Differential cross section of electron-positron bremsstrahlung

Eberhard Haug
Lehrstuhl für Theoretische Astrophysik, Universität Tübingen, Auf der Morgenstelle 12, D-7400 Tübingen, Federal Republic of Germany
(Received 29 August 1984)

A formula is given for the doubly differential cross section of electron-positron bremsstrahlung which is exact in lowest-order perturbation theory. Angular distributions and energy spectra of the emitted photon are computed in the center-of-mass system and in the laboratory system. The results are compared with available approximations and with the cross sections of electron-electron and electron-proton bremsstrahlung.

I. INTRODUCTION

The calculation of the fully differential cross section for the production of bremsstrahlung in collisions between free electrons and positrons is a straightforward application of quantum electrodynamics. However, even in lowest-order perturbation theory eight Feynman diagrams contribute to the matrix element, four of them representing scattering graphs and four representing virtual annihilation graphs.[1] Therefore the evaluation of the traces is very laborious and the resulting cross-section formula is extremely lengthy. It is most simply derived from the corresponding expression for electron-electron bremsstrahlung[2,3] by means of the well-known substitution law.[4]

Stimulated by experiments with colliding electron-positron beams of high energy, most calculations of the angular distribution and spectrum of electron-positron (e^-e^+) bremsstrahlung were performed at ultrarelativistic energies[1,5–10] where only two of the eight Feynman diagrams give an appreciable contribution. Besides, it can be shown that all the interference terms of the matrix element can be neglected within the high-energy approximation. The resulting cross sections were given either in the center-of-mass system or in the laboratory system, i.e., the rest system of one of the incident particles.

Recent interest in the process of e^-e^+ bremsstrahlung arose from the study of hot astrophysical plasmas which are expected to exist in active galactic nuclei and in gamma-ray bursters.[11–14] At semirelativistic temperatures electron-positron pairs are created through photon-photon, photon-particle, and particle-particle interactions.[15] For a full understanding of all the physical processes occurring in hot astrophysical plasmas it is indispensable to know the cross sections of the contributing reactions in a wide energy range. So far, the lack of the e^-e^+ bremsstrahlung spectrum has been most serious.[16]

In the present paper a manageable formula for the cross section of e^-e^+ bremsstrahlung differential in photon energy and angles is given. It is obtained by integrating analytically the fully differential cross section over the angles of the outgoing positron, without any approximations. Since the formula is expressed as a function of invariant products, it can be specialized to any frame of reference, e.g., the c.m. or the rest system of one of the incoming particles. This property is essential for the astrophysical applications. The angle-independent photon spectrum can be easily computed by numerical integration. The results are compared with various approximations and with the corresponding processes of electron-proton (ep) and electron-electron (e^-e^-) bremsstrahlung.

II. CROSS SECTIONS

In the elementary process of e^-e^+ bremsstrahlung (Fig. 1) an electron with four-momentum $p=(\epsilon_-,\mathbf{p})$ and a positron with four-momentum $q=(\epsilon_+,\mathbf{q})$ collide under the emission of a photon with four-momentum $k=(k,\mathbf{k})$.[17] The outgoing particles have the four-momenta $p'=(\epsilon'_-,\mathbf{p}')$ and $q'=(\epsilon'_+,\mathbf{q}')$. The differential cross section for unpolarized particles is given by[1,4]

$$d\sigma=\frac{\alpha r_0{}^2}{\pi^2}\frac{\delta^4(p+q-p'-q'-k)}{[(pq)^2-1]^{1/2}}A\frac{d^3p'}{\epsilon'_-}\frac{d^3q'}{\epsilon'_+}\frac{d^3k}{k}\,, \tag{1}$$

where $\alpha\approx\frac{1}{137}$ is the fine-structure constant and $r_0=e^2/mc^2$ is the classical electron radius. A is the absolute square of the matrix element summed over the spin directions of the electrons and positrons and the polarization directions of the photons; it is a complicated function of invariant products between the four-momenta of the

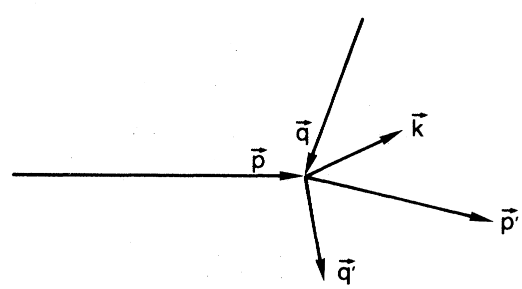

FIG. 1. Elementary process of electron-positron bremsstrahlung.

particles and photon which is too lengthy to be reproduced here. A was derived by means of the substitution rule from the traces evaluated by Anders[18] for the corresponding process of e^-e^- bremsstrahlung.

By squaring the conservation law,

$$p+q=p'+q'+k \tag{2}$$

the relation,

$$(p'q')=(pq)-(pk)-(qk) \tag{3}$$

can be derived. Using Eqs. (2) and (3), only 5 of the 10 possible invariant products are independent. The integration of (1) over d^3p' is easily performed by means of the δ function resulting in

$$d\sigma=\frac{\alpha r_0{}^2}{\pi^2}\frac{A}{[(pq)^2-1]^{1/2}}\frac{\delta(\epsilon_++\epsilon_--\epsilon'_+-\epsilon'_--k)}{\epsilon'_-\epsilon'_+k}\times d^3q'\,d^3k\ , \tag{4}$$

where ϵ'_- is now defined by

$$\epsilon'_-=[(\mathbf{p}+\mathbf{q}-\mathbf{q}'-\mathbf{k})^2+1]^{1/2}\ . \tag{5}$$

With the aid of the relations $d^3q'=q'^2dq'\,d\Omega_{q'}=\epsilon'_+q'\,d\epsilon'_+d\Omega_{q'}$, $d^3k=k^2dk\,d\Omega_k$, and

$$\int\delta(\epsilon_++\epsilon_--\epsilon'_+-\epsilon'_--k)d\epsilon'_+=\frac{\epsilon'_-q'^2}{|\,\epsilon'_-q'^2-\epsilon'_+(\mathbf{p}'\mathbf{q}')\,|}=\frac{\epsilon'_-q'^2}{|\,\epsilon'_+(p'q')-\epsilon'_-\,|} \tag{6}$$

the fully differential cross section becomes

$$\frac{d^3\sigma}{dk\,d\Omega_kd\Omega_{q'}}=\frac{\alpha r_0{}^2}{\pi^2}\frac{A}{[(pq)^2-1]^{1/2}}\frac{kq'^3}{|\,\epsilon'_+(p'q')-\epsilon'_-\,|}\ . \tag{7}$$

In order to express energy and momentum of the outgoing positron by the momenta $\mathbf{p}$, $\mathbf{q}$, and $\mathbf{k}$, Eq. (2) is multiplied by the four-vector q' yielding

$$q'\cdot(p+q-k)=(p'q')+1 \tag{8}$$

or

$$C\epsilon'_+-Bq'=D \tag{9}$$

with the notations

$$B=\hat{\mathbf{q}}'\cdot(\mathbf{p}+\mathbf{q}-\mathbf{k}),\quad C=\epsilon_-+\epsilon_+-k\ ,\qquad D=(p'q')+1\ . \tag{10}$$

$\hat{\mathbf{q}}'=\mathbf{q}'/q'$ is the unit vector in the direction of the outgoing positron. Using $\epsilon'^2_+-q'^2=1$, the solution of (9) is

$$\epsilon'_+=\frac{CD\pm BW}{C^2-B^2},\quad q'=\frac{BD\pm CW}{C^2-B^2} \tag{11}$$

with

$$W=(D^2+B^2-C^2)^{1/2}=\frac{1}{q'}\,|\,\epsilon'_+(p'q')-\epsilon'_-\,|\ . \tag{12}$$

The choice of the signs in (11) depends on the frame of reference considered. With the aid of (3) D can be expressed by the three invariant products (pq), (pk), and (qk). Using (12) the fully differential cross section is given by

$$\frac{d^3\sigma}{dk\,d\Omega_kd\Omega_{q'}}=\frac{\alpha r_0{}^2}{\pi^2}\frac{A}{[(pq)^2-1]^{1/2}}\frac{kq'^2}{W}\ . \tag{13}$$

It is easy to see that the function W takes a very simple form if one specializes to the center-of-mass system S' of the outgoing electron and positron where

$$\mathbf{p}'+\mathbf{q}'=\mathbf{p}+\mathbf{q}-\mathbf{k}=0\ ,\qquad \epsilon'_+=\epsilon'_-\equiv\epsilon'=\tfrac{1}{2}(\epsilon_++\epsilon_--k)=\frac{C}{2}\ ,\qquad (p'q')=2\epsilon'^2-1\ , \tag{14}$$

and

$$W=2\epsilon'p'=2\epsilon'q'\ . \tag{15}$$

In the system S' the cross section takes the form

$$\left[\frac{d^3\sigma}{dk\,d\Omega_kd\Omega_{q'}}\right]_{S'}=\frac{\alpha r_0{}^2}{2\pi^2}\frac{kq'}{\epsilon'}\frac{A}{[(pq)^2-1]^{1/2}}\ . \tag{16}$$

This equation is the most convenient starting point for the integration over the solid angle $\Omega_{q'}$ since only the function A depends on the angles of the final positron, and this expression can be integrated exactly. As the momenta $\mathbf{p}$, $\mathbf{q}$, and $\mathbf{k}$ form a triangle in the system S' [cf. Eq. (14)], the orientation of the vectors $\mathbf{p}'$ and $\mathbf{q}'=-\mathbf{p}'$ is quite arbitrary. So the full solid angle $\Omega_{q'}=4\pi$ is allowed kinematically, i.e., the limits of integration are independent of the momenta $\mathbf{p}$, $\mathbf{q}$, and $\mathbf{k}$.

The laborious but elementary integration yields a cross section which again can be expressed in covariant form, namely, by the three invariant products (pq), (pk), and (qk). Introducing the notations

$$(pq)=\tau,\quad (pk)=\kappa_1,\quad (qk)=\kappa_2\ , \tag{17}$$

and

$$\rho^2=(p'+q')^2=2[(p'q')+1]=2(\tau-\kappa_1-\kappa_2+1) \tag{18}$$

the e^-e^+ bremsstrahlung cross-section differential with respect to the photon energy and photon angles can be written as

$$\sigma(k,\theta)\equiv\frac{d^2\sigma}{dk\,d\Omega_k}=\frac{\alpha r_0{}^2}{2\pi}\frac{k}{\rho(\tau^2-1)^{1/2}}\frac{(\rho^2-4)^{1/2}}{\pi}\int A\,d\Omega_{q'}\ . \tag{19}$$

The expression for $(1/\pi)(\rho^2-4)^{1/2}\int A\,d\Omega_{q'}$ is given in the Appendix. It is not possible to further integrate $\sigma(k,\theta)$ over the photon angles analytically. However, the numerical computation of the cross section

$$\sigma(k)\equiv\frac{d\sigma}{dk}=\int\frac{d^2\sigma}{dk\,d\Omega_k}d\Omega_k \tag{20}$$

is quite easy in the c.m. and in the laboratory systems: Because of the rotational symmetry around the momentum vector $\mathbf{p}$ or $\mathbf{q}$, respectively, the cross section is not dependent on the azimuth angle; so the solid angle of the outgoing photon is simply given by $d\Omega_k = 2\pi \sin\theta \, d\theta$ and one has to perform only a single integration over the angle θ.

In the general case of arbitrary directions of the initial momenta $\mathbf{p}$ and $\mathbf{q}$, as, for instance, in a thermal astrophysical plasma, the computation of $\sigma(k)$ requires two numerical integrations.

III. COULOMB CORRECTION

The cross sections derived in the preceding section are exact within lowest-order perturbation theory. Whereas radiative corrections are assumed to be small in the energy region considered, the Coulomb correction may be significant at low energies. Correct results can be expected only if the conditions

$$a = \frac{\alpha}{\beta} \ll 1, \quad a' = \frac{\alpha}{\beta'} \ll 1 \tag{21}$$

are satisfied, where

$$\beta = \frac{(\tau^2-1)^{1/2}}{\tau}, \qquad \beta' = \frac{[(p'q')^2-1]^{1/2}}{(p'q')} = \frac{\rho(\rho^2-4)^{1/2}}{\rho^2-2}, \tag{22}$$

are the relative velocities (in units of the speed of light) of the electrons and positrons in the initial and final state, respectively. For the bremsstrahlung process in the field of a nucleus, the cross section in the Born approximation can be corrected by a simple factor derived by Elwert[19] for nonrelativistic energies. This factor which is given by the ratio of probabilities for finding the final and initial electron, respectively, at the position of the nucleus, has been shown to yield accurate results in the full energy range for nuclei with low atomic numbers Z.[20,21] In the case of electron-positron bremsstrahlung a corresponding correction factor can be obtained by calculating the ratio of probabilities of finding the two initial and the two final particles, respectively, at the same position. It has the form

