Zitterbewegung of the electron in external fields

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A new method is given to find the exact solutions of the Heisenberg equations in proper time of the first-order Dirac Hamiltonian in constant electromagnetic fields. We also give the exact *Zitterbewegung* of spin in a magnetic field.

I. INTRODUCTION

By *Zitterbewegung* we mean the internal dynamics of the electron (or neutrino) in which the additional (spin) degrees of freedom appear as a finite quantum system of the oscillator type in the Heisenberg picture. This phenomenon also occurs for a relativistic rotator, for extended or composite systems, and in general for any relativistic system with internal degrees of freedom. Its study is important for an understanding of the foundations of relativistic quantum mechanics, and of the possible excited states of the electron.

In a previous paper[1] we studied in a covariant way how the position operator of the free electron separates into a "center of mass" moving in proper time with constant velocity plus an oscillatory *Zitterbewegung*. The internal position and momentum operators of the *Zitterbewegung* generate the Lie algebra of SO(3,2). Acting on a particular Fourier component of the Dirac wave function with center-of-mass momentum p, the vector and tensor operators of this algebra lie in the hyperplane orthogonal to p. The *Zitterbewegung* has the dynamics of a three-dimensional harmonic oscillator confined to this hyperplane. We also generalized this dynamical system to the larger system with symmetry group SO(4,2) by including the pseudoscalar and axial-vector operators γ^5 and $\gamma^5\gamma^\mu$.

The purpose of this work is to investigate, in the Heisenberg picture, the motion of the Dirac electron in external electromagnetic fields, and to determine the influence of external fields on the dynamical system described in Ref. 1. An external field should affect both the center of mass of the electron, causing it to accelerate, and the internal oscillation of the charge around the center of mass. We investigate this phenomenon by solving the Heisenberg equations in proper time for the velocity, position, and spin of the electron in homogeneous, time-independent electric and magnetic fields. Although the constant electromagnetic field is a very special case, it has some interest because the amplitude of the *Zitterbewegung* is small (about 10^{-11} centimeters) and many fields in nature are effectively constant over this distance scale. The exact solutions obtained here show that the *Zitterbewegung* and center-of-mass motion of the electron are inextricably interconnected in the presence of external fields. Nevertheless, we can obtain a picture of the additional features of the electron's motion introduced by the *Zitterbewegung*, by comparing with the motