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Fractional charge and spectral asymmetry in one dimension: A closer look

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The physics of charge fractionalization is studied using a simple and physical approach. The normal-ordered charge is related to the Atiyah-Patodi-Singer invariant, and the physical interpretation of the spectral asymmetry is clarified in the presence of a continuous spectrum. By introducing the quantity $B(E)$ which is a ratio of Jost-type determinants we relate the asymmetry to the phase and zeros or poles of $B(E)$. The fractional part of the charge is determined by the high-energy behavior of the phase and the integer part is related to the spectral flow. We give simple examples showing that only the fractional part of the charge is a topological invariant; the integer part is determined by local properties of the background fields.

I. INTRODUCTION AND PHYSICAL MOTIVATION

Since the original paper by Jackiw and Rebbi[1] where it had been noted that fermions interacting with solitons give rise to fractional-charge states, this interesting effect has attracted attention from several different disciplines.

In condensed-matter physics it has been realized that certain quasi-one-dimensional materials have a broken-(discrete) symmetry ground state and consequently solitonic excitations.[2] Orbital electrons are coupled to the solitons giving rise to fractional-charge states in much the same way as in the Jackiw and Rebbi example.[3]

In particle physics it has been recognized that fractional quantum numbers arise in several situations in which fermions are coupled to external background fields with nontrivial behavior at spatial infinity (solitons or kinks, monopoles).

It has been suggested that the physics of fractionalization can be thought of as a "vacuum polarization" effect. Indeed the background fields distort the Dirac sea in such a way that the ground state ("vacuum") in the presence of this external field has very unusual features. An intuitive (and rough) argument for this picture is the following: suppose a soliton-antisoliton ($S\bar{S}$) is created; this configuration has trivial behavior at infinity. However, as the pair is separated, the electronic states are rearranged, the local density of states is modified, and states "pile up" or "thin out" (controlled by the phase shift) near the region where the fields are rapidly changing. As the $S\bar{S}$ separation becomes infinite and we only "see" one soliton for example, we find that the charge has been accumulated near its center (this also happens near $\bar{S}$). Of course the total charge of the $S\bar{S}$ system is an integer. It has been proven that in the limit when the $S\bar{S}$ distance is very large, the charge measured near each one of them is an observable.[4–6]

When the Hamiltonian for the fermions interacting with an external field has a charge-conjugation symmetry a simple counting-of-states argument yields the fractional-charge result.[3] Goldstone and Wilczek[7] have introduced a method that allows one to compute the induced vacuum charge for general interactions; this method involves an adiabatic approximation (slowly varying fields). The results found with this method have been reproduced using very different approaches, among them exact solution of the scattering problem for certain solitons profile,[8,9] anomalous commutators techniques,[10,11] "twisted" boundary conditions,[12] etc.

From a more mathematical point of view, the fractional charge has been related to index theorems and concepts in topology.[13–15] Topological methods have been used to compute the induced vacuum charge in different dimensionalities for different topological background fields.[16,17]

This paper is a modest attempt to try to understand the underlying physics of fractional charge in one space dimension with simple techniques and to try to offer a unifying yet simple view of the phenomena involved. We will use a simple counting argument. The main observation is that static background fields produce a distortion in the density of states in the positive- and negative-energy continuum (conduction and valence band) and may also induce the formation of bound states.[18] In Sec. II we show by keeping account of the states that the ground-state charge (obtained by filling all the negative-energy states) is related to the asymmetry in the spectrum (spectral asymmetry) and a quantity called η or Atiyah-Patodi-Singer (APS) invariant.[14,19]

As a fundamental measure of the asymmetry of the spectrum of H we introduce the quantity

$$B(E)=\det\left[\frac{H+E}{H-E}\right] \tag{1.1}$$

with $B(0)=1$. That this is a simple but interesting measure of the spectral asymmetry (and hence of η) can be seen as follows: suppose the spectrum of H is discrete and define the ordered positive and negative eigenvalues to be λ_k^+ and $-\lambda_l^-$, respectively. Then

$$B(E)=\prod_{k,l=1}^{K,L} B_{k,l}(E)\,,\qquad B_{k,l}(E)=\left[\frac{\lambda_k^+ +E}{\lambda_l^- +E}\right]\left[\frac{\lambda_l^- -E}{\lambda_k^+ -E}\right]. \tag{1.2}$$

Clearly if the spectrum is symmetric then $B(E)=1$. If the eigenvalues are not symmetric but there are as many positive eigenvalues as negative, then $L=K$ and

$$\frac{1}{2\pi i}\oint dE\frac{d}{dE}\ln B(E)=L-K \tag{1.3}$$

vanishes, where the integral is around a closed contour[20] enclosing only the positive real axis. In the next section we will reconsider the above properties when H possesses a continuous spectrum.

It will be proved that $B(E)$ is a ratio of well-defined Jost functions,[21] which are, in turn, simply related to the transmission coefficients of an associated scattering process. From $B(E)$, the odd part of the density of states can be computed, leading to a simple evaluation of η.

In Sec. III we evaluate $B(E)$ in some special cases in which the existence of an operator that maps positive-energy states onto negative-energy ones ensures the topological invariance of $B(E)$.

In Sec. IV we compute $B(E)$ in two examples where the aforementioned operator does not exist. In this section we offer examples of the concept of spectral flow energy levels crossing zero) and how it is related to the integral part of the charge. We learn that the fractional part is related to the high-energy behavior of phase shifts. We also argue that in general $B(E)$ is *not a topological invariant,* and that only η is invariant. We are surprised that seemingly general discussions of this problem using topological methods have missed important and physical features exposed in our examples.

Finally in Sec. V we analyze the general case in view of the features learned from the examples of Secs. III and IV, and summarize our conclusions.

II. GROUND-STATE CHARGE, SPECTRAL ASYMMETRY, AND JOST FUNCTIONS

As promised in the Introduction, in this section we relate the *ground-state charge* to the spectral asymmetry of Atiyah, Patodi, and Singer (APS) the η invariant of the Dirac Hamiltonian.[19]