$$F_{e^-e^+} = \frac{a'}{a} \frac{1-e^{-2\pi a}}{1-e^{-2\pi a'}} . \tag{23}$$

This factor is always larger than 1 as a consequence of the Coulomb attraction between electron and positron. That is, the true values of the cross sections are always higher than those given by the formulas of Sec. II. Due to the small factor $\alpha \simeq \frac{1}{137}$ in the quantities a and a', however, $F_{e^-e^+}$ is approximately equal to 1, especially at high energies ($\beta, \beta' \approx 1$). An important exception is the short-wavelength limit given by $\rho \to 2$. Here $a' \to \infty$ and $F_{e^-e^+} \to \infty$. As can be seen from Eqs. (19) and (A1), $\sigma(k,\theta)$ tends to zero for $\rho \to 2$. By applying the factor $F_{e^-e^+}$ the quantity $(\rho^2-4)^{1/2}$ cancels out resulting in a finite cross section at the short-wavelength limit. This fact is well known in the theory of electron-nucleus bremsstrahlung.[20]

Generally, the factor (23) is dependent on the momenta $\mathbf{p}$, $\mathbf{q}$, and $\mathbf{k}$, i.e., it is different for various photon angles. In the c.m. system, however, ρ^2 is independent of the photon angle so that $F_{e^-e^+}$ is only a function of the photon energy k for given momenta $\mathbf{p}$ and $\mathbf{q}$.

In the following sections the Coulomb factor is not taken into account because its effect is not significant at the energies considered.

IV. RESULTS

The formula (A1) can easily be programmed for the computation of the doubly differential cross section $\sigma(k,\theta)$. One should, however, pay attention to the fact that roundoff errors may occur in the calculation of cross sections at very high energies which may even lead to negative values of $\sigma(k,\theta)$. In these cases it is necessary to employ variables with double precision. The following results for $\sigma(k,\theta)$ and $\sigma(k)$ are given in the c.m. system and in the laboratory system where one of the initial particles is at rest.

A. Angular distributions in the c.m. system

The c.m. system is defined by

$$\mathbf{p}+\mathbf{q} = \mathbf{p}'+\mathbf{q}'+\mathbf{k} = 0, \quad \epsilon_+ = \epsilon_- \equiv \epsilon . \tag{24}$$

The invariants τ and ρ^2 have the form

$$\tau = 2\epsilon^2 - 1, \quad \rho^2 = 4\epsilon(\epsilon - k) , \tag{25}$$

i.e., they are independent of angles. Because of the symmetry of Eq. (A1) with respect to the products

$$\kappa_1 = (pk) = k(\epsilon - p\cos\theta), \qquad \kappa_2 = (qk) = k(\epsilon + p\cos\theta), \tag{26}$$

where θ is the photon angle relative to $\mathbf{p}$, the photon distributions in the c.m. system are symmetric about $\theta = \pi/2$, and all angles θ are allowed kinematically.

Figure 2 shows the cross section $\sigma(k,\theta)$ as a function of θ for the kinetic energy $E = (\epsilon - 1)mc^2 = 10$ keV of the initial particles and for various photon energies $h\nu = mc^2 k$. At these nonrelativistic energies the photon angular distributions form smooth curves. The minimum at $\theta = \pi/2$ for low photon energies changes into a maximum at higher values of $h\nu$. The corresponding cross sections for e^-e^- bremsstrahlung[3] are smaller by a factor of 20 to 30 resulting from their quadrupole nature. The maximum values of $\sigma(k,\theta)$ for e^-e^- bremsstrahlung are given in Fig. 2 by marks at the vertical axes.

In Fig. 3 are plotted the angular distributions for mildly relativistic particle energies, $E = 300$ keV, and various photon energies. Due to the relativistic beaming most of the photons are emitted near $\theta = 0$ and $\theta = \pi$. The angular distributions are similar to the corresponding curves for e^-e^- bremsstrahlung,[3] however, the latter cross sections are still smaller, in particular for low photon energies.

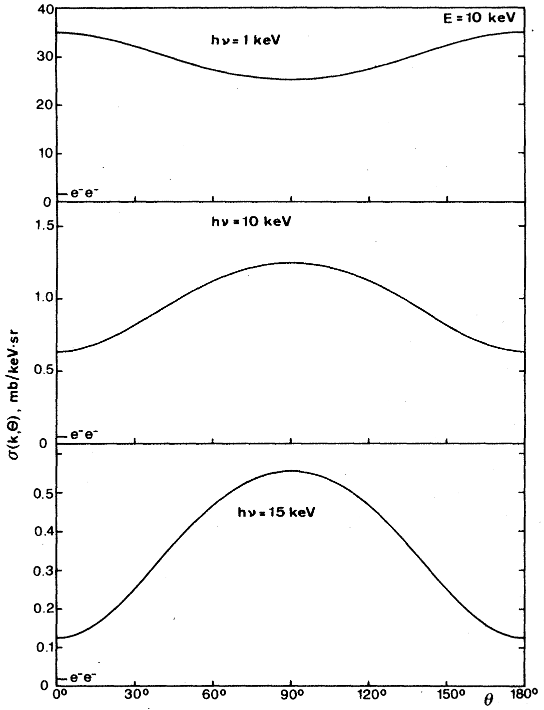

FIG. 2. Differential cross section $\sigma(k,\theta)$ of e^-e^+ bremsstrahlung in the c.m. system for $E=10$ keV and various photon energies $h\nu$. The maximum cross sections of e^-e^- bremsstrahlung are given by marks on the ordinates.

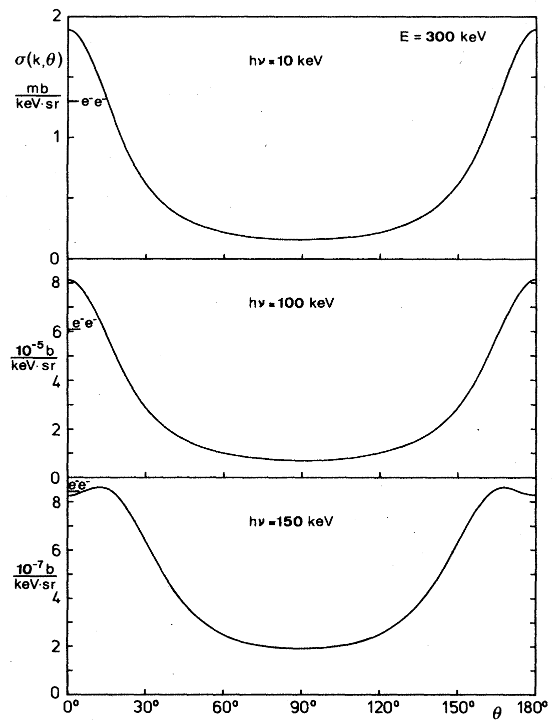

FIG. 3. Differential cross section $\sigma(k,\theta)$ of e^-e^+ bremsstrahlung in the c.m. system for $E=300$ keV and various photon energies $h\nu$. The maximum cross sections of e^-e^- bremsstrahlung are given by marks on the ordinates.

At relativistic energies ($\epsilon \gg 1$) the cross sections for e^-e^+ and e^-e^- bremsstrahlung are virtually equal. For $E=50$ MeV (cf. Fig. 3 of Ref. 3) the relative differences are of the order of 10^{-3} for the important angles around 0 and π. Here the relativistic beaming is most pronounced, the width at half maximum of the curves being $\theta_{1/2} \simeq 1/(2\epsilon)$. Around $\theta=\pi/2$ the cross section has decreased by many orders of magnitude.

An excellent approximation for the cross section $\sigma(k,\theta)$ in the c.m. system at ultrarelativistic energies which holds both for e^-e^+ and e^-e^- bremsstrahlung has been derived by Baier, Fadin, and Khoze[6] (BFK) who systematically expanded all quantities in powers of $1/\epsilon^2$ and retained the leading term of the expansion. The equality of the cross sections for the two processes follows from the fact that at these high energies the exchange-type Feynman graphs for e^-e^- collisions make the same contribution as the diagrams of direct type, whereas the contribution of the annihilation graphs in the case of e^-e^+ collisions can be neglected. The agreement between the BFK formula and the present results is better than 0.1% for $E=50$ MeV and the important photon angles. The only distinct differences occur at photon energies very close to the high-frequency limit $\rho^2=4$ or $k=p^2/\epsilon$ at large angles θ. Here the cross section for e^-e^+ bremsstrahlung is higher than that given by the BFK formula, in contrast to the e^-e^- case where it is lower.[3] The inaccuracy of the approximation is, however, not significant because the contribution to the total cross section $\sigma(k)$ from these large angles is fully negligible.

B. Angular distributions in the laboratory system

In the laboratory system one of the initial particles is at rest. Choosing $\mathbf{q}=0$, the quantities τ and ρ^2 have the form

$$\tau=\epsilon_-, \quad \rho^2=2[\epsilon_-+1-k(\epsilon_-+1-p\cos\theta)] \; . \tag{27}$$

Here θ is the photon angle relative to the direction of the incoming electron. The maximum photon energy occurs for $\rho^2=4$ and is given by

$$k_{\max}(\epsilon_-,\theta)=\frac{\epsilon_- -1}{\epsilon_- +1-p\cos\theta} \; . \tag{28}$$

The absolute maximum of k is reached in the forward direction $\theta=0$:

$$k_{\max}(\epsilon_-)=\frac{\epsilon_- -1}{\epsilon_- -p+1} \; . \tag{29}$$

Photons with energies $(\epsilon_- -1)/(\epsilon_- +p+1) < k \le k_{\max}(\epsilon_-)$ can be emitted only into a cone with half apex angle θ_0 given by

$$\cos\theta_0=\frac{(\epsilon_- +1)k-(\epsilon_- -1)}{pk} \; . \tag{30}$$

Figure 4 shows the cross sections $\sigma(k,\theta)$ for the kinetic energy $E_-=(\epsilon_- -1)mc^2=10$ keV of the incoming electron for various photon energies. For comparison the angular distributions of electron-proton (*ep*) bremsstrahlung are depicted. The *ep* cross sections are smaller at low

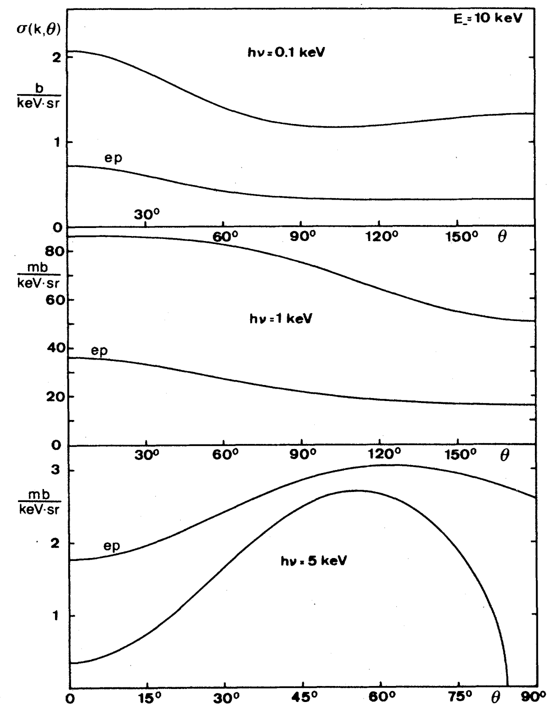

FIG. 4. Differential cross section $\sigma(k,\theta)$ of e^-e^+ bremsstrahlung in the laboratory system for $E_-=10$ keV and various photon energies $h\nu$, compared with the cross section of electron-proton bremsstrahlung (ep).

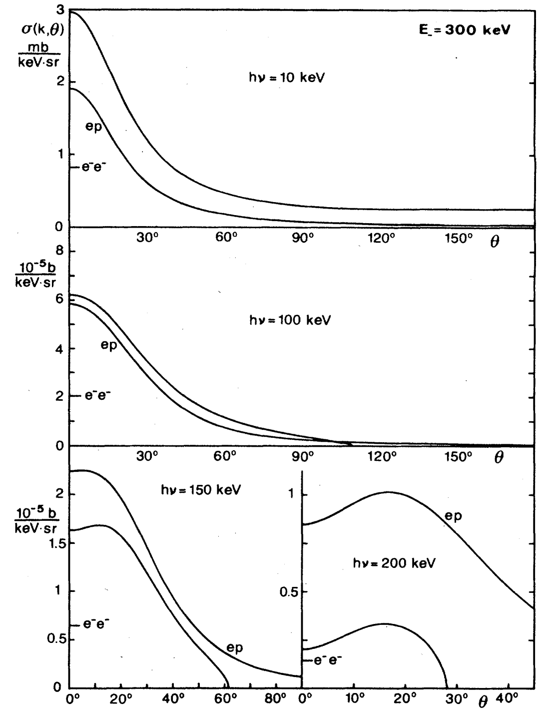

FIG. 5. Differential cross section $\sigma(k,\theta)$ of e^-e^+ bremsstrahlung in the laboratory system for $E_-=300$ keV and various photon energies $h\nu$, compared with the cross section of electron-proton bremsstrahlung (ep). The maximum cross sections of e^-e^- bremsstrahlung are given by marks on the ordinates.

photon energies, whereas they exceed the e^-e^+ cross sections at higher photon energies.

