Explicit decoupling gauge condition

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The gauge condition for the decoupling of fermions in two-dimensional QCD is given explicitly for the gauge group SU(2). The condition involves in this case a second-order differential operator, but in general the order depends on the gauge group.

Massive quantum chromodynamics in two space-time dimensions has been treated in a variety of approximation schemes and a rough picture of the particle spectrum has emerged. For the case of QCD_2 with massless fermions one might have expected to do somewhat better following the techniques known from QED_2.[1,2]

Some progress in this direction has been done, based essentially on the observation that the non-Abelian gauge field in two dimensions has a particularly simple representation.[3] This gives rise to a decoupling of the fermions and an evaluation of the fermion determinant, which in turn produces an effective interaction in the gauge field sector.[4–6]

The decoupling phenomenon is also useful in connection with the external-source problem of QCD_2.[7,8] Recently, however, much attention has been focused on the fact that the effective action can be shown to contain a so-called Wess-Zumino term which is related to the anomaly.[9–12]

Let us recall that in two-dimensional Minkowski space any non-Abelian gauge field A can be represented as

$$ig\gamma^{\mu}A_{\mu}=\gamma^{\mu}(\partial_{\mu}T)T^{\dagger} \ , \tag{1}$$

where T is of the form

$$T=\exp(i\lambda)\exp(i\gamma_5\phi) \ , \tag{2}$$

with λ and ϕ belonging to the Lie algebra of the gauge group. In the following only SU(2) will be considered and for simplicity we may put $g=1$.

Equation (1) can also be written explicitly as

$$A_{\mu}=-(i/2)\,\mathrm{Tr}\gamma_{\mu}\gamma_{\nu}\partial^{\nu}TT^{\dagger} \ , \tag{3}$$

where the trace is over Dirac matrices. In the Abelian case Eq. (3) would simply read

$$A_{\mu}=\partial_{\mu}\lambda-\epsilon_{\mu\nu}\partial^{\nu}\phi \ . \tag{4}$$

Going to light-cone coordinates $x_{\pm}=(x^0\pm x^1)/2$, $\partial_{\pm}=\partial_0\pm\partial_1$, $A_{\pm}=A_0\pm A_1$ and using the γ-matrix representation

$$\gamma^0=\begin{pmatrix}0&1\\1&0\end{pmatrix},\quad \gamma^1=\begin{pmatrix}0&1\\-1&0\end{pmatrix},\quad \gamma_5=\gamma^0\gamma^1=\begin{pmatrix}-1&0\\0&1\end{pmatrix} \ , \tag{5}$$

Eq. (1) becomes

$$iA_{\pm}=\partial_{\pm}T_{\mp}T_{\mp}^{\dagger} \ . \tag{6}$$

Here, the matrices $T_{\pm}\in SU(2)$ are defined by

$$T=\begin{pmatrix}T_+&0\\0&T_-\end{pmatrix} \ . \tag{7}$$

Using the Baker-Campell-Hausdorff formula this representation of T can be seen to be equivalent to Eq. (2). From Eq. (6) the existence of a T with the desired properties is clear, since given A we have in general

$$T=P\exp\left(i\int^{x_{\mp}}dx_{\pm}A_{\mp}\right)V_{\pm}(x_{\pm}) \ , \tag{8}$$

where P stands for path ordering and $V_{\pm}$ are arbitrary SU(2) matrices depending on $x_{\pm}$.

The decoupling gauge is defined by making the gauge choice $\lambda=0$ in Eq. (2). With this choice the fermion part of the Lagrangian becomes

$$L=\bar{\psi}\gamma^{\mu}(i\partial_{\mu}+A_{\mu})\psi=\bar{\chi}i\gamma^{\mu}\partial_{\mu}\chi \ , \tag{9}$$