The basic observation is that the topological background fields distort the local density of states, however the total number of states remains constant. The ground-state (vacuum) charge is defined as

$$Q=\int_{-\infty}^{0}[\rho^S(E)-\rho^0(E)]dE=\int_{-\infty}^{0}\Delta\rho(E)dE\ , \tag{2.1}$$

where $\rho^S(E)$ ($\rho^0(E)$) is the density of states in the presence (absence) of background fields (soliton). This definition of the charge is properly normal ordered. We shall assume there are no $E=0$ states (we can always add a parameter to the Hamiltonian to achieve this situation and study the limiting behavior as this parameter goes to zero). Suppose that there are N^- (N^+) bound states of negative (positive) energy and that the continuum starts at the threshold energy E_T. The ground-state charge obtained by filling all the negative-energy states is

$$Q=N^-+\int_{-\infty}^{-E_T}\Delta\rho(E)dE\ . \tag{2.2}$$

The background fields modify the density of states in the positive and negative continuum. If the Hamiltonian has a charge-conjugation symmetry the density of continuum states is equal for positive and negative energy. However, in the most general case when there is no charge-conjugation symmetry, the density of states for positive and negative energy are no longer equal. There is an asymmetry in the spectrum and we write[18]

$$\int_{-\infty}^{-E_T}\Delta\rho(E)dE=-\frac{N_B}{2}+\Delta\ , \tag{2.3a}$$

$$N_B=N^++N^-\ ,$$

$$\int_{E_T}^{\infty}\Delta\rho(E)dE=-\frac{N_B}{2}-\Delta\ , \tag{2.3b}$$

where Δ is a function of the charge-conjugation symmetry-breaking fields in the Hamiltonian. Clearly the sum of (2.3a) and (2.3b) is $-N_B$ by conservation of the number of states. Combining (2.2) with (2.3a) and (2.3b) we obtain

$$Q=\tfrac{1}{2}\left[\int_{-\infty}^{0}\Delta\rho(E)dE-\int_{0}^{\infty}\Delta\rho(E)dE\right]. \tag{2.4}$$

In the free case in which (the external fields are constant) $\rho^0(E)=\rho^0(-E)$, one finds

$$Q=-\tfrac{1}{2}\int_0^\infty[\rho(E)-\rho(-E)]dE=-\int_0^\infty\rho_{\rm odd}(E)dE\ , \tag{2.5}$$

where $\rho_{\rm odd}(E)$ is the odd part of the density of states.

We recognize that the spectral asymmetry (APS invariant)[14,19] is given by

$$\eta=\int_0^\infty[\rho(E)-\rho(-E)]=\sum_{E_n\neq 0}\mathrm{sign}(E_n) \tag{2.6}$$

and therefore

$$Q=-\tfrac{1}{2}\eta\ . \tag{2.7}$$

Formally Eq. (2.6) should be defined as a properly regularized quantity in the limit of taking the regulator to zero (see Ref. 19 for details). However, if the spatial variation of the background fields only introduce a compact perturbation, we expect the density of states to be changed by a finite amount; Q in Eq. (2.1) is a measure of this small change. Therefore we do not introduce any regulator (the uneasy reader will be comforted by looking at the well-defined result). We will study the following Hamiltonian for fermions interacting with external static background fields in *one* spatial dimension [for a more general case involving metric modifications, see Lott (Ref. 14)]:

$$H[K,\phi]=-i\sigma_2\frac{d}{dx}+\sigma_1\phi(x)+\sigma_3K(x)\ , \tag{2.8}$$

where the σ's are the usual Pauli matrices. Since the semiclassical approximation amounts to solving this Hamiltonian and filling up all the negative-energy states to define the vacuum, we would like to understand the properties of the ground-state charge and its relation to the topology of the soliton fields by studying general properties of the spectrum. This is achieved by introducing the resolvent of the Hamiltonian

$$G(E)=\mathrm{Tr}\frac{1}{H-E}\ , \tag{2.9}$$

where the trace is over spin and spatial indices. The density of states is related to $G(E)$ by

$$\rho(E)=\frac{1}{2\pi i}[G(E+i\eta)-G(E-i\eta)]\ . \tag{2.10}$$

Writing $G(E)$ in terms of its even G_e and odd G_0 parts, we find

$$\rho_{\mathrm{odd}}(E)=\frac{1}{\pi}\mathrm{Im}G_e(E)\ . \tag{2.11}$$

Finally we write the even part of the resolvent in terms of the B function introduced in the previous section:

$$\begin{aligned} G_e(E)&=\tfrac{1}{2}\mathrm{Tr}\left[\frac{1}{H+E}+\frac{1}{H-E}\right]=\frac{1}{2}\frac{d}{dE}\ln B(E)\ ,\\ B(E)&=\det\left[\frac{H+E}{H-E}\right]\ . \end{aligned} \tag{2.12}$$

As noted before if the spectrum of H is symmetric then $B(E)=1$.

The expression for $B(E)$ is reminiscent of that of Jost functions[21] in scattering theory, however the numerator and denominator have the same operator H but different signs of E, hence they do not satisfy the requirements that guarantee the existence of $B(E)$. To ensure the existence of the Jost functions we need to introduce a suitable comparison Hamiltonian H_0 such that H and H_0 only differ locally. For simplicity we also impose the condition that the spectrum of H_0 be symmetric. To fulfill these two conditions we notice that if $H\psi=E\psi$, we can introduce the quantities

$$\begin{aligned} &H=e^{i\sigma_2\theta/2}H_\chi e^{-i\sigma_2\theta/2}\ ,\quad H_\chi=H_0+\tfrac{1}{2}\theta'(x)\ ,\\ &\psi=e^{i\sigma_2\theta/2}\chi\ ,\\ &H_0=-i\sigma_2\frac{d}{dx}+\sigma_1\rho(x)\ ,\quad H_\chi\chi=E\chi\ ,\\ &\phi(x)=\rho(x)\cos\theta(x)\ ,\quad K(x)=\rho(x)\sin\theta(x)\ . \end{aligned} \tag{2.13}$$

This chiral transformation does not modify $B(E)$; it amounts to a change of basis states. Any possible change in the numerator is compensated by the same change in the denominator.

If we assume that θ' vanishes fast enough as $x\to\pm\infty$, H_χ and H_0 only differ locally. Furthermore since $\{H_0,\sigma_3\}=0$ the spectrum of H_0 is symmetric with respect to $E=0$ and since $\rho(x)$ is a positive semidefinite function, there are no $E=0$ states in H_0. Hence $\det[(H_0+E)/(H_0-E)]=1$. Therefore we can choose H_0 as the comparison Hamiltonian and write

$$B(E)=\det\left[\frac{H_\chi+E}{H_0+E}\right]\Big/\det\left[\frac{H_\chi-E}{H_0-E}\right]=\frac{J(-E)}{J(E)}\ . \tag{2.14}$$

With this choice of H_0 and boundary conditions on θ', each of the determinants in $B(E)$ is guaranteed to exist and is Fredholm. To relate the Fredholm determinants to the Jost functions and scattering matrix elements we proceed as follows.