In Fig. 5 are plotted the angular distributions for $E_-=300$ keV, again in comparison with the cross sections for ep bremsstrahlung. With increasing photon energy the allowed angular region of e^-e^+ bremsstrahlung is more and more restricted due to the kinematical effect characterized by the limiting angle θ_0 [Eq. (30)]. The shapes of the e^-e^+ angular distributions are similar to those of e^-e^- bremsstrahlung (cf. Fig. 4 of Ref. 3), however, the values of $\sigma(k,\theta)$ are higher in the e^-e^+ case by a factor of about 3. At low and medium photon energies the e^-e^+ cross sections exceed the ep cross sections (except for the small contribution at angles $\theta \gtrsim \theta_0$), whereas at higher photon energies the ep bremsstrahlung dominates.

At extreme-relativistic energies the e^-e^+ bremsstrahlung is emitted essentially into a narrow cone in the forward direction. For $E_-=200$ MeV the e^-e^+ cross section is virtually equal to the e^-e^- cross section (cf. Fig. 5 of Ref. 3) at most photon energies and angles, the relative differences being generally less than 1%. Hence the formula of BFK[6] is a fairly good approximation for small angles θ and sufficiently far away from the short-wavelength limit, as in the e^-e^- case.[3] Since the momentum transfer to the recoiling particle is very low at the important angles $\theta \lesssim 1/\epsilon_-$, the cross sections for e^-e^+ and e^-e^- bremsstrahlung are approximately equal to the cross section for ep bremsstrahlung at very high energies.[22]

C. Photon spectra in the c.m. system

The photon spectra are calculated by numerical integration of the angular distributions $\sigma(k,\theta)$ in the c.m. system. Figure 6 shows the spectra for kinetic energies $E=10$, 100, and 300 keV. At low energies $E \ll mc^2$, i.e., $p=q \ll 1$, the short-wavelength limit of the bremsstrahlung spectrum is given by

$$k_{\max} \simeq p^2 \simeq 2(\epsilon-1) \quad \text{or} \quad h\nu_{\max} \simeq 2E \;.$$

In the nonrelativistic dipole approximation Garibyan[23] has derived a simple formula for the total cross section of e^-e^+ bremsstrahlung in the c.m. system:[22,24]

$$\sigma_{\rm NR}(k) = \frac{16}{3}\frac{\alpha r_0^{\,2}}{p^2 k}\ln\frac{p+(p^2-k)^{1/2}}{p-(p^2-k)^{1/2}} \;. \tag{31}$$

As can be seen from Fig. 6, this expression agrees quite well with the present results for kinetic energies $E \lesssim 10$ keV where the error is of the order of a few percent. In the region of intermediate energies E, however, the approximation (31) may only be used as a rough estimate of the true cross section.

For extreme-relativistic energies the expression[6]

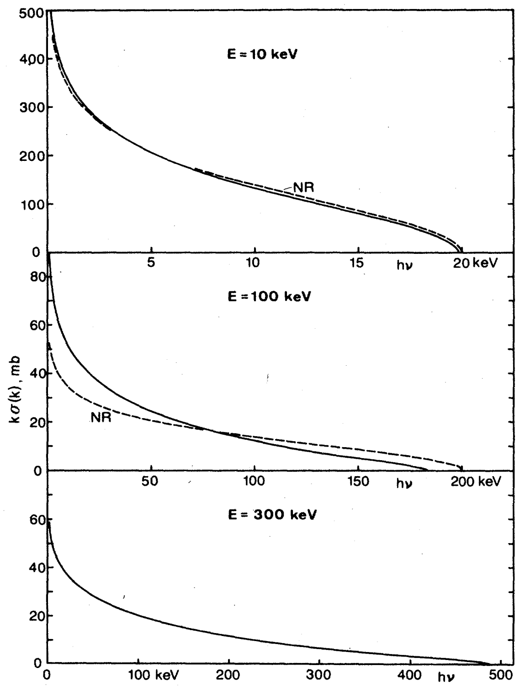

FIG. 6. Cross section $\sigma(k)$ of e^-e^+ bremsstrahlung in the c.m. system for various initial energies E, compared with the nonrelativistic approximation (NR).

$$\sigma_{\rm ER}(k)=\frac{8\alpha r_0{}^2}{\epsilon k}\left[\frac{4}{3}(\epsilon-k)+\frac{k^2}{\epsilon}\right]\left[\ln\left[4\epsilon^2\frac{\epsilon-k}{k}\right]-\frac{1}{2}\right] \tag{32}$$

is a good approximation for the cross sections of both e^-e^+ and e^-e^- bremsstrahlung as long as the condition $\epsilon-k\gg1$ is satisfied, i.e., the photon energy is not near its maximum $k_{\max}=p^2/\epsilon$. Baier, Fadin, and Khoze[7] have found, however, that the bremsstahlung cross sections for electron-positron and electron-electron collisions differ appreciably very close to the high-frequency end of the spectrum ($p^2-\epsilon k\lesssim1$). They gave a cross-section formula which is valid for all photon energies including the hard end of the spectrum. As the corresponding expression[7] for the differential cross section $\sigma(k,\theta)$ this is an excellent approximation down to energies E of a few MeV. It describes also the sharp drop of $\sigma(k)$ near the short-wavelength limit (see Fig. 7 of Ref. 3), in contrast to Eq. (32).

D. Photon spectra in the laboratory system

Figure 7 shows the photon spectra in the laboratory system ($q=0$) for kinetic electron energies $E_-=10$, 100, and 300 keV. For comparison the cross sections of ep bremsstrahlung are also plotted. The latter are considerably smaller at low photon energies but they exceed the e^-e^+ cross sections at higher photon energies, i.e., the ep spectrum is harder than the e^-e^+ spectrum. The intersection of the two curves is shifted to lower values of $k/k_{\max}$ for increasing energies E_-.

