)]^2}{m_K{}^2}\ ,$$

and

$$x_{\min}=\frac{2m_{\pi^0}}{m_K}\ . \tag{7}$$

The variation of the decay probability with finite $m(\nu_3)$ for $l=e,\mu$ are shown in Figs. 3 and 4. Mixing effects are observable in the case of $K^+\to e^+\nu_i\pi^0$, but cannot be deciphered for $K^+\to\mu^+\nu_i\pi^0$ from those of finite $m(\nu_3)$.

Using experimental data[29] for $K^+$ decay time $=(1.2371\pm0.0026)\times10^{-8}$ sec, and the branching fraction $=(4.82\pm0.05)$ percent, the experimental value of decay probability is found to be

$$W_{\text{expt}}(K^+\to e^+\nu_i\pi^0)=(3.90\pm0.041)\times10^6\ \text{sec}^{-1}\ . \tag{8}$$

Giving different values to $m(\nu_3)$ in its theoretical expression (6) and comparing them with the experimental value [Eq. (8)], we calculate the value of $m(\nu_3)$ for which the two decay probabilities are equal. The values so obtained with the inclusion of hierarchical and KM mixing are given in Table I.

The corresponding value for the decay $K^+\to\mu^+\nu_i\pi^0$, with the use[29] of $K^+$ decay time $=(1.2371\pm0.0026)\times10^{-8}$ sec and the branching fraction $=(3.20\pm0.09)$ percent, is found to be

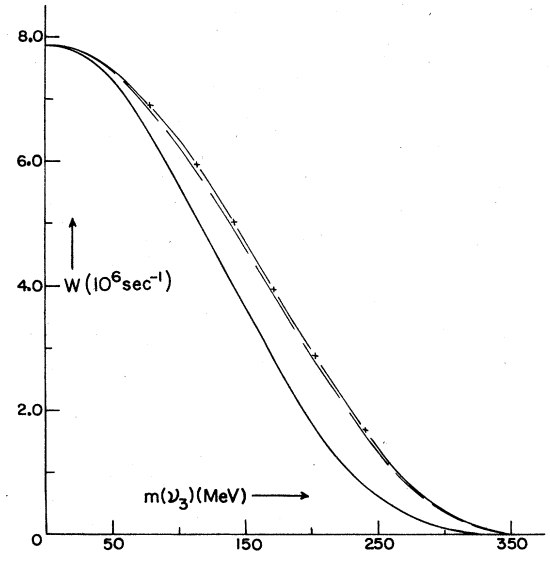


FIG. 3. Variation of decay probability with $m(\nu_3)$ in the decay $K^+\to e^+\nu_i\pi^0$. Solid curve, dashed curve, and cross curve are for without mixing, KM mixing, and hierarchical mixing, respectively.

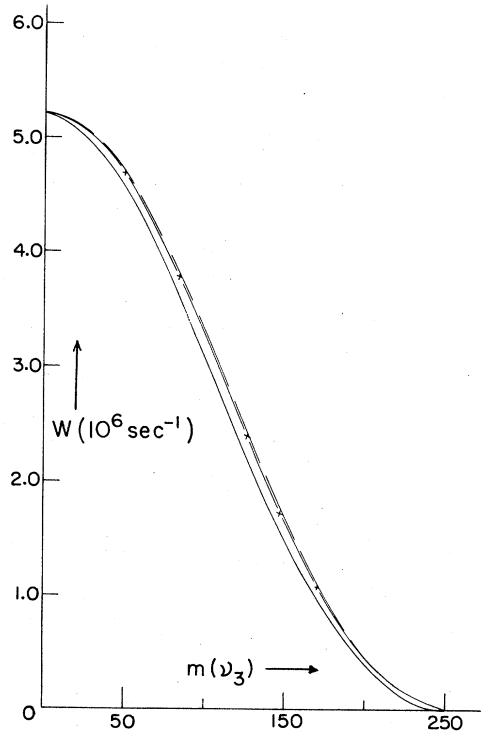


FIG. 4. Variation of decay probability with $m(\nu_3)$ in the decay $K^+\to\mu^+\nu_i\pi^0$. Description of the curves is identical to that given in Fig. 3.

$$W_{\text{expt}}(k^+\to\mu^+\nu_i\pi^0)=(2.59\pm0.07)\times10^6\ \text{sec}^{-1}\ , \quad (9)$$

and the values of $m(\nu_3)$ calculated are given in Table I.