Consider two independent scattering solutions to H_0, namely, f_0, f_1, with the asymptotic behavior

$$\begin{aligned} &f_0\underset{x\to+\infty}{\sim}T_0e^{ik_+x}\chi_+(k)\ ,\\ &f_0(x)\underset{x\to-\infty}{\sim}e^{ik_-x}\chi_-(k)+R_0\chi_-(-k)e^{-ik_-x}\ ,\\ &f_1(x)\underset{x\to-\infty}{\sim}T_1e^{-ik_+x}\chi_-(-k)\ ,\\ &f_1(x)\underset{x\to+\infty}{\sim}e^{-ik_+x}\chi_+(-k)+R_1\chi_+(k)e^{ik_+x} \end{aligned} \tag{2.15}$$

with

$$\chi_\pm(k)=\frac{1}{\sqrt{2}}\begin{bmatrix}1\\ \dfrac{\rho_\pm+ik_\pm}{E}\end{bmatrix}$$

being the asymptotic states of H_0 with energy

$$E=(k_+{}^2+\rho_+{}^2)^{1/2}=(k_-{}^2+\rho_-{}^2)^{1/2}\ . \tag{2.16}$$

Consider the Jost solution for H_χ (Ref. 22),

$$f(x)=f_0(x)+\frac{1}{W}\int_x^\infty H(x,x')\tfrac{1}{2}\theta'(x')f(x')dx' \tag{2.17}$$

with $H(x,x')$ being the matrix (Green's function)

$$H_{\alpha\beta}(x,x')=[f_{0\alpha}(x)f_{1\beta}^T(x')-f_{1\alpha}(x)f_{0\beta}^T(x')] \tag{2.18}$$

and W the Wronskian

$$W=\det\{f_{0\alpha}(x),f_{1\beta}(x)\}\ . \tag{2.19}$$

Then

$$f(x)\underset{x\to\infty}{\sim}f_0\approx T_0e^{ik_+x}\chi_+(k) \tag{2.20}$$

and

$$\begin{aligned} f(x)\underset{x\to-\infty}{\sim}&f_0(x)\left[1+\frac{1}{W}\left\langle f_1\frac{\theta'}{2}f\right\rangle\right]-\frac{1}{W}f_1(x)\left\langle f_0\frac{\theta'}{2}f\right\rangle\\ \simeq&e^{ik_-x}\chi_-(k)\left[1+\frac{1}{W}\left\langle f_1\frac{\theta'}{2}f\right\rangle\right]+\chi_-(k)e^{-ik_-x}\\ &\times\left[R_0\left[1+\frac{1}{W}\left\langle f_1\frac{\theta'}{2}f\right\rangle\right]-\frac{T_0}{W}\left\langle f_1\frac{\theta'}{2}f\right\rangle\right]\ , \end{aligned} \tag{2.21}$$

where

$$\left\langle f\frac{\theta'}{2}g\right\rangle=\int_{-\infty}^{\infty}f^T(x')\tfrac{1}{2}\theta'(x')g(x')dx'\ . \tag{2.22}$$

Since the normalized solution has the asymptotic conditions

$$f_N(x)=\begin{cases}T\,e^{ik_+x}\chi_+(x), & x\to+\infty\\ e^{ik_-x}\chi_-(k)+R\,e^{-ik_-x}\chi_-(-k), & x\to-\infty\end{cases}$$

we find the Jost function to be given by

$$J(E)=1+\frac{1}{W}\left\langle f_1\frac{\theta'}{2}f\right\rangle=\frac{T_0E}{T(E)} . \quad (2.23)$$

The proof that $J(E)$ is the Fredholm determinant is now just a slight modification of the standard arguments. Multiply the potential by a constant γ, expand the integral equation (2.17) and the determinant in powers of γ, and compare the expressions. An alternative derivation is found in the literature.[21] Therefore we conclude that

$$\det\left[\frac{H_\chi\pm E}{H_0\pm E}\right]=\frac{T_0(\mp E)}{T(\mp E)} , \quad (2.24)$$

where T (T_0) is the transmission coefficient of the scattering states of H_χ (H_0).

Because the spectrum of H_0 is symmetric, $T_0(E)=T_0(-E)$, and therefore

$$B(E)=\frac{T(E)}{T(-E)} . \quad (2.25)$$

The transmission coefficients have poles at the bound-state energies and are complex above thresholds, their phase being the phase shifts of the scattering states. For E above thresholds ($E>E_T$),

$$\frac{T(E)}{T(-E)}=\left|\frac{T(E)}{T(-E)}\right|e^{i\delta(E)} , \quad (2.26)$$

then

$$\rho_{\text{odd}}(E>E_T)=\frac{1}{2}\frac{1}{\pi}\frac{d}{dE}\delta(E) . \quad (2.27)$$

Therefore,

$$\eta=2\int_0^\infty \rho_{\text{odd}}(E)dE$$
$$=N^+-N^-+\frac{1}{\pi}[\delta(\infty)-\delta(E=E_T)] . \quad (2.28)$$

N^+ (N^-) is the number of positive (negative) bound states. If the ratio $T(E)/T(-E)$ only depends upon the topological properties of the background fields so does $\rho_{\text{odd}}(E)$, however, we will see in the next sections that this is true in *very special* cases; in general $\rho_{\text{odd}}(E)$ will depend upon local details of the external fields.

This remark generalizes statements in Ref. 15 where it is claimed that $\rho_{\text{odd}}(E)$ does not depend upon local details of the soliton fields for those special cases. The argument given there was that $\rho_{\text{odd}}(E)$ can be obtained as an inverse Mellin transform of the regulated APS[19] invariant $\eta(S)$. However this transform clearly involves the eigenvalues E which in a general case do depend upon local details. This will be demonstrated in Sec. IV in some examples where $\rho_{\text{odd}}(E)$ is computed exactly. In the next section we solve for $\rho_{\text{odd}}(E)$ and η in two simple models for which $\rho_{\text{odd}}(E)$ is invariant.

However η is *not* sensitive to the numerical values of the energies but only involves the number and sign of these eigenvalues. Therefore η will be invariant under local variations of the background fields that *do not* change the signs of the eigenvalues but just move them around slightly. Indeed when an eigenvalue changes sign, η jumps by ± 2. This is associated with the "spectral flow" of the Dirac Hamiltonian.[19] When this happens the ground-state charge changes by one as an energy level crosses zero, essentially if an $E>0$ state crosses zero and becomes an $E<0$ state our definition of the charge immediately fills up this state.

Whether or not this state is filled as it crosses zero is a dynamical question, if the process proceeds adiabatically this state will remain empty and the ground-state charge will differ from the adiabatic charge by one.

III. SOME SPECIAL CASES

In Sec. II we have proven that the ground-state charge and η can be computed from the ratio of transmission coefficients for positive and negative scattering states.

The results of Refs. 14 and 15 suggest that this ratio is only a function of the asymptotic values of the background fields. This, in turn, suggests that the positive- and negative-energy continuum states are related.