For nonrelativistic energies E_- a general formula can be derived from Eq. (31) taking into account that $\sigma_{\rm NR}(k)$ has to be dependent on $|\mathbf{p}-\mathbf{q}|$. The argument of the logarithm in (31) can be expressed in terms of $x=k/k_{\max}$. If the initial momenta $\mathbf{p}$ and $\mathbf{q}$ are arbitrary, the highest possible photon energy is

$$k_{\max}(\mathbf{p},\mathbf{q})=\frac{(pq)-1}{\epsilon_++\epsilon_--|\mathbf{p}+\mathbf{q}|}\ . \tag{33}$$

In the nonrelativistic limit this expression reduces to $k_{\max}(\mathbf{p},\mathbf{q})\simeq\frac{1}{4}(\mathbf{p}-\mathbf{q})^2$, i.e., it is a function of $|\mathbf{p}-\mathbf{q}|$. Therefore, in any frame of reference the nonrelativistic cross section for e^-e^+ bremsstrahlung can be written as

$$\sigma_{\rm NR}(k)=\frac{16}{3}\frac{\alpha r_0{}^2}{k^2}x\ln\frac{1+(1-x)^{1/2}}{1-(1-x)^{1/2}}\ , \tag{34}$$

where $x=k/k_{\max}(\mathbf{p},\mathbf{q})\simeq4k/(\mathbf{p}-\mathbf{q})^2$. In the c.m. system this is equivalent to Eq. (31), and in the laboratory system ($\mathbf{q}=0$),

$$\sigma_{\rm NR}(k)=\frac{64}{3}\frac{\alpha r_0{}^2}{p^2k}\ln\frac{p+(p^2-4k)^{1/2}}{p-(p^2-4k)^{1/2}}\ . \tag{35}$$

For kinetic energies $E_-\lesssim20$ keV Eq. (35) is a good approximation except for the neighborhood of the short-wavelength limit. The shape of the spectrum depends a little on the choice of x which can be taken as $x=4k/p^2$

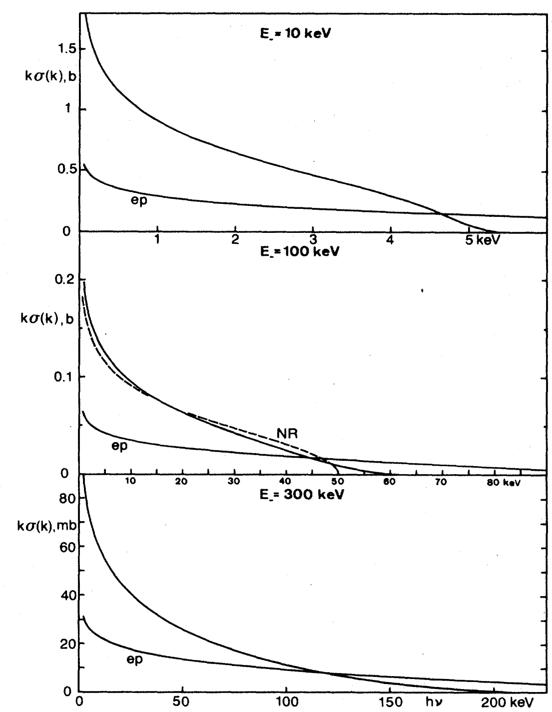

FIG. 7. Cross section $\sigma(k)$ of e^-e^+ bremsstrahlung in the laboratory system for various electron energies E_-, compared with the nonrelativistic approximation (NR) and with the cross section of electron-proton bremsstrahlung (ep).

or $x=2h\nu/E_-$. For intermediate energies E_- the formula (35) can still serve as a rough approximation as can be seen in Fig. 7 for $E_-=100$ keV.

For the energies of Fig. 7 the cross sections of e^-e^- bremsstrahlung[3] are considerably lower than the e^-e^+ cross sections, the ratios being about 150 for $E_-=10$ keV, 10 to 15 for $E_-=100$ keV, and 3 to 5 for $E_-=300$ keV.

In the extreme-relativistic energy region $\epsilon_- \gg 1$ the cross sections for e^-e^+ and e^-e^- bremsstrahlung coincide. The approximation formula of Baier, Fadin, and Khoze[6] for the laboratory system is not as accurate as the expression for the c.m. system since the reciprocal expansion parameter (pq) is equal to ϵ_- in the laboratory system $(\mathbf{q}=0)$ but it is $\simeq 2\epsilon^2$ in the c.m. system. As a consequence the agreement between the present results and the BFK formula is better than 1% only at energies E_- beyond 1000 MeV (see Fig. 10 of Ref. 3).

V. CONCLUSIONS

Angular distributions and photon spectra for e^-e^+ bremsstrahlung were computed in the c.m. system and the laboratory system in the whole range between nonrelativistic and extreme-relativistic energies. From the comparison with available approximation formulas and with the cross sections of other bremsstrahlung processes the following conclusions can be drawn.

(i) The cross sections of e^-e^+ bremsstrahlung are considerably higher than the e^-e^- cross sections up to energies of a few MeV. At extreme-relativistic energies the cross sections for the two processes coincide except for a very small region near the high-energy end of the spectrum.

(ii) In the laboratory system the cross sections of e^-e^+ and ep bremsstrahlung are of the same order of magnitude. The ep spectra are harder than the e^-e^+ spectra.

(iii) The analytical formulas for the cross sections in the nonrelativistic and extreme-relativistic limits were found to be good approximations, in particular in the c.m. system. At intermediate energies, between $\simeq 20$ keV and $\simeq 10$ MeV, there is no alternative to the formula derived in this paper.

(iv) In hot astrophysical plasmas with a positron component the e^-e^+ bremsstrahlung will give a significant contribution to the total x-ray emission.

APPENDIX

The integration of the fully differential cross section over the angles of the outgoing positron results in the formula:

$$\begin{aligned}
\frac{1}{\pi}(\rho^2-4)^{1/2}\int A\,d\Omega_{q'} = {} & (\rho^2-4)^{1/2}\left\{\frac{9\tau+4}{4\kappa_1\kappa_2}-\frac{3}{4\kappa_1^2}(\rho^2+3)-\frac{\kappa_2\rho^2}{4\kappa_1^3}+\frac{\rho^2}{\kappa_1(\tau^2-1)}\left[\frac{\kappa_2^2}{\kappa_1^3}-\frac{2\kappa_2\tau}{\kappa_1^2}+\frac{\kappa_2\tau}{2\kappa_1}-\frac{1}{2}\right]\right.\\
&+\frac{1}{2(\tau+1)}\left[\frac{\kappa_2}{\kappa_1}-\frac{1}{\tau+1}\right]-\frac{1}{\kappa_1+\kappa_2}+\frac{1}{(\kappa_1+\kappa_2)^2}\left[\frac{\kappa_1\kappa_2-2.5\rho^2}{\tau+1}-\frac{\rho^2}{2(\tau+1)^2}-4\rho^2\right]\\
&+\frac{\kappa_1\kappa_2\rho^2}{(\kappa_1+\kappa_2)^4}\left[4-\frac{1}{\tau+1}\right]+\frac{1}{\rho^2}\left[\frac{4\tau}{\kappa_1\kappa_2}-\frac{4}{\kappa_1^2}+\frac{4\kappa_2}{\kappa_1}+3\right]\\
&\left.+\frac{\rho^2-4}{12\kappa_1\rho^4}\left[(\rho^2-4)\left[\frac{\tau}{\kappa_2}-\frac{1}{\kappa_1}\right]-4\kappa_2\right]\right\}\\
&+X_1\left[\frac{2\tau+3}{\kappa_2\rho^2}-\frac{2}{\kappa_1\kappa_2\rho^2}-\frac{1}{\kappa_1\rho^2}+\frac{3\tau-1}{2\kappa_1\kappa_2}+\frac{\kappa_2}{2\kappa_1(\tau+1)}-\frac{1}{\kappa_1}\right]+X_2\left[\tau+1-\frac{1}{2(\tau+1)}\right]\\
&-\frac{X_3}{2\kappa_1}-\frac{\tau+1-\kappa_2}{2\kappa_1\kappa_2\rho^2}X_4-\tau^2X_5+\rho\frac{L_1}{W_1}\left\{\frac{1}{2(\tau+1)}\left[\frac{1}{\kappa_1}(3\kappa_2^2+2\kappa_2-4)+3\kappa_2+6\right]\right.\\
&+\frac{5\tau-11\kappa_1+1}{2\kappa_1}-\frac{13\tau+7}{2\kappa_2}+\frac{2\rho^2}{\kappa_1^2}-\frac{2\tau\rho^2}{\kappa_1\kappa_2}\\
&+\frac{1}{2\rho^2}\left[7\kappa_1+4\kappa_2-10\tau+2+\frac{1}{\kappa_1}(\kappa_2^2-\kappa_2\tau+\kappa_2+4)\right.\\
&\left.\left.-\frac{1}{\kappa_2}(11\kappa_1\tau+4\tau+9\kappa_1-8)-\frac{8}{\kappa_1^2}+\frac{8\tau}{\kappa_1\kappa_2}\right]\right\}
\end{aligned}$$

$$+4\rho\frac{L_2}{W_2}\left\{\kappa_1-\tau+2-\frac{2\tau}{\kappa_2}+\frac{1}{\kappa_1}[(\tau-\tfrac{5}{4}\kappa_2)\rho^2+4\tau^2-3\kappa_2\tau+2\kappa_2^2]+\frac{\kappa_2}{2(\tau+1)}(\tau-\kappa_1-2\kappa_2)\right.$$
$$-\frac{\kappa_2^3-3\kappa_2+\tau}{\kappa_1(\tau+1)}+\frac{1}{2(\kappa_1+\kappa_2)}(3\kappa_2\tau+\kappa_1\kappa_2-4\tau^2-4\tau-\kappa_2+2)-\frac{2\tau^3}{\kappa_1(\kappa_1+\kappa_2)}$$
$$\left.+\frac{\kappa_1-1}{(\kappa_1+\kappa_2)(\tau+1)}\right\}$$
$$+\rho L_3\left\{\frac{1}{\tau+1}\left[1-\frac{3\kappa_2}{\kappa_1}+\frac{1}{\kappa_1\kappa_2}-\frac{8}{\kappa_1+\kappa_2}+2\frac{2\tau-\kappa_1\kappa_2}{(\kappa_1+\kappa_2)^2}+\frac{4\kappa_1\kappa_2}{(\kappa_1+\kappa_2)^3}+\frac{4\kappa_1\kappa_2}{(\kappa_1+\kappa_2)^4}\right]\right.$$
$$+\frac{1}{(\tau+1)^2}\left[1-\frac{2}{\kappa_1+\kappa_2}-\frac{2}{(\kappa_1+\kappa_2)^2}\right]-\frac{6}{\kappa_1}-\frac{8}{\kappa_1^2}-\frac{4\rho^2}{\kappa_1\kappa_2}$$
$$\left.+\frac{5\rho^2}{(\kappa_1+\kappa_2)^2}-(\rho^2+6)\frac{2\kappa_1\kappa_2}{(\kappa_1+\kappa_2)^4}+\frac{2}{\kappa_1\rho^2}\left[\frac{1}{\kappa_2}-\kappa_2-2(\tau+1)\right]\right\}$$
$$+\frac{\rho L_4}{\kappa_1\kappa_2(\tau^2-1)^{1/2}}\left\{2\kappa_1^2+\tfrac{5}{2}\kappa_1\kappa_2-\tau(\rho^2-2)+\frac{\kappa_1}{2}\rho^2+2+2\frac{\kappa_2}{\kappa_1}(2\tau^2+\kappa_2^2-2\kappa_2\tau-\rho^2-1)\right.$$
$$+\frac{1}{\kappa_1+\kappa_2}(4\tau^3-3\kappa_1\kappa_2\tau+\kappa_1\kappa_2)$$
$$+\frac{1}{\tau+1}\left[\kappa_1^2-2\kappa_1-1+\frac{2\kappa_1\kappa_2}{\kappa_1+\kappa_2}+2\kappa_2\frac{\kappa_2+1}{\kappa_1}\right]$$
$$+\frac{1}{\rho^2}\left[3\tau(\kappa_1+\kappa_2)^2-2\kappa_1\kappa_2(\tau+1)+4\tau^2-2\tau-2-4\tau\frac{\kappa_2}{\kappa_1}\right]$$
$$\left.+\kappa_2\frac{\kappa_1+\kappa_2}{\tau^2-1}\left[\tau-\frac{\kappa_2}{\kappa_1}\right]\left[4\tau\frac{\tau-\kappa_2}{\kappa_1^2}-1\right]\right\}+\{\kappa_1\leftrightarrow\kappa_2\} \tag{A1}$$

with the notations

$$L_1=\ln\frac{\rho(\tau-\kappa_2-1)+(\rho^2-4)^{1/2}W_1}{2\kappa_1},\quad L_2=\ln\left[1+\frac{\kappa_2\rho^2(\rho^2-4)+\rho(\rho^2-4)^{1/2}W_2}{4\kappa_1(\kappa_1+\kappa_2)}\right],\quad L_3=\ln\frac{\rho+(\rho^2-4)^{1/2}}{2},$$
$$L_4=\ln\left[\frac{\tau^2-1+(\rho/2)[(\rho^2-4)(\tau^2-1)]^{1/2}}{\kappa_1+\kappa_2}-\tau\right],\quad W_1=[(\tau-\kappa_2)^2+2\kappa_1-1]^{1/2},\quad W_2=\{\kappa_2[\kappa_2\rho^2(\rho^2-4)+8\kappa_1(\kappa_1+\kappa_2)]\}^{1/2},$$
$$X_1=\frac{R}{W_1^2}\left[(\rho^2-4)^{1/2}+(\kappa_2-\tau+1)\frac{\rho L_1}{W_1}\right],\quad X_2=\frac{2\rho}{\kappa_1^2(\tau^2-1)}\left[\rho(\rho^2-4)^{1/2}\frac{\kappa_1\tau-\kappa_2}{\kappa_1+\kappa_2}+(\tau^2-\kappa_2\tau+\kappa_1-1)\frac{2L_4}{(\tau^2-1)^{1/2}}\right], \tag{A2}$$
$$X_3=\frac{\rho R}{W_1^2}\left[2\frac{L_1}{W_1}+\frac{\kappa_2-\tau+1}{2\kappa_1^2}\rho(\rho^2-4)^{1/2}\right],$$
$$X_4=\frac{1}{W_1^4}\left[(\kappa_2-\tau+1)(\rho^2-4)^{1/2}\left[R^2-\frac{\rho^2}{2}S\right]+[(\kappa_2-\tau+1)^2R^2-2\kappa_1^2S]\frac{\rho L_1}{W_1}\right],$$
$$X_5=\frac{\rho}{\kappa_1^2(\tau^2-1)}\left\{\rho(\rho^2-4)^{1/2}\left[1-\frac{3S}{\kappa_1^2(\tau^2-1)}\right]+\left[2(\tau+1)-\frac{\kappa_2}{\kappa_1}\rho^2-\frac{3S}{\kappa_1^2(\tau^2-1)}\left[\kappa_1+\kappa_2-\frac{\tau-1}{2}\rho^2\right]\right]\frac{2L_4}{(\tau^2-1)^{1/2}}\right\},$$
$$R=(\tau+1)(\kappa_2-\kappa_1)-\kappa_2(\kappa_1+\kappa_2),\quad S=2\kappa_1\kappa_2\tau-\kappa_1^2-\kappa_2^2\,.$$

The complicated function A in the integrand was derived from the corresponding expression for e^-e^- bremsstrahlung[3] by means of the substitution rule.[4] The correctness of A was verified by a comparison of e^-e^- bremsstrahlung cross sections with an independent calculation of Mack and Mitter.[25] The two results agreed excellently. The correctness of the formula (A1) was checked as follows.

(1) All terms of (A1) including those which are obtained by interchanging κ_1 and κ_2 have been calculated separately. Thus all the errors violating symmetry could be discovered easily.

(2) The cross sections computed with the aid of (A1) in the c.m. and laboratory systems were compared with the results from numerical integration of the triply differential cross section over $d\Omega_{q'}$ yielding full agreement within the accuracy of the numerical integration in all energy regions.

An additional test for the correctness of (A1) provides the fact that the resulting cross sections in the c.m. and laboratory systems agree with the available approximations both in the nonrelativistic and in the extreme-relativistic limits.

---

[1] G. Altarelli and F. Buccella, Nuovo Cimento **34**, 1337 (1964).

[2] I. Hodes, Ph.D. thesis, University of Chicago, 1953.

[3] E. Haug, Z. Naturforsch. **30a**, 1099 (1975).

[4] J. M. Jauch and F. Rohrlich, *The Theory of Photons and Electrons,* 2nd edition (Springer, New York, 1976).

[5] S. M. Swanson, Phys. Rev. **154**, 1601 (1967).

[6] V. N. Baier, V. S. Fadin, and V. A. Khoze, Zh. Eksp. Teor. Fiz. **51**, 1135 (1966) [Sov. Phys.—JETP **24**, 760 (1967)].

[7] V. N. Baier, V. S. Fadin, and V. A. Khoze, Zh. Eksp. Teor. Fiz. **53**, 2194 (1967) [Sov. Phys.—JETP **26**, 1238 (1968)].

[8] V. N. Baier, V. S. Fadin, V. A. Khoze, and E. A. Kuraev, Phys. Rep. **78**, 293 (1981).

[9] F. A. Berends, R. Kleiss, P. de Causmaecker, R. Gastmans, and Tai Tsun Wu, Nucl. Phys. **B206**, 61 (1982).

[10] F. A. Berends and R. Kleiss, Nucl. Phys. **B228**, 537 (1983).

[11] G. Cavallo and M. J. Rees, Mon. Not. R. Astron. Soc. **183**, 359 (1978).

[12] A. P. Lightman, R. Giacconi, and H. Tananbaum, Astrophys. J. **224**, 375 (1978).

[13] B. McBreen, Astron. Astrophys. **71**, L19 (1979).

[14] M. J. Rees, in *Plasma Astrophysics,* edited by T. D. Guyenne and G. Lévy (European Space Agency, Paris, 1981), p. 297.

[15] R. Svensson, Astrophys. J. **258**, 335 (1982).

[16] S. Stepney and P. W. Guilbert, Mon. Not. R. Astron. Soc. **204**, 1269 (1983).

[17] Relativistic units mc^2 and mc for energy and momentum are used. The metric is such that the product of two four-vectors $a=(a_0,\mathbf{a})$ and $b=(b_0,\mathbf{b})$ is defined by $(ab)=a_0b_0-\mathbf{a}\cdot\mathbf{b}$.

[18] T. Anders, Ph.D. thesis, Universität Freiburg, Germany, 1961; Nucl. Phys. **59**, 127 (1964).

[19] G. Elwert, Ann. Phys. (Leipzig) **34**, 178 (1939).

[20] G. Elwert and E. Haug, Phys. Rev. **183**, 90 (1969).

[21] R. H. Pratt and H. K. Tseng, Phys. Rev. A **11**, 1797 (1975).

[22] J. Joseph and F. Rohrlich, Rev. Mod. Phys. **30**, 354 (1958).

[23] G. M. Garibyan, Zh. Eksp. Teor. Fiz. **24**, 617 (1953).

[24] A. I. Akhiezer and V. B. Berestetskii, *Quantum Electrodynamics* (Interscience, New York, 1965).

[25] D. Mack and H. Mitter, Phys. Lett. **44A**, 71 (1973).