TABLE I. Values of $m(\nu_3)$ in the decays $K^+\to l^+\nu_i\pi^0$.

| Decay mode | $m(\nu_3)$ (MeV) Hierarchical mixing | KM mixing | Without mixing |
|---|---|---|---|
| 1. $K^+\to e^+\nu_i\pi^0$ | 174±1 | 172±1 | 144±1 |
| 2. $K^+\to\nu^+\nu_i\pi^0$ | 120±2 | 120±2 | 115±2 |

### C. $\pi$-$l$ angular correlations

The expression for the $\pi$-$l$ angular correlation is obtained as

$$\frac{dW}{dx\,dz}=\frac{G'^2V_{12}^2m_K^5(x^2-4\delta_\pi^2)^{1/2}}{512\pi^3}$$
$$\times\sum_{i=1}^{3}\frac{|U_{li}|^2(y^2-4\delta_l^2)f_+^2}{[(2-x)(y^2-4\delta_l^2)^{1/2}+(x^2-4\delta_\pi^2)^{1/2}yz]}$$
$$\times([(2+x)(1-\delta_\pi^2)+(x^2-4\delta_\pi^2)]2y-y^2(2+x)^2$$
$$+(x^2-4\delta_\pi^2)^{1/2}(y^2-4\delta_l^2)^{1/2}2z[y(2+x)-(3-\delta_\pi^2+x)]-(x^2-4\delta_\pi^2)(y^2-4\delta_l^2)z^2$$
$$+8\delta_l^2(k+x)+2\,\text{Re}\xi[(2+x)(k-x)2y-y^2(4-x^2)-(x^2-4\delta_\pi^2)^{1/2}(y^2-4\delta_l^2)^{1/2}2z(xy+k-x)$$
$$+(y^2-4\delta_l^2)(x^2-4\delta_\pi^2)z^2+8\delta_l^2(1-\delta_\pi^2)]$$
$$+|\xi|^2\{(2-x)(k-x)2y-y^2(2-x)^2-(x^2-4\delta_\pi^2)^{1/2}(y^2-4\delta_l^2)^{1/2}2z[(2-x)y-k+x]$$
$$-(x^2-4\delta_\pi^2)(y^2-4\delta_l^2)z^2+8\delta_l^2(k-x)\})\ , \quad (10)$$

where $z=\cos\theta_{\pi l}$,

$$y=\frac{2E_l}{m_K}$$
$$=\frac{2((k+\delta_l^2-\delta_i^2-x)(2-x)-(x^2-4\delta_\pi^2)^{1/2}z\{(k+\delta_l^2-\delta_i^2-x)^2-\delta_l^2[(2-x)^2-(x^2-4\delta_\pi^2)z^2]\}^{1/2})}{(2-x)^2-(x^2-4\delta_\pi^2)z^2}\ , \quad (11)$$

$\theta_{\pi l}$ will be in the range 0 to $\pi/2$. With the retention of dominant $\delta_3$ contributions, the expression (10) reduces to the following:

$$\frac{dW}{dx\,dz}=\frac{(1-U_{l3}^2)G'^2V_{12}^2m_K^5(x^2-4\delta_\pi^2)^{1/2}(y_1^2-4\delta_l^2)f_+^2(0)[1+\lambda_+(k-x)/\delta_\pi^2]^2}{512\pi^3[(2-x)(y_1^2-4\delta_l^2)^{1/2}+(x^2-4\delta_\pi^2)^{1/2}y_1z]}$$
$$\times\Bigg[[(2+x)(1-\delta_\pi^2)+(x^2-4\delta_\pi^2)]2y_1-y_1^2(2+x)^2+(x^2-4\delta_\pi^2)^{1/2}(y_1^2-4\delta_l^2)^{1/2}2z[y_1(2+x)-(3-\delta_\pi^2+x)]$$
$$-(x^2-4\delta_\pi^2)(y_1^2-4\delta_l^2)z^2+8\delta_l^2(k+x)$$
$$+2\xi(0)\left[1-\lambda_+\frac{k-x}{\delta_\pi^2}\right][(2+x)(k-x)2y_1-y_1^2(4-x)^2-(x^2-4\delta_\pi^2)^{1/2}(y_1^2-4\delta_l^2)^{1/2}2z(xy_1+k-x)$$
$$+(y_1^2-4\delta_l^2)(x^2-4\delta_\pi^2)z^2+8\delta_l^2(1-\delta_\pi^2)]$$

$$+\xi^2(0)\left[1-\lambda_+\frac{k-x}{\delta_\pi{}^2}\right]^2\{(2-x)(k-x)2y_1-y_1{}^2(2-x)^2-(x^2-4\delta_\pi{}^2)^{1/2}(y_1{}^2-4\delta_l{}^2)^{1/2}2z[(2-x)y_1-k+x]$$

$$-(x^2-4\delta_\pi{}^2)(y_1{}^2-4\delta_l{}^2)z^2+8\delta_l{}^2(k-x)\}\Bigg]$$

$$+\frac{U_{l3}{}^2G'^2V_{12}{}^2m_K{}^5(x^2-4\delta_\pi{}^2)^{1/2}(y^2-4\delta_l{}^2)f_+{}^2(0)}{512\pi^3[(2-x)(y^2-4\delta_l{}^2)^{1/2}+(x^2-4\delta_\pi{}^2)^{1/2}yz]}\left[1+\lambda_+\frac{k-x}{\delta_\pi{}^2}\right]^2$$

$$\times[\text{all the above terms of this expression with the replacement } y_1\to y]\,, \tag{12}$$

where

$$y_1=\frac{2((k+\delta_l{}^2-x)(2-x)-(x^2-4\delta_\pi{}^2)^{1/2}z\{(k+\delta_l{}^2-x)^2-\delta_l{}^2[(2-x)^2-(x^2-4\delta_\pi{}^2)z^2]\}^{1/2})}{(2-x)^2-(x^2-4\delta_\pi{}^2)z^2}\,,$$

and (13)

$$y=\frac{2((k+\delta_l{}^2-\delta_3{}^2-x)(2-x)-(x^2-4\delta_\pi{}^2)^{1/2}z\{(k+\delta_l{}^2-\delta_3{}^2-x)-\delta_l{}^2[(2-x)^2-(x^2-4\delta_\pi{}^2)z^2]\}^{1/2})}{(2-x)^2-(x^2-4\delta_\pi{}^2)z^2}\,.$$

The $\pi$-$l$ angular correlations for $l=e$ and $\mu$ are shown, respectively, in Figs. 5 and 6 for $m(\nu_3)=0$, 100, 150, and 200 MeV. Finite-$m(\nu_3)$ effects are precisely transparent in numerical values, whereas the qualitative variation is identical to that with $m(\nu_3)=0$. Mixing effects are, however, very small.

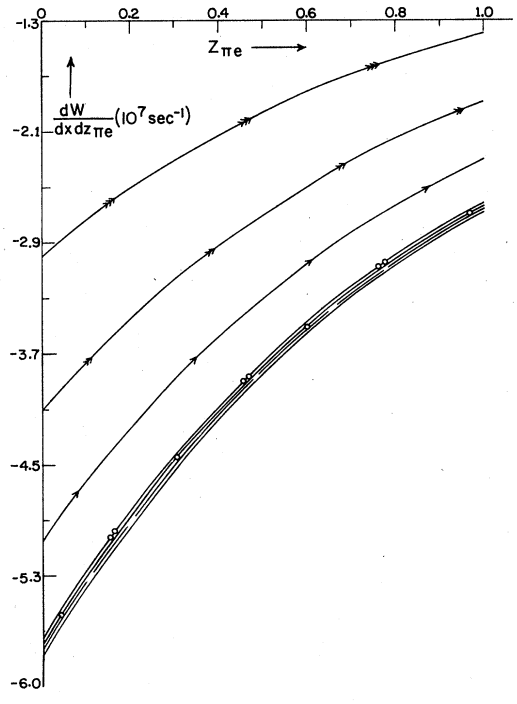


FIG. 5. $\pi$-$e$ angular correlation in the decay $K^+\to e^+\nu_i\pi^0$. Solid curve, single-arrow curve, double-arrow curve, and triple-arrow curve, are, respectively, for $m(\nu_3)=0$, 100, 150, and 200 MeV without mixing. Dashed curve, single-circle curve, and double-circle curve are, respectively, for $m(\nu_3)=100$, 150, and 200 MeV with KM mixing. Curves for $m(\nu_3)=100$, 150, and 200 MeV with hierarchical mixing (not shown) are almost coincident with the solid curve for $m(\nu_3)=0$.

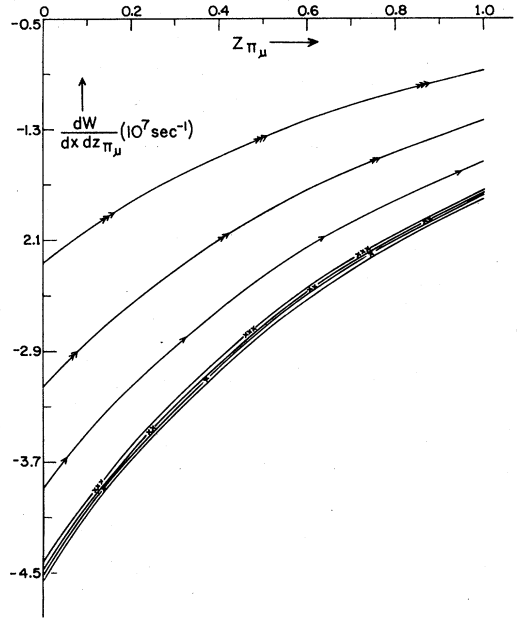


FIG. 6. $\pi$-$\mu$ angular correlation in the decay $K^+\to\mu^+\nu_i\pi^0$. Single-cross, double-cross, and triple-cross curves are, respectively, for $m(\nu_3)=100$, 150, and 200 MeV, with hierarchical mixing. The description of other curves is identical to that given in Fig. 5. Curves, not shown in figure, for $m(\nu_3)=100$, 150, and 200 MeV, with KM mixing, almost coincide with the solid curve.

