Rotational anomalies without anyons

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A specific field theory is proposed in two spatial dimensions which has anomalous rotational properties. Although this might be expected to lead to a concrete realization of what Wilczek refers to as the anyon, it is shown by utilizing the transformation properties of the system and the statistics of the underlying charge fields that anyonic interpolations between bosons and fermions do not occur. This leads to the suggestion that anyons inferred from semiclassical considerations will not survive the transition to a fully relativistic field theory.

There has recently been a considerable interest in two-dimensional models[1-3] which display anomalous contributions to the angular momentum operator. These additional terms have peculiar effects upon the quantum mechanics of multiparticle systems, leading allegedly to the possibility of any phase (hence anyon) upon exchange of identical particles. The work on such models has generally looked at two-dimensional slices of three-dimensional theories, a procedure which, though plausible to denizens of a three-dimensional world, is not entirely satisfactory. In particular, the logic of using intrinsically three-dimensional concepts such as solenoids and monopoles is somewhat less than compelling and indeed has certain pitfalls, as detailed in Ref. 3.

The present work starts from the premise that properties of two-dimensional anyons are of some interest and that their study would be significantly enhanced by embedding them in a theory which is intrinsically two dimensional. The model advanced here toward that end is a gauge theory, but one which is not QED or one of its non-Abelian generalizations. It is described by the Lagrange density[4]

$$\mathscr{L}=\tfrac{1}{2}\phi^\mu\epsilon_{\mu\nu\alpha}\partial^\alpha\phi^\nu+g\phi^\mu j_\mu+\mathscr{L}' \quad , \tag{1}$$

where j^μ is a conserved current whose precise structure is not relevant for the moment. The quantity $\mathscr{L}'$ refers to the charge-bearing fields which give rise to this current. The $\mathscr{L}$ of Eq. (1) is clearly invariant under

$$\phi^\mu \rightarrow \phi^\mu+\partial^\mu\Lambda$$

up to a total divergence provided that

$$\mathscr{L}'\rightarrow\mathscr{L}'-gj_\mu\partial^\mu\Lambda$$

in the usual way. It is of some interest to compare the standard formulation of three-dimensional QED in the first-order form with the theory being considered here. The former requires six components (a vector and antisymmetrical tensor) whereas one utilizes here only a single three-vector. This stinginess relative to QED has important consequences—namely, that while QED_3 has a true photon which persists in the limit of zero coupling, the field ϕ^μ has the property of vanishing for $g=0$. This peculiarity makes the model in some respects a closer relative of the Schwinger model than QED_3.

The validity of these claims follows from a consideration of the field equations

$$\epsilon_{\mu\nu\alpha}\partial^\nu\phi^\alpha=gj_\mu \quad , \tag{2}$$

which are most conveniently analyzed in the radiation gauge

$$\partial_i\phi^i=0, \quad i=1,2 \quad . \tag{3}$$

The temporal component of (2) is

$$-\nabla\times\boldsymbol{\phi}=gj^0 \quad , \tag{4}$$

where one is reminded that in two dimensions the curl is a rotational scalar. The solution to (4) in the gauge (3) is

$$\phi_i(x)=-\epsilon_{ij}\partial_j g\int d^2x'\mathscr{D}(\mathbf{x}-\mathbf{x}')j^0(x') \quad ,$$

where ϵ_{ij} is the Levi-Civita tensor in two-space and $\mathscr{D}(\mathbf{x})$ is defined by

$$-\nabla^2\mathscr{D}(\mathbf{x})=\delta(\mathbf{x}) \quad . \tag{5}$$

As is well known the solution of (5) is

$$\mathscr{D}(\mathbf{x})=-\frac{1}{4\pi}\ln\mathbf{x}^2+\text{const} \quad . \tag{6}$$

Here, and in what follows, the function $\mathscr{D}(\mathbf{x})$ is always differentiated, so that the constant in (6) effectively can be ignored.

Upon taking the curl of the spatial components of (2) and using current conservation we obtain

$$-\nabla^2\phi^0=g\nabla\times\mathbf{j} \quad , \tag{7}$$

which has solution

$$\phi^0(x)=g\int d^2x'\mathbf{j}(x')\times\nabla'\mathscr{D}(\mathbf{x}-\mathbf{x}') \quad . \tag{8}$$

It is to be noted that the naive solution of (7) differs from (8) by an integration by parts. In fact (8) is the correct result as it is demonstrably the solution of (2) with no assumptions being necessary as to the legitimacy of integrations by parts.[5]

This completes the demonstration of the absence of a photonlike excitation of the theory as all components ϕ^μ are seen to vanish in the $g=0$ limit. Alternatively one can show by explicit calculation that for no coupling the generator of variations of ϕ^μ vanishes and consequently that there are no nonvanishing commutators in the theory. Since these and other aspects of the model are more fully explored elsewhere, the concern here will be with the properties of the theory with respect to rotations.

To display the rotational features one computes the generator J of rotations in the x-y plane. It is given by

$$J=\int d^2x\,\mathbf{x}\times\mathbf{T} \quad ,$$

where $(\mathbf{T})_k \equiv T^{0k}$ is given for the case of a spin-$\frac{1}{2}$ field ψ by

$$T^{0k} = -\frac{i}{2}\psi\partial^k\psi + \frac{i}{8}\psi[\gamma^k,\gamma^l]\partial_l\psi - g\phi^k j^0 \ . \quad (9)$$