Indeed if there is a local operator U that anticommutes with H at every point x, then

$$U(x)\psi_E\propto\psi_{-E} \quad (3.1)$$

or

$$\tilde U\chi_E\propto\chi_{-E} \quad (3.2)$$

with

$$\tilde U(x)=e^{-i\sigma_2\theta(x)/2}U(x)e^{i\sigma_2\theta(x)/2} . \quad (3.3)$$

The existence of the operator U ($\tilde U$) automatically guarantees that the ratio $T(E)/T(-E)$ is a topological invariant. The reasoning behind this statement is as follows: the scattering solutions with energy E of H_χ have the asymptotic behavior

$$\chi^E(x)\underset{x\to+\infty}{\sim} e^{ik_-x}\chi_-^E(k)+R(E)\chi_-^E(-k)e^{-ik_-x} ,$$
$$\chi^E(x)\underset{x\to+\infty}{\sim} T(E)e^{ik_+x}\chi_+^E(k) . \quad (3.4)$$

Now we apply the operator $\tilde U$ to the above conditions and recognize that as $x\to\pm\infty$, $\tilde U(x)\chi^E(x)\to\mathscr{F}_\pm\chi_\mp^{-E}$ since $\chi_\pm$ are the asymptotic solutions of H_χ. We find

$$\tilde U\chi(x)\underset{x\to-\infty}{\sim}\mathscr{F}_-(k)\chi_-^{-E}(k)e^{ik_-x}$$
$$+R(E)\mathscr{F}_-(-k)\chi_-^{-E}(-k)e^{-ik_-x} ,$$
$$\tilde U\chi(x)\underset{x\to+\infty}{\sim}\mathscr{F}_+(k)\chi_+^{-E}(k)e^{ik_+x}T(E) . \quad (3.5)$$

Therefore $R(-E)$ and $T(-E)$ can be read off:

$$R(-E)=R(E)\mathscr{F}_-(-k)/\mathscr{F}_-(k) ,$$
$$T(-E)=T(E)\mathscr{F}_+(k)/\mathscr{F}_-(k) , \quad (3.6)$$

where $\mathscr{F}_\pm$ are only functions of $k_\pm$ and the asymptotic values of the background fields ϕ and K. The existence of this operator has far-reaching consequences. If by a modification of the parameters in the Hamiltonian, energy levels cross zero, positive- and negative-energy eigenvalues have to do so in pairs and in opposite directions, the net change or spectral flow is therefore zero. This is the reason behind the topological invariance of $B(E)$ (see example 2 for a counterexample to this statement).

We were able to construct the operator $\tilde{U}$ explicitly in only two cases, when either $K(x)$ or $\phi(x)$ is a constant. Indeed, if an operator commutes with H^2 then its commutator with H anticommutes with H. It can then be seen that in the case mentioned above there is a simple operator that commutes with H^2.

Case a: $K=$constant. H^2 commutes with σ_3.[14,15] And

$$\{H,[H,\sigma_3]\}=0 \text{ and } U=(\sigma_3 H-K)\,, \qquad \tilde{U}(x)=(\sigma_3 e^{i\sigma_2\theta(x)}E-K)\,. \tag{3.7}$$

If we apply $\tilde{U}$ to the free spinors $\chi_{\pm}^{E}(k)$ we find

$$\tilde{U}(\pm\infty)\chi_{\pm}^{E}(k)=(E^2-K^2)^{1/2}e^{i\alpha_\pm}\chi_{\pm}^{-E}(k)\,, \tag{3.8}$$

where

$$\tan\alpha_\pm=\frac{Kk_\pm}{\phi_\pm E}\,. \tag{3.9}$$

Therefore from Eq. (3.6),

$$R(-E)=R(E)e^{-2i\alpha_-}\,, \qquad T(-E)=T(E)e^{i(\alpha_+-\alpha_-)}\,, \tag{3.10}$$

where

$$B(E)=e^{i\delta(E)}\,,$$

$$\delta(E)=-\alpha_+(E)+\alpha_-(E)\,.$$

From this expression we can evaluate $G_e(E)$ and $\rho_{\rm odd}(E)$ using Eqs. (2.11) and (2.12), and find

$$G_e(E)=-\frac{i}{2}\frac{K}{E^2-K^2}\left[\frac{\phi_+}{k_+}-\frac{\phi_-}{k_-}\right]. \tag{3.11}$$

If $\phi_+\neq\phi_-$ there are two thresholds at $E=\rho_\pm$ where $\rho_\pm=(\phi_\pm{}^2+K^2)^{1/2}$. Below both thresholds and for $E>0$ we find $\rho_{\rm odd}=\rho_1+\rho_2$, where ρ_1 is a discrete contribution

$$\rho_1(E)=\tfrac{1}{4}\text{sign}(K)[\text{sign}(\phi_+)-\text{sign}(\phi_-)]\delta(E-|K|)\,, \tag{3.12}$$

and ρ_2 arises from the continuum

$$\rho_2(E)=-\frac{1}{2\pi}\left[\theta(E-\rho_+)\frac{d}{dE}\tan^{-1}\left[\frac{k_+K}{\phi_+E}\right]-\theta(E-\rho_-)\frac{d}{dE}\tan^{-1}\left[\frac{k_-K}{\phi_-E}\right]\right]. \tag{3.13}$$

Before going any further let us analyze the expressions for G_e and $\rho_{\rm odd}$ given above. $G_e(E)$ agrees with the results given in Refs. 14 and 15. Indeed from expression (2.12) $G_e(E)$ can be written as

$$G_e(E)=\text{Tr}\frac{H}{H^2-E^2} \tag{3.14}$$

and for *constant* K it coincides with the expression given by Callias for the regulated index[23,13,24,14] [up to the factor $K/(K^2-E^2)$], however this is only true in this special case.

The result for $\rho_{\rm odd}(E)$ below threshold has the correct features. Indeed from several examples it is known that when $\text{sign}(\phi_+)\neq\text{sign}(\phi_-)$ there is a bound state[8,25] (of topological origin) at $E=\pm K$ (depending on the sign difference). This is the same bound state as the one found by Jackiw and Rebbi[1] in the charge-conjugate case but shifted by K.

The phase of $B(E)$ is related to the phase shifts of the scattering states. As is seen in Eq. (3.10) above, these phase shifts have a finite limit at $E\to\infty$. Indeed unlike the nonrelativistic case where the phase shifts go to zero as $E\to\infty$ (because the velocity goes to infinity) in the relativistic case they approach a constant[22] (the velocity goes to 1) which is proportional to the integral of the potential over all space (notice that the scattering "potential" for H_χ is θ' since it is compared to H_0). It is interesting to point out that η is related to the phase shifts of the spinors χ (eigenstates of H_χ) not of ψ. Reference 26 seems to be ambiguous on this point. Using Eq. (2.28) we find

$$\eta=\tfrac{1}{2}\text{sign}(K)[\text{sign}(\phi_+)-\text{sign}(\phi_-)]+\frac{1}{\pi}[\delta(\infty)-\delta(0)]\,, \tag{3.15}$$

where

$$\delta(0)=-\alpha_+(E=\rho_+)+\alpha_-(E=\rho_-)$$

and $\delta(\infty)$ is the limit of $\delta(E)$ as $E\to\infty$. Both quantities $\delta(0)$ and $\delta(\infty)$ depend on the branches of the inverse tangent function. The difference $\delta(\infty)-\delta(0)$ is, however, branch independent. Once the branch of $\delta(\infty)$ [or $\delta(E)$ for any value of E] is fixed the branch of $\delta(0)$ is determined by following the analytic function $\delta(E)$ down to threshold. Any branch dependence cancels in the difference. Therefore the expression given above for η is unambiguous. After a careful analysis of the branches (for $K=$const), we find that if $K>0$ and $\phi(x)$ has an even number of zeros then $0\le\delta(\infty)\le\pi/2$ and $\delta(0)=0$ (there are no bound states). If $\phi(x)$ has an odd number of zeros $0\le\delta(\infty)\le\pi$ and $\delta(0)=\pi$ (there is one bound state at K). The signs of δ are reversed for $K<0$. Then

$$\delta(0)=\frac{\pi}{2}\text{sign}(K)[\text{sign}(\phi_-)-\text{sign}(\phi_+)]\,. \tag{3.16}$$