### D. $\pi$-$l$ energy correlations

The expression for $\pi$-$l$ energy correlations is

$$\frac{dW}{dx\,dy}=\frac{\sum_{i=1}^{3}|U_{li}|^2 G'^2 V_{12}^2 m_K^5 f_+^2(0)\left[1+\lambda_+\frac{k-x}{\delta_\pi^2}\right]^2}{128\pi^3}$$

$$\times\Bigg[[(2+x)(1-\delta_\pi^2)+(x^2-4\delta_\pi^2)]2y-y^2(2+x)^2+2(x^2-4\delta_\pi^2)^{1/2}(y^2-4\delta_l^2)^{1/2}\cos\theta_{\pi l}[y(2+x)-3+\delta_\pi^2-x]$$

$$-(x^2-4\delta_\pi^2)(y^2-4\delta_l^2)\cos^2\theta_{\pi l}+8\delta_l^2(k+x)$$

$$+2\xi(0)\left[1-\lambda_+\frac{k-x}{\delta_\pi^2}\right][2y(2+x)(k-x)-y^2(4-x^2)-2(x^2-4\delta_\pi^2)^{1/2}(y^2-4\delta_l^2)^{1/2}\cos\theta_{\pi l}(k-x+xy)$$

$$+(x^2-4\delta_\pi^2)(y^2-4\delta_l^2)\cos^2\theta_{\pi l}+8\delta_l^2(1-\delta_\pi^2)]$$

$$+\xi^2(0)\left[1-\lambda_+\frac{k-x}{\delta_\pi^2}\right]^2\{(k-x)(2-x)2y-y^2(2-x)^2$$

$$+2(x^2-4\delta_\pi^2)^{1/2}(y^2-4\delta_l^2)^{1/2}\cos\theta_{\pi l}[k-x-y(2-x)]$$

$$-(x^2-4\delta_\pi^2)(y^2-4\delta_l^2)\cos^2\theta_{\pi l}+8\delta_l^2(k-x)\}\Bigg], \tag{14}$$

with $x=2E_{\pi^0}/m_K$, $y=2E_l/m_K$, and

$$\cos\theta_{\pi l}=\frac{2(k-x+\delta_l^2-\delta_i^2)-y(2-x)}{(x^2-4\delta_\pi^2)^{1/2}(y^2-4\delta_l^2)^{1/2}}\ . \tag{15}$$

Again retaining the contributions from the $\delta_3$ term in Eq. (14), we plot $dW/dx\,dy$ versus $x$ in Figs. 7 and 8 for $e$ and $\mu$ decays, respectively. Here again the finite-$m$ ($\nu_3$) effects are observable and mixing contributions are negligible.

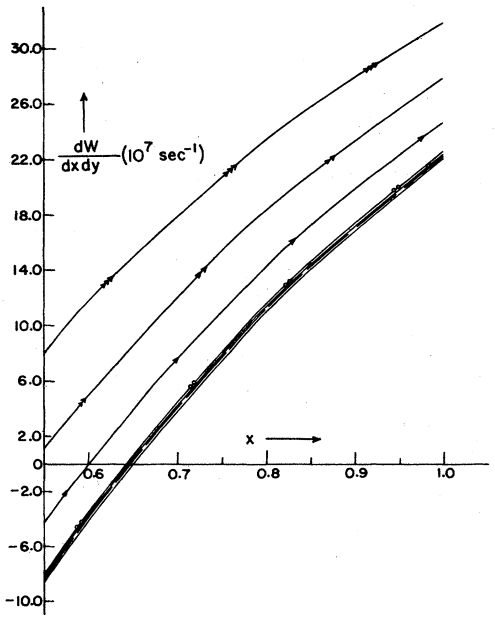


FIG. 7. $\pi$-$e$ energy correlation in the decay $K^+\to e^+\nu_i\pi^0$. Description of the curves is identical to that given in Fig. 5.

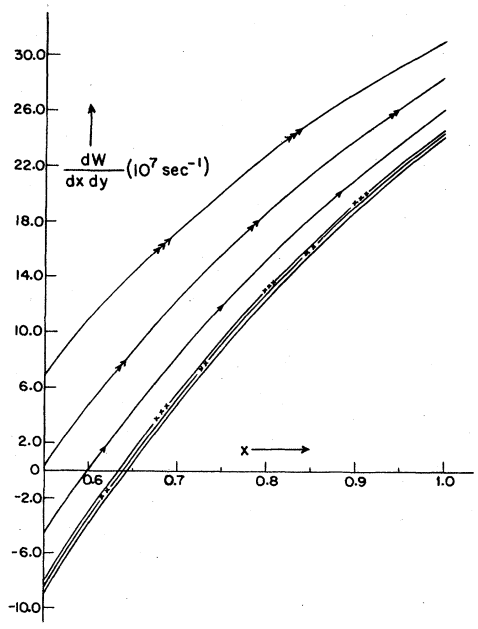


FIG. 8. $\pi$-$\mu$ energy correlation in the decay $K^+\to\mu^+\nu_i\pi^0$. Description of the curves is identical to that in Fig. 6.