where the chirally rotated fermion $\chi=T^{\dagger}\psi$ $[T=\exp(i\gamma_5\phi)]$ is seen to "decouple." The implications of this chiral transformation in a functional-integral framework has been discussed in detail elsewhere. Here, I will simply show how to find the explicit gauge condition on A which corresponds to the gauge choice $\lambda=0$. In the Abelian case (4), $\lambda=0$ means $A_{\mu}=-\epsilon_{\mu\nu}\partial^{\nu}\phi$ which is equivalent to the Lorentz gauge $\partial^{\mu}A_{\mu}=0$, or in light-cone coordinates

$$\partial_-A_++\partial_+A_-=0 \ . \tag{10}$$

In the non-Abelian case the situation is more complicated. From $\lambda=0$ we find $T_+=T_-^{\dagger}$, but the constraint implied by this relation on A following, for example, from the representation (8) does not seem to be too transparent. However, it was inferred by Roskies[3] that the gauge condition on A, for the case of SU(2) at hand, must be a local one, involving at most second-order differential operators. That this is indeed correct will be shown explicitly below.

Suppose that we have made the choice $\lambda=0$ so that $T_+=T_-^{\dagger}$; then Eq. (6) becomes

$$\begin{aligned}A_+&=-i\partial_+T_-T_+ \ ,\\ A_-&=-i\partial_-T_+T_- \ .\end{aligned} \tag{11}$$

Taking derivatives $\partial_{\pm}$ of these equations one finds easily

$$\partial_-A_+=-T_-\partial_+A_-T_+ \ . \tag{12}$$

In the Abelian case this is of course nothing else than the Lorentz gauge condition (10). Now, T depends on A in a complicated way and Eq. (12) is not precisely what we are looking for. From Eq. (12) we can, however, get two new equations by taking ∂ derivatives once more:

$$\begin{aligned}\partial_-{}^2&=-T_-(\partial_+\partial_-A_--i[A_-,\partial_+A_-])T_+ \ ,\\ \partial_+\partial_-A_+-i[A_+,\partial_-A_+]&=-T_-\partial_+{}^2A_-T_+ \ .\end{aligned} \tag{13}$$

Introduce now the two sets of vectors a_i and b_i, $i=1,2,3$ by

$$a=(\partial_-A_+,\partial_-^2A_+,\partial_+\partial_-A_+-i[A_+,\partial_-A_+])\ ,$$
$$b=(-\partial_+A_-,-\partial_+\partial_-A_-+i[A_-,\partial_+A_-],-\partial_+^2A_-)\ . \tag{14}$$

From Eqs. (12) and (13) it transpires that a and b are related by a similarity transformation

$$a=T_-bT_+\ . \tag{15}$$

A necessary condition for being in the decoupling gauge is therefore that

$$\mathrm{Tr}a_ia_j=\mathrm{Tr}b_ib_j,\quad i,j=1,2,3\ , \tag{16}$$

where the trace is over SU(2) indices. For $i=j=1$ this is just the condition found previously by Roskies.[3] The conditions (16) are all local and involve at most differential operators of order two acting on A.

The given conditions are also sufficient to characterize the decoupling gauge. In general, for given $A_\pm$, the three a vectors are linearly independent elements of the SU(2) Lie algebra (or equivalently, linearly independent three-dimensional vectors in R^3), and similarly for the b vectors; so if Eq. (16) is fulfilled, there will always exist a unitary T $(=T_-=T_+^\dagger)$ such that Eq. (15) is true. In fact, to determine T one would only need Eq. (16) for $i,j=1,2$. Using now the definitions (14) it is easy to work backwards and find that the gauge field A must be given by (11); i.e., A is represented in the decoupling gauge.

For a gauge group bigger than SU(2) the situation is somewhat more complicated. In principle one can, however, follow the above procedure taking more and more derivatives of the basic relation (12) until one gets enough elements to span the Lie algebra. The gauge condition will then of course involve higher derivatives of a maximal order depending on the particular group at hand.

The fact that the decoupling gauge condition depends on the group (contrary for example to the Lorentz condition), and that for the simplest case of SU(2) it is already quite nonlinear, seems to indicate that any direct use of it, say in a functional-integral approach, is deemed to be rather involved.

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