Thus the phase shifts at threshold cancel the bound-state contribution as it should by Levinson's theorem[27] and the final answer is

$$\eta=-\frac{1}{\pi}\left[\tan^{-1}\left[\frac{K}{\phi_+}\right]-\tan^{-1}\left[\frac{K}{\phi_-}\right]\right] \tag{3.17}$$

which agrees with Refs. 14 and 15, however the above (necessary) analysis of the branches indicates that $-1\le\eta\le1$ and $-\frac{1}{2}\le Q\le\frac{1}{2}$. In Refs. 14 and 15 this is not explicitly indicated, and without the above analysis the unwary reader may be misled to believe that in this case Q can acquire *any* fractional value. The reason for this lies in the existence of the operator $U(x)$. This analysis will be much less obvious in the examples worked in Sec. IV.

In the case of the soliton profile $\phi_+=\phi$, $\phi_-=-\phi$ with $\phi>0$, it can be easily seen that

$$\delta(\infty)-\delta(E)=\tan^{-1}\left[\frac{\phi K(E-K)}{\phi^2 E+K^2 k}\right] . \quad (3.18)$$

Since the branch of this formula cannot change, $\delta(\infty)-\delta(E)$ must be between $\pm\pi/2$ for any value of $E>E_T$. Hence η is given by

$$\eta=\text{sign}(K)-\frac{2}{\pi}\tan^{-1}\left[\frac{K}{\phi}\right]-\frac{\pi}{2}\le\tan^{-1}\left[\frac{K}{\phi}\right]\le\frac{\pi}{2} \quad (3.19)$$

which can be compared to the result obtained in Refs. 18 and 25. The above expression for η has the correct spectral flow behavior. When $\phi_-=-\phi_+$ there is a bound state with energy $E=K$. If K is adiabatically changed from a positive value to a negative one η jumps by 2 when K crosses zero. This is the correct behavior for η, as discussed in Sec. II. The ground-state charge has changed by -1, but the adiabatic charge has not changed, as was pointed out in Ref. 9.

Case b: $\phi=$ constant. H^2 commutes with σ_1. In this case $\{H,[H,\sigma_1]\}=0$ and

$$\tilde{U}(x)=-(\sigma_1 e^{i\sigma_2\theta(x)}-\phi) . \quad (3.20)$$

Following the steps of the previous case

$$\tilde{U}(x=\pm\infty)\chi_\pm^E(k)=(E^2-\phi^2)^{1/2}e^{-i\beta_\pm}\chi_\pm^{-E}(k) , \quad (3.21)$$

where $\tan\beta_\pm=\phi k_\pm/K_\pm E$.

Indeed this case can be obtained from the former by the change $K\to-\phi$, $\phi_\pm\to K_\pm$, following the steps for case a we find

$$B(E)=e^{i(\beta_+-\beta_-)} ,$$
$$G_e(E)=\frac{i}{2}\frac{\phi}{(E^2-\phi^2)}\left[\frac{K_+}{k_+}-\frac{K_-}{k_-}\right] , \quad (3.22)$$

and below thresholds:

$$\rho_{\text{odd}}(E>0)=-\tfrac{1}{4}\text{sign}(\phi)[\text{sign}(K_+)-\text{sign}(K_-)]\delta(E-|\phi|) . \quad (3.23)$$

All the results of case a can be applied to this situation with the above exchange of K, ϕ. That this is so is no surprise, it is just the result of a $\pi/2$ rotation around σ_2 with the consequent exchange $\phi\to-K$. Since case b is equivalent to case a we will not explore it any further.

As we have seen in these examples, the fact that the operator $\tilde{U}$ exists is crucial for the topological invariance of $\rho_{\text{odd}}(E)$ and $G_e(E)$. In the general case when *both* ϕ and K are functions of position this operator *may not* exist as it will be shown explicitly in the next section for some interesting solvable examples.

This in turn means that $G_e(E)$ and $\rho_{\text{odd}}(E)$ will *in general* depend on local details of the soliton fields. The spectral asymmetry will be insensitive to "small" changes in local features. However as the local properties are changed there may be levels crossing zero energy and this will be associated with the corresponding jumps in η (spectral flow).

IV. TWO EXAMPLES

In this section we analyze two simple examples for which the T matrix can be computed exactly and yet they are rich enough to contain interesting physical information relevant to charge fractionalization.

Example 1: Infinitely thin soliton. (This is a slightly modified version of the problem studied in Refs. 9 and 12.) For this problem we choose

$$\phi(x)=\begin{cases}\phi_-, & x<0\\ \phi_+, & x>0 ,\end{cases}$$
$$K(x)=\begin{cases}K_-, & x<0\\ K_+, & x>0 .\end{cases} \quad (4.1)$$

The eigenstates of H are easily shown to be continuous across the origin. The spinor-wave function

$$\psi(x)=\begin{cases}\psi_<(x), & x<0\\ \psi_>(x), & x>0\end{cases} \quad (4.2)$$

obeys the following boundary condition at the origin

$$\psi_<(0)=\psi_>(0) . \quad (4.3)$$

This in turn means that the eigenfunctions of H_χ obey

$$e^{i\sigma_2\theta_-/2}\chi_<=e^{i\sigma_2\theta_+/2}\chi_> \quad (4.4)$$

or

$$\chi_<=e^{i\sigma_2\Delta\theta/2}\chi_> , \quad (4.5)$$

with $\theta_\pm=\tan^{-1}(K_\pm/\phi_\pm)$ and $\Delta\theta=\theta_+-\theta_-$. The scattering solutions have the following behavior,

$$\chi_<(x)=\chi_-(k)e^{ik_-x}+R\chi_-(-k)e^{-ik_-x}, \quad x<0$$
$$\chi_>(x)=T\chi_+(k)e^{ik_+x}, \quad x>0 \quad (4.6)$$

where the notation is the same as Eqs. (2.15) and (2.16) in Sec. II.

The transmission coefficient T can be easily found for any $\rho_\pm$ and $\theta_\pm$, but for simplicity and to illustrate the physics more clearly we quote the answer for $\rho_+=\rho_-=\rho$ $(k_+=k_-=k)$:

$$\frac{1}{T(E)}=C+iS\frac{E}{k} , \quad (4.7)$$

where $C=\cos(\Delta\theta/2)$, $S=\sin(\Delta\theta/2)$. Below threshold [where $k=i\mathbf{k}=i(\rho^2-E^2)^{1/2}$] $T(E)$ has a bound-state pole at $E=-\text{sign}(S)\rho C$. As $E\to\infty$ the phase of $T(E)$ (phase shift) approaches $-\Delta\theta/2$ as was pointed out in Sec. III. From Eqs. (2.25) and (4.7) we find

$$B(E)=\frac{C-iSE/k}{C+iSE/k}\ ,$$
$$G_e(E)=i\frac{\rho^2SC}{k(E^2-\rho^2C^2)}\ . \tag{4.8}$$

Below threshold ($0<E<\rho$) the odd density of states is

$$\rho_{\rm odd}(E)=-\tfrac{1}{2}{\rm sign}(S){\rm sign}(C)\delta(E-\rho\,|\,C\,|\,)\ , \tag{4.9}$$

and above threshold

$$\rho_{\rm odd}(E)=+\frac{1}{\pi}\frac{d}{dE}\delta(E)\ ,$$

where (4.10)

$$\tan\frac{\delta(E)}{2}=-\frac{SE}{Ck}\ ,$$

therefore

$$\eta=-{\rm sign}(S){\rm sign}(C)+\frac{1}{\pi}[\delta(\infty)-\delta(0)] \tag{4.11}$$

which can be written as

$$\eta=-\frac{\Delta\theta}{\pi}+{\rm sign}(S)[1-{\rm sign}(C)]\ ,\ -\pi\le\frac{\Delta\theta}{2}\le\pi\ . \tag{4.12}$$

When $\Delta\theta$ is adiabatically changed from slightly below π to slightly above π the bound state (at $E=-\rho C$) crosses zero and η jumps by $+2$ and the charge changes by one unit. As was pointed out before $\delta(\infty)-\delta(0)$ is independent of the branches of the function $\tan^{-1}(x)$ and so is η. It would then seem that in expression (4.12) η depends on the definition of the branches, however the reader can be readily convinced that it is not. η is a discontinuous, periodic function of $\Delta\theta$ with period 2π and $-1\le\eta\le1$; it can be written as

$$\eta=-\frac{\Delta\theta}{\pi}+2n\ ,$$

where

$$\pi(2n-1)\le\Delta\theta\le\pi(2n+1)\ .$$

Therefore the ground-state charge $-\frac{1}{2}\le Q=-\frac{1}{2}\eta\le\frac{1}{2}$. We see that the fractional part of the charge $Q_F=\Delta\theta/2\pi$ ($-\pi\le\Delta\theta\le\pi$) is a smooth function and is given by the *high-energy* behavior of the phase shifts mod π. The integer part is related to *low-energy* features; namely, bound states and phase shifts at thresholds (see next example).

To compare with the next example we quote the results for the case $\phi_+=\phi$, $\phi=-\phi$ ($\phi>0$), and $K=$constant ($K>0$):

$$\theta(x)=\begin{cases}\theta_+=\tan^{-1}\left[\dfrac{K}{\phi}\right], & x>0\\[2ex] \theta_-=\pi-\tan^{-1}\left[\dfrac{K}{\phi}\right], & x<0\ ,\end{cases}$$

$$\Delta\theta=\theta_+-\theta_-=2\tan^{-1}\left[\frac{K}{\phi}\right]-\pi\ ,$$

$$-\frac{\pi}{2}\le\tan^{-1}\left[\frac{K}{\phi}\right]\le\frac{\pi}{2}\ ,$$

and η is given by expression (4.12).

Example 2: Three steps (wide soliton). Although example 1 sheds light on the physics of charge fractionalization and allowed us to understand better the high- and low-energy aspects, we cannot draw conclusions regarding the dependence of η on local details of the external fields. To study this aspect consider the following soluble example,

$$\phi(x)=\begin{cases}\phi_-, & x<0\\ \phi_+, & x>0\ ,\end{cases}$$
$$K(x)=\begin{cases}K_-, & x<-d_1\\ K_0, & -d_1<x<d_2\\ K_+, & x>d_2\ .\end{cases} \tag{4.13}$$

However, to simplify the final formulas and to expose the physics clearly we will analyze and quote the results for the simple case $\phi_-=-\phi_+=-\phi$ ($\phi>0$), $K_-=K_+=-K_0=K$ ($K>0$) (this implies $\rho=$constant), and $d_1=d_2=d$. Therefore, $\theta(x)$ is obtained by following the branches

$$\theta(x)=\begin{cases}\theta_+=\tan^{-1}\left[\dfrac{K}{\phi}\right], & x>d\\[2ex] \theta_2=-\tan^{-1}\left[\dfrac{K}{\phi}\right], & 0\le x\le d\\[2ex] \theta_1=-\pi+\tan^{-1}\left[\dfrac{K}{\phi}\right], & -d\le x\le 0\\[2ex] \theta_-=-\pi-\tan^{-1}\left[\dfrac{K}{\phi}\right], & x<-d\ .\end{cases} \tag{4.14}$$

Using the matching conditions Eq. (4.5) at $x=-d;0;d$ and after some algebra we find

$$\frac{1}{T(E)}=A+i\frac{E}{k}B\ , \tag{4.15}$$

where

$$A=C+\frac{2iK}{k^2\rho}[2\phi^2\sin z\,e^{iz}+K^2\sin 2z\,e^{2iz}]\ ,$$
$$B=S+\frac{4K^2\phi}{k^2\rho}\sin^2 z\,e^{2iz}\ , \tag{4.16}$$

and

$$C=\cos\left[\frac{\theta_+-\theta_-}{2}\right]=-K/\rho\ ,$$
$$S=\sin\left[\frac{\theta_+-\theta_-}{2}\right]=\phi/\rho\ , \tag{4.17}$$
$$z=kd=d(E^2-\rho^2)^{1/2}\ .$$

From the above expressions we learn several important features. First we see that $T(E)$ *does depend* on local details, here the width of the soliton d. This in turn implies that $B(E)$, G_e, and $\rho_{\rm odd}(E)$ will depend on d nontrivially. Second, the high-energy behavior is the same as the infinitely thin soliton example since the second terms in A and B vanish as $(1/E^2)$ in the limit $E\to\infty$. Therefore $\delta(\infty)$ *will be the same* in both cases. When $d=0$ (example 1) there is a bound state at $E=K$ and $\delta(\infty)-\delta(0)=2\tan^{-1}(\phi/K)-\pi$, therefore $\eta=(2/\pi)\tan^{-1}(\phi/K)$. As d is increased the bound-state energy decreases. For very small d, one finds

$$E_b\approx K(1-4\phi d)+O(d^2)\ . \tag{4.18}$$

As d is increased further several things happen. The bound state initially at $E_b=K$ crosses $E=0$, but also more bound states peel off from the negative continuum. Indeed it is easy to see from Eqs. (4.15) and (4.16) (below threshold) that at $d\simeq\infty$ there is a bound state at $E=-K$ and two nearly degenerate at $E=-\phi$ (the splitting being of order e^{-2Kd}).

The critical values of d_0 at which the bound state (initially at K) crosses zero and d_1 and d_2 for which new bound states just peel off from the negative continuum are easily obtained analytically. The first, d_0, corresponds to the solution of $1/T(0)=0$ and the other ones correspond to $1/T(-\rho)=0$. From the same analysis we also learn that at the critical values d_1 and d_2, the phase shifts (at $k=0$) of $B(E)$ decrease by π each time a bound state arises from the negative continuum and increase by π if it arises from the positive continuum. This is the usual behavior of phase shifts at threshold whenever new bound states appear. All these features can be understood analytically from Eqs. (4.15) and (4.16) and they lead to the following scenario as d increases (for fixed K and ϕ): at $d=0$ there is one bound state at K and

$$\eta=1+\frac{1}{\pi}\left[2\tan^{-1}\left[\frac{\phi}{K}\right]-\pi\right]=\frac{2}{\pi}\tan^{-1}\left[\frac{\phi}{K}\right]. \tag{4.19}$$

As d increases, η remains constant until $d=d_0$ at which point the bound state crosses $E=0$. Of course the phase shifts remain unchanged since the bound state came from the positive continuum and therefore $\delta(0)=\pi$. For $d>d_0$, the bound state has negative energy and

$$\eta=-1+\frac{1}{\pi}\left[2\tan^{-1}\left[\frac{\phi}{K}\right]-\pi\right]$$
$$=-2+\frac{2}{\pi}\tan^{-1}\left[\frac{\phi}{K}\right]. \tag{4.20}$$

As d increases further and passes d_1 another bound state appears from the *negative* continuum and $\delta(0)$ drops by π. There are now two bound states with negative energy, with $\delta(0)=0$, and

$$\eta=-2+\frac{2}{\pi}\tan^{-1}\left[\frac{\phi}{K}\right]. \tag{4.21}$$

The appearance of the new bound state does not modify η, because the phase shifts at threshold change by π whenever a new bound state appears. When d is increased further and reaches d_2 there is another bound state peeling off the negative continuum and $\delta(0)$ drops by another π. Now we have

$$\eta=-3+\frac{1}{\pi}\left[2\tan^{-1}\left[\frac{\phi}{K}\right]+\pi\right]$$
$$=-2+\frac{2}{\pi}\tan^{-1}\left[\frac{\phi}{K}\right]. \tag{4.22}$$

Therefore η has changed only when the bound state initially at K crossed $E=0$. The above scenario is modified for different values of K and ϕ in the sense that the ordering of d_0,d_1,d_2 may be changed. Bound states may emerge from the continuum before the one at $E\sim K$ crosses the origin, but the picture is the same. The index η changes only when there is spectral flow. This behavior can be understood with the following argument. Let us imagine our system in a very large box, then the spectrum is discrete and we can use the formal expression for $\eta=\sum_{E_n\neq 0}\text{sign}(E_n)$. As d is varied the eigenvalues move but as long as their sign does not change η remains invariant.

An anomalous case. A peculiar situation arises when d is exactly d_1 or d_2. The phase shift at threshold has dropped only by $\pi/2$. However there is a state at threshold that is about to become bound; it is the "half bound state" noticed in Ref. 24 in another context. In this situation the integral of the density of states along the continuum cut [see Eq. (2.28)] has to be performed carefully because there is a contribution from the edge of the cut. The ratio of Jost functions $B(E)$ has a zero (or pole) linear in k and thus produces a pole in the logarithmic derivative weighted with a factor $\frac{1}{2}$ [arising from $(E^2-\rho^2)^{1/2}$]. Proper account of this behavior yields the following result for η:

$$\eta=N^+-N^-\pm\tfrac{1}{2}+\frac{1}{\pi}[\delta(\infty)-\delta(0)]\ , \tag{4.23}$$

where $\delta(0)=\delta(0)\pm\pi/2$, and $\delta(0)$ is the value of $\delta(0)$ just before d reaches d_1,d_2. The term $+\frac{1}{2}$ $(-\frac{1}{2})$ arises from the edge contribution of the positive- (negative-)energy continuum. The $-\pi/2$ $(+\pi/2)$ corresponds to the increase (decrease) of the phase shifts at threshold when a half bound state forms.

We see that η *does not* change when d reaches these critical values. Indeed, when d *passes* a critical value, a new bound state is formed but its contribution to η is canceled by the change in the phase shifts at threshold. When d is *exactly* one of these critical values, the same cancellation takes place but with a factor $\frac{1}{2}$; therefore η *is continuous* at these values of d. This behavior corresponds to the anomalous Levinson's theorem of potential scattering. However Eq. (4.23) is *not* Levinson's theorem.

Remarks. From the examples worked above we learn several important features. As an illustration of the relation between charge fractionalization and topological concepts like spectral flow and indices, let us observe the following.

(1) Adiabatic spectral flow. In example 1 the Hamil-

tonian is invariant under the shift $\theta \to \theta + 2\pi$ therefore the spectrum of the theory is unchanged. Let us introduce a parameter τ and suppose that $\Delta\theta$ is a function of τ such that $\Delta\theta(\tau = -\infty) = 0$ and $\Delta\theta(\tau = +\infty) = 2\pi$. Therefore the spectrum of H at $\tau = -\infty$ is the same as the one at $\tau = +\infty$. However, as τ evolves and $\Delta\theta$ evolves adiabatically, the spectrum changes. A bound state comes from the negative continuum moving up in energy,[9] and the density of continuum states changes as this happens. When $\Delta\theta = \pi$ the bound states cross $E = 0$ and η jumps by 2. As τ evolves further, the bound state moves towards positive threshold and at $\tau = +\infty$ it reaches $E = +\rho$. The spectrum is the same as $\tau = -\infty$ ($\Delta\theta = 0$) but now there is a filled positive-energy state. The state reached adiabatically is therefore an excited state which differs from the ground state by one unit of charge, i.e., the adiabatic and ground-state charge differ by unity.[9]

(2) The integer. Comparing example 2 to the case K=constant ($K > 0$) in example 1, we learn several features. If one follows the angle $\theta(x)$ from $x = +\infty$ to $x = -\infty$ in both cases, keeping track of the branches, $\Delta\theta$ in the second case is 2π bigger than the one in the first case. At first one may be tempted to conclude from Eq. (4.12) that this difference would account for the *integer* part of η, however we have seen that *this* 2π has *nothing* to do with the integer part since, for example, it is *independent of d*. The integer change in η corresponds to the spectral flow that occurs whenever $d > d_0$. The *fractional part* is entirely given by the step functions used in example 1 and is a *topological invariant*; the integer part has to do with levels crossing zero and depends on the local details of the fields in agreement with the conclusions of Refs. 9 and 12.

(3) Charge additivity. An important physical attribute of charge is the property of additivity. The adiabatic charge is truly additive, whereas the ground-state charge can change by one when a level crosses zero energy.

It is interesting to check that the index η in our example 2 does indeed have this last feature. Let us begin by noting that as $d \to \infty$, example 2 consists of three widely separated solitons. Near each soliton there is a localized state with energies $-\phi$, $-K$, and $-\phi$, respectively. The corresponding indices can be evaluated individually in this limit (see example 1) and are $\eta_1 - 1$, $-\eta_1$, $\eta_1 - 1$ with $\eta_1 = 1 - (2/\pi)\tan^{-1}(K/\phi)$. Thus the total index, η_3, is $\eta_1 - 2$, and this is the exact result for $d > d_0$.

As d decreases, these three localized states start to overlap and to interact. Two of the levels are forced into the negative continuum, and one is pushed above zero energy. At this point η_3 jumps by two, and $\eta_3 = \eta_1$. Finally as $d \to 0$, the configuration is that of a simple soliton and η_1 is the correct value.

Therefore we have learned that the total charge is the sum of the charges induced by each of the solitons (to within spectral flow effects). This property can be traced to the fact that the wave functions overlap of the separated bound states vanishes exponentially. This is crucial to show that the induced charge is a sharp observable.[4–6]

(4) Charge-conjugate case. Perhaps at this point the reader is confused by the following conflict: In the charge-conjugate case ($K = 0$), the usual argument[1,3] suggests that the zero-energy states are responsible for the fractionalization, and yet our arguments indicate that the fractional part comes entirely from high energy. Let us analyze this case more carefully. Consider $K > 0$ and constant, and the soliton profile $\phi_+ = \phi > 0$, and $\phi_- = -\phi$.

We find (looking carefully at the branches) $\delta(E) = \pi - 2\tan^{-1}(kK/\phi E)$ and $\delta(\infty) = \pi - 2\tan^{-1}(K/\phi)$, $\delta(0) = \pi$ (there is a positive-energy bound state). Now take $K \to 0^+$. We find

$$\eta = 1 + \frac{1}{\pi}[+\pi - \pi] .$$

The factor 1 corresponds to the ($E = 0^+$) bound state, the factor $+\pi$ inside the brackets is the high-energy piece captured by the adiabatic method,[7] and the factor $-\pi$ is dictated by Levinson's theorem. The excess of states at zero energy is compensated for by a deficit of states at infinite energy. This behavior neatly reconciles the high- and low-energy aspects of the problem.

V. GENERAL CASE

We expect the results obtained from analysis of the models of Secs. III and IV to hold in the general case when the soliton fields are arbitrary functions. Given a Hamiltonian H in which the fields ϕ and K [or $\theta(x)$ and $\rho(x)$] have a definite behavior at spatial infinity, we can always form a Hamiltonian H_V like the one in example 1 with step functions for $\phi(x)$ and $K(x)(\theta,\rho)$ with the *same asymptotic values as the fields in H.* Then we can write the quantity $B(E)$ for H as

$$B(E) = B_V(E) B_R(E), \quad B_R = B(E)/B_V(E) , \tag{5.1}$$

where $B_V(E)$ contains H_V. This choice of $B_V(E)$ ensures that $B_R(E)$ has *zero* phase as $E \to \infty$ and its only contribution to η arises from possible bound states and the value of $\delta_R(0)$ (δ_R = relative phase shift). Therefore we can write

$$\eta = \eta_V + (\eta - \eta_V) , \tag{5.2}$$

where η_V contains *all* the topological features of the background fields and completely describes the *high-energy* behavior of the theory. It accounts for the *fractional* part of η and consequently the *fractional* part of the charge

$$Q_F = \frac{1}{2\pi}[\theta(x = +\infty) - \theta(x = -\infty)] \qquad (-\pi \le \Delta\theta \le \pi) .$$

The part $(\eta - \eta_V)$ is an even integer (or zero) arising from the spectral flow (levels crossing $E = 0$) that occurs when H_V is locally deformed onto H. Thus we have isolated the topological (asymptotic) properties of the background fields in H_V. Since the high-energy behavior is only sensitive to these topological features and *not* to local details, H_V describes completely the fractional part of the charge. Consequently, $(\eta - \eta_V)$ only depends on local features of the background fields and accounts for the *integer* part of η and the charge.

Conclusions. We have related the (properly normal-ordered) ground-state charge to the asymmetry η between the positive- and negative-energy parts of the Dirac spectrum $Q = -\frac{1}{2}\eta$. As a measure of this spectral asymmetry we introduced the fundamental quantity

$$B(E)=\det\left[\frac{H+E}{H-E}\right]$$

that allowed us to write an exact expression for the APS invariant:

$$\eta=N^+-N^-+\frac{1}{\pi}[\delta(E=\infty)-\delta(E=E_T)]\ ,$$

where $\delta(E)$ is the *phase* of $B(E)$ (E_T=threshold energy) and $N^{\pm}$ are the number of positive- (negative-)energy *bound states.* We point out that $\delta(E)$ is related to the (relativistic) phase shifts of the scattering states of the chirally rotated Hamiltonian. If threshold resonances exist, the above formula is slightly modified. Given an interacting Hamiltonian H with arbitrary background fields it may be very difficult to compute η as given above.

However we can define a *very simple* Hamiltonian H_V in which the external fields have the *same asymptotic properties* as the ones in H, and for which η_V can be computed exactly and we write $\eta=\eta_V+(\eta-\eta_V)$. Since the *high-energy behavior* of $\delta(E)$ is only sensitive to the asymptotic properties of the fields, H_V completely describes the high-energy features and therefore yields the *fractional* part of the charge exactly. It is only this *fractional* part that is a topological invariant and related to the high-energy behavior of the theory. The quantity $(\eta-\eta_V)$ depends on the local details, it is an even integer or zero and corresponds to the spectral flow (levels crossing $E=0$) that occurs when H_V is locally deformed onto H and accounts for the integer part of the charge. The fractional part of the charge is shown to be given by

$$Q_F=\frac{1}{2\pi}[\theta(x=+\infty)-\theta(x=-\infty)]\ ,\quad -\pi\leq\Delta\theta\leq\pi$$

and it is a high-energy feature of the theory. The formalism and examples of this paper offer a unifying view of the physics of charge fractionalization using familiar concepts.

Since the fractional part of the charge arises from the *high-energy* behavior of the theory, we expect the adiabatic approximation to accurately describe it since it corresponds to the external fields being probed at very short wavelengths and in this regime the approximation is reliable. This high-energy behavior is at the heart of the anomalous commutator method, and the fact that the twisted boundary conditions of Ref. 12 reproduce the fractional charge correctly comes as no surprise since the phase shifts can be determined from these conditions.

The integer part of the charge is nontopological and is related to local details of the background fields and in particular to energy levels crossing zero, hence it is a *low-energy* feature of the theory. Although this integer may not be seen in field theory approaches to the physics of the charge fractionalization (we can always fill these states and redefine the vacuum) its properties allowed us to understand and expose the beauty of the concept of spectral flow. It requires a thorough analysis of the specific problem, and may be particularly interesting in a condensed-matter context. With the simple methods introduced and developed here, we hope to study the physics of charge fractionalization and its relation to anomalies in higher-dimensional theories; work on these lines is in progress.

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