### E. Lepton energy spectrum in $K^+ \to l^+ \nu_i \pi^0$

Integration of Eq. (14) for $x$ gives the expression for lepton energy spectrum

$$\frac{dW}{dy} = \frac{\sum_{i=1}^{3} |U_{li}|^2 G'^2 V_{12}^2 f_+^2(0) m_K^5}{128\pi^3} \times \int_{x_{\min}}^{x_{\max}} \left[1+\lambda_+ \frac{k-x}{\delta_\pi^2}\right]^2 \Big[ \{(2+x)(1-\delta_\pi^2)+(x^2-4\delta_\pi^2)\}2y - y^2(2+x)^2$$
$$+[2(k-x+\delta_l^2-\delta_i^2)-y(2-x)][y(6+x)-2(4+\delta_l^2-\delta_i^2)]+8\delta_l^2(k+x)$$
$$+2\xi(0)\left[1-\lambda_+\frac{k-x}{\delta_\pi^2}\right]\{2y(2+x)(k-x)-y^2(4-x^2)$$
$$+[2(k-x+\delta_l^2-\delta_i^2)-y(2-x)](2\delta_l^2-2\delta_i^2-xy-2y)$$
$$+8\delta_l^2(1-\delta_\pi^2)\}$$
$$+\xi^2(0)\left[1-\lambda_+\frac{k-x}{\delta_\pi^2}\right]^2\{(k-x)(2-x)2y-y^2(2-x)^2$$
$$+[2(k-x+\delta_l^2-\delta_i^2)-y(2-x)](xy-4y+2\delta_l^2+2\delta_i^2)$$
$$+8\delta_l^2(k-x)\}\Big] dx \,, \tag{16}$$

where $x_{\max}$ and $x_{\min}$ are given by Eq. (7). Retaining again the dominant contribution from the $\delta_3$ term, we show the variation of $dW/dy$ with $y$, $l=e$ and $\mu$, respectively, in Figs. 9 and 10. The finite-$m(\nu_3)$ effects are distinct and that of mixing can be observable for $0.45 < y < 0.65$, and $y < 0.15$. The distinction between hierarchical and KM mixings is, however, not possible.

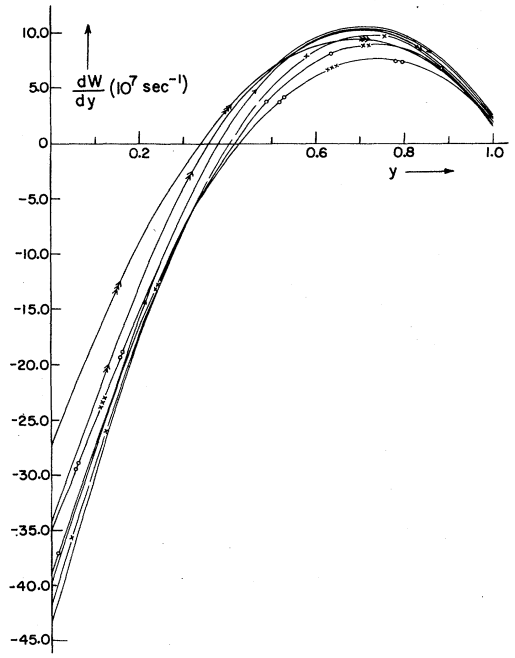


FIG. 9. Electron energy spectrum in the decay $K^+ \to e^+ \nu_i \pi^0$. Description of the curves is identical to that given in Figs. 5 and 6.

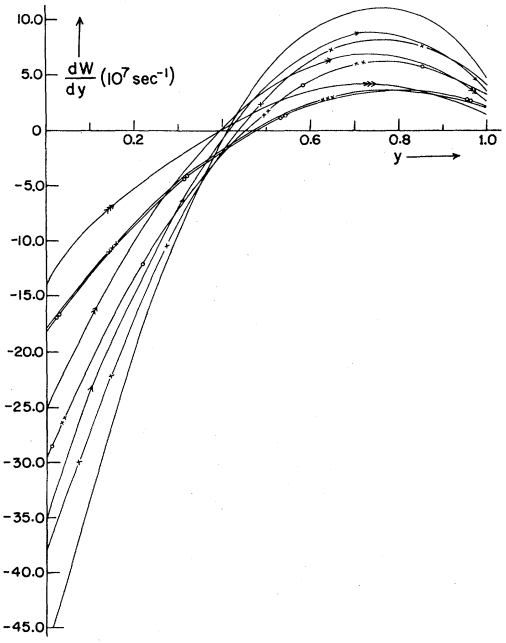


FIG. 10. Muon energy spectrum in the decay $K^+ \to \mu^+ \nu_i \pi^0$. Description of the curves is identical to that given in Figs. 5 and 6.

### F. $\pi$-$\nu_i$ angular correlations

The expressions for pion-neutrino angular correlations are obtained from Eq. (10) by making the replacements $m_l \leftrightarrow m(\nu_i)$, $E_l \rightarrow E_{\nu_i}$, $\theta_{\pi l} \rightarrow \theta_{\pi\nu_i}$, and $\sum |U_{li}|^2 \rightarrow |U_{li}|^2$:

$$\frac{dW}{dx\,dz} = \frac{|U_{li}|^2 G'^2 V_{12}^2 f_+^2(0)(x^2-4\delta_\pi^2)^{1/2}(y^2-4\delta_i^2)m_K^5\left[1+\lambda_+\frac{k-x}{\delta_\pi^2}\right]^2}{512\pi^3[(2-x)(y^2-4\delta_i^2)^{1/2}+(x^2-4\delta_\pi^2)^{1/2}zy]}$$

$$\times\Bigg[[(1-\delta_\pi^2)(2+x)+(x^2-4\delta_\pi^2)]2y-y^2(2+x)^2$$

$$+2(x^2-4\delta_\pi^2)^{1/2}(y^2-4\delta_i^2)^{1/2}z[y(2+x)-3+\delta_\pi^2-x]-(x^2-4\delta_\pi^2)(y^2-4\delta_i^2)z^2+8\delta_i^2(k+x)$$

$$+2\xi(0)\left[1-\lambda_+\frac{k-x}{\delta_\pi^2}\right][(k-x)(2+x)2y-y^2(4-x^2)-2(x^2-4\delta_\pi^2)^{1/2}(y^2-4\delta_i^2)^{1/2}z(xy+k-x)$$

$$+(x^2-4\delta_\pi^2)(y^2-4\delta_i^2)z^2+8\delta_i^2(1-\delta_\pi^2)]$$

$$+\xi^2(0)\left[1-\lambda_+\frac{k-x}{\delta_\pi^2}\right]^2\{(k-x)(2-x)2y-y^2(2-x)^2+2(x^2-4\delta_\pi^2)^{1/2}(y^2-4\delta_i^2)^{1/2}z[k-x-y(2-x)]$$

$$-(x^2-4\delta_\pi^2)(y^2-4\delta_i^2)z^2+8\delta_i^2(k-x)\}\Bigg], \tag{17}$$

where

$$y=\frac{2E_{\nu_i}}{m_K}=\frac{2((k+\delta_i^2-\delta_l^2-x)(2-x)\pm(x^2-4\delta_\pi^2)^{1/2}z\{(k+\delta_i^2-\delta_l^2-x)^2-\delta_i^2[(2-x)^2-(x^2-4\delta_\pi^2)z^2]\}^{1/2})}{(2-x)^2-(x^2-4\delta_\pi^2)z^2} \tag{18}$$

and

$$z=\cos\theta_{\pi\nu_i}\ .$$

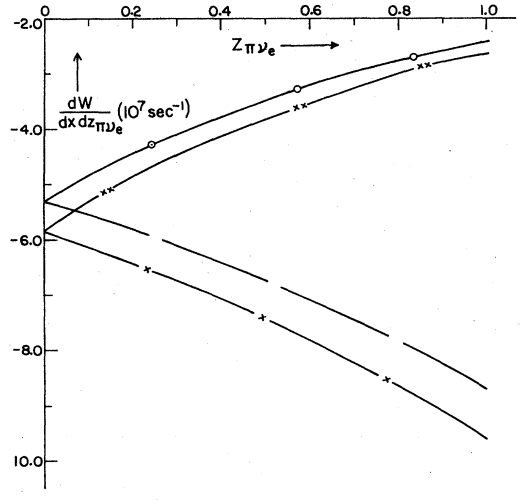


FIG. 11. $\pi$-$\nu_e$ angular correlation in the decay $K^+ \rightarrow e^+\nu_i\pi^0$. Single-cross and double-cross curves are for hierarchical mixing with $y_{max}$ and $y_{min}$, respectively. Dashed curve and single-circle curve are for KM mixing with $y_{max}$ and $y_{min}$, respectively (see text).

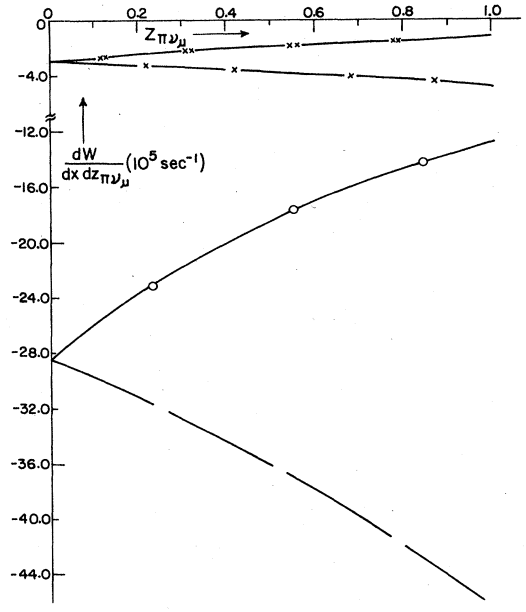


FIG. 12. $\pi$-$\nu_\mu$ angular correlation in the decay $K^+ \rightarrow e^+\nu_i\pi^0$. Description of the curves is identical to that given in Fig. 11.

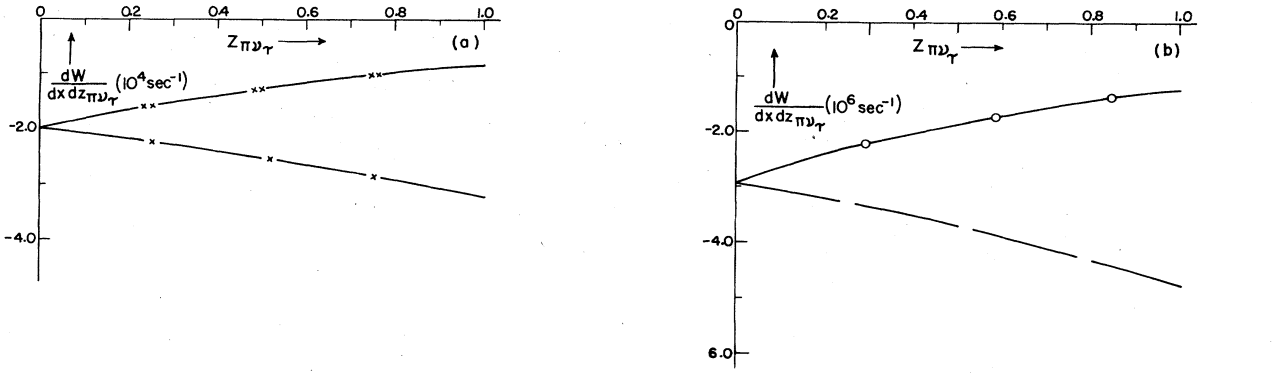


FIG. 13. $\pi$-$\nu_\tau$ angular correlation in the decay $K^+ \to e^+ \nu_i \pi^0$. (a) Single-cross curve and double-cross curve are for hierarchical mixing with $y_{\max}$ and $y_{\min}$, respectively (see text). (b) Dashed curve and single-circle curve are for KM mixing with $y_{\max}$ and $y_{\min}$, respectively (see text).

Variations of $dW/dx\,dz$ with $z = z_{\pi\nu_e}$, $z_{\pi\nu_\mu}$, and $z_{\pi\nu_\tau}$ in the decay $K^+ \to e^+ \nu_i \pi^0$ are shown in Figs. 11—13, respectively, for the dominant mode taking $m(\nu_e) = m(\nu_\mu) = 0$, $m(\nu_3) = 150$ MeV, and $x = 0.6$. It is seen that mixings contribute significantly towards these correlations. In this case one may also discern between the types of mixing used.

Corresponding angular correlations in the decay $K^+ \to \mu^+ \nu_i \pi^0$, give almost identical qualitative spectrum but with obvious numerical differences. However, it is very difficult to differentiate between KM and hierarchical mixing in $\pi$-$\nu_\mu$ angular correlation of this decay.

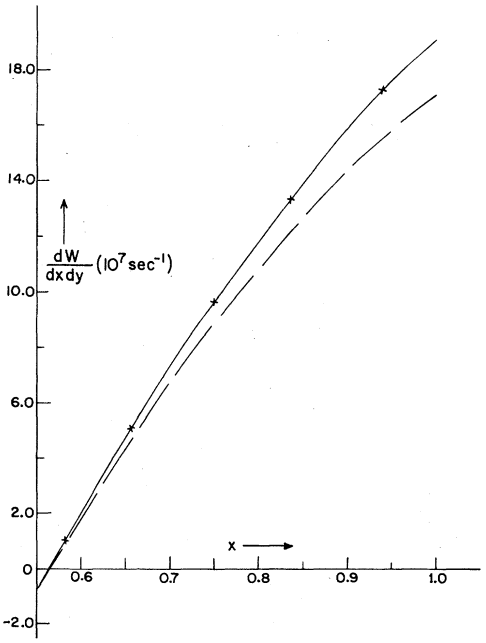


FIG. 14. $\pi$-$\nu_e$ energy correlation in the decay $K^+ \to e^+ \nu_i \pi^0$. Single-cross curve and dashed curve are for hierarchical and KM mixing, respectively (see text).