All fields are taken to be Hermitian and the current j^μ is consequently given by

$$j^\mu = \tfrac{1}{2}\psi\beta\gamma^\mu q\psi \ ,$$

where q is an imaginary antisymmetric charge matrix. The important term in (9) is the last one as is clearly seen by its contribution to J:

$$-g\int d^2x\mathbf{x}\times\boldsymbol{\phi}j^0 = -g^2\int d^2xd^2x'j^0(x)\mathbf{r}\cdot\nabla\mathscr{D}(\mathbf{x}-\mathbf{x}')j^0(x')$$
$$= \frac{g^2}{2\pi}\int d^2xd^2x'j^0(x)\frac{\mathbf{x}\cdot(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^2}j^0(x')$$
$$= \frac{g^2}{4\pi}Q^2 \ , \quad (10)$$

where

$$Q \equiv \int d^2xj^0(x) \ .$$

The result (10) easily leads to the desired commutator of J with ψ,

$$[J,\psi] = i(\mathbf{r}\times\nabla)\psi - \frac{i}{2}(\tfrac{1}{2}\epsilon_{kl}\gamma_k\gamma_l)\psi - \frac{g^2}{2\pi}qQ\psi \ ,$$

where operator symmetrization of Q and ψ is understood.[6] This implies in terms of the (non-Hermitian) eigenfields $\psi_\pm$ of q,

$$q\psi_\pm = \pm\psi_\pm \ ,$$

the finite rotation result

$$e^{i\pi J}\psi_\pm(\mathbf{x})e^{-i\pi J}$$
$$= \tfrac{1}{2}\epsilon_{kl}\gamma_k\gamma_l\exp\left(\mp\frac{ig^2}{4}Q\right)\psi_\pm(-\mathbf{x})\exp\left(\mp\frac{ig^2}{4}Q\right)$$
$$= \tfrac{1}{2}\epsilon_{kl}\gamma_k\gamma_l\exp\left(\mp\frac{ig^2}{2}Q - \frac{ig^2}{4}\right)\psi_\pm(-\mathbf{x}) \ . \quad (11)$$

Evidently the term $\frac{1}{2}\epsilon_{kl}\gamma^k\gamma^l$ is a spin term which would be altered upon taking a different choice for the charge field. The g^2-dependent terms, however, can be expected to persist for all cases in view of the generality of (10) and, more importantly, cannot be eliminated by simple redefinition of J without destroying Poincaré invariance.

Although (11) and its obvious extension to other spin values would seem to imply the desired field-theoretical basis for introducing the anyon, one can now show that the wave functions of multiparticle systems have the same statistics as the underlying charged field operators. To this end one recalls that the wave function Ψ of a two-particle system in the center-of-mass frame can be defined (somewhat crudely) by

$$|2\rangle = \int d^2x\Psi^*(\mathbf{x})\psi_\pm(\mathbf{x})\psi_\pm(-\mathbf{x})|0\rangle \ , \quad (12)$$

where for purposes of simplicity $\psi_\pm$ will be assumed bosonic, thereby allowing the spin term in (11) to be dropped. This condition leads to

$$|2\rangle = \int d^2x\Psi^*(\mathbf{x})\psi_\pm(-\mathbf{x})\psi_\pm(\mathbf{x})|0\rangle \ , \quad (13)$$

which by (11) can be recast into

$$|2\rangle = e^{-ig^2}\int d^2x\Psi^*(\mathbf{x})e^{i\pi J}\psi_\pm(\mathbf{x})\psi_\pm(-\mathbf{x})|0\rangle \ , \quad (14)$$

where use has been made of the fact that J and Q each annihilate the vacuum. Evidently (12) and (13) imply

$$\Psi(\mathbf{x}) = \Psi(-\mathbf{x}) \ ,$$

while (14) requires

$$e^{i\pi(J-g^2/\pi)}|2\rangle = |2\rangle \ . \quad (15)$$

Since the operator $J-g^2/\pi$ can be replaced here by $J-(g^2/4\pi)Q^2$, Eq. (15) is evidently the statement that the eigenvalues of the orbital part of J must be the even integers.[7] This leads to the condition that for Ψ of the form $e^{im\phi}$, an identical condition must be placed on the allowed values of m. Thus, there follows the claimed result—namely, that the anomaly in the transformation law of the charged field does not manifest itself as a modification of the usual symmetry properties of wave functions. While the possibility that such a phenomenon could occur in other field theories cannot be excluded, its absence in the model considered here—the first field-theoretical manifestation of a rotational anomaly[8]—strongly suggests its absence quite generally in fully quantized relativistic theories.

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[1]F. Wilczek, Phys. Rev. Lett. **48**, 1144 (1982); **49**, 957 (1982).

[2]D. H. Kobe, Phys. Rev. Lett. **49**, 1592 (1982).

[3]H. J. Lipkin and M. Peshkin, Phys. Lett. **118B**, 385 (1982).

[4]We take $\hbar = c = 1$ and the trace of the metric tensor to be $+1$.

[5]The reader should perhaps be reminded that it is precisely this point, namely, surface terms which plague the Schwinger model and lead to its noncovariance in the charged sectors.

[6]It should be noted here that a rotational anomaly has also been claimed to exist in a related model in $(2+1)$-space by S. Deser, R. Jackiw, and S. Templeton [Ann. Phys. (N.Y.) **140**, 372 (1982)]. However, as is pointed out by the author [Ann. Phys. (N.Y.) **157**, 342 (1984)], the anomaly which they find for the free field case is a result of their introduction of an auxiliary field not present in the original formulation of the model. The latter is anomaly free.

[7]The possibility of fractional angular momentum has also been considered by R. Jackiw and A. M. Redlich [Phys. Rev. Lett. **50**, 555 (1983)]. Their approach is a kinematical one in the framework of nonrelativistic wave mechanics as contrasted with the dynamical arguments presented here in the context of a fully relativistic quantized field theory.

[8]Topological arguments for fractional statistics in $(2+1)$-space have been advanced by F. Wilczek and A. Zee, Phys. Rev. Lett. **51**, 2250 (1983).