### G. $\pi$-$\nu_i$ energy correlations in $K^+ \to l^+ \nu_i \pi^0$

The expression for $\pi$-$\nu_i$ energy correlations are obtained from Eqs. (14) and (15) by making the replacements $m_l \leftrightarrow m(\nu_i)$, $E_l \to E_{\nu_i}$, $\theta_{\pi l} \to \theta_{\pi\nu_i}$, and $\sum_{i=1}^{3} |U_{li}|^2 = |U_{li}|^2$.

$\pi$-$\nu_e$, $\pi$-$\nu_\mu$, and $\pi$-$\nu_\tau$ energy correlations in the decay $K^+ \to e^+ \nu_i \pi^0$ are shown, respectively, in Figs. 14—16, taking $m(\nu_e) = m(\nu_\mu) = 0$, $m(\nu_3) = 150$, and $y = 0.65$. Here again one notices the distinct contributions from mixing and also the distinction between hierarchical and KM mixing contributions in $\pi$-$\nu_\mu$ and $\pi$-$\nu_\tau$ energy correlations. However, this is very small in $\pi$-$\nu_e$ energy correlations.

For the decay $K^+ \to \mu^+ \nu_i \pi^0$, the qualitative nature of $\pi$-$\nu_i$ energy correlations is almost identical to that for electronic decay. But in this case the difference in the nature of mixing is very small for $\pi$-$\nu_\mu$ energy correlation and is distinct in $\pi$-$\nu_e$ and $\pi$-$\nu_\tau$ energy correlations.

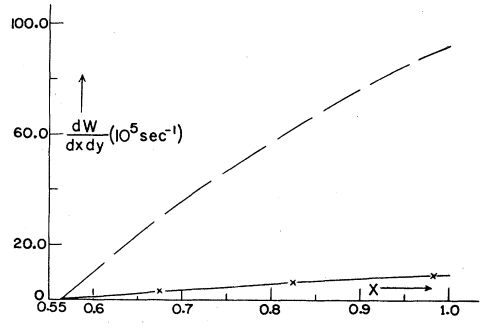


FIG. 15. $\pi$-$\nu_\mu$ energy correlation in the decay $K^+ \to e^+ \nu_i \pi^0$. Description of the curves is identical to that given in Fig. 14.

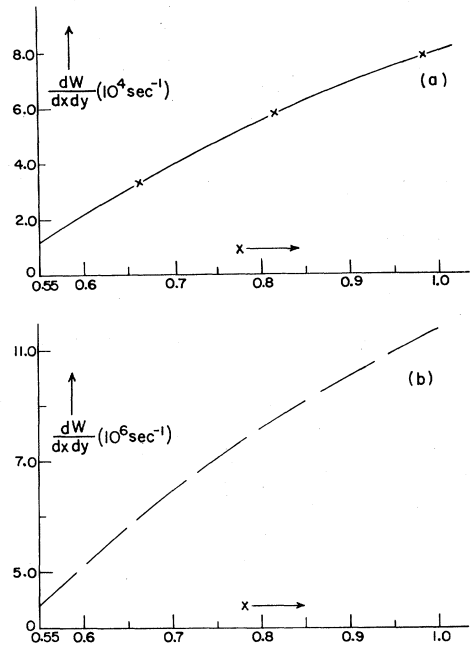


FIG. 16. $\pi$-$\nu_\tau$ energy correlation in the decay $K^+ \to e^+ \nu_i \pi^0$, (a) with hierarchical mixing, (b) with KM mixing.

## III. SUMMARY AND CONCLUSIONS

(1) The contributions of finite $m(\nu_3)$ are substantial in (i) pion energy spectrum, (ii) $\pi$-$l$ angular correlations, (iii) $\pi$-$l$ energy correlations, and (iv) lepton energy spectrum. The effect becomes pronounced in the pion energy spectrum towards the high-energy tail ($x > 0.85$).

(2) The neutrino-mass-mixing effects are, in general, very small in (i) pion energy spectrum, (ii) $\pi$-$l$ angular correlations, and (iii) $\pi$-$l$ energy correlations. However, in the pion energy spectrum it could be reasonable towards the high-energy end ($x > 0.9$) and for higher value of $m(\nu_3)$. The mixing effects are pronounced in $\pi$-$\nu_i$ angular and energy correlations. As such, correlation studies[31] could be a useful place for ascertaining such contributions, and possibly the values of mixing parameters.

(3) The $\nu_3$ mass limits, calculated for the two decays by using the present values of experimental parameters, are consistent with the recently reported upper-bound $m(\nu_3) < 164$ MeV. It may, however, be pointed out that in models involving massive neutrinos, neutrinos involving still higher-mass species[13] are not ruled out, notwithstanding the constraints provided by cosmological considerations.[32]

Finally, the discussion reported in this work does point towards the fact that $K_{l3}^{+}$ decays may be a useful place to look for finite-neutrino-mass contributions and possibly the mass-mixing effects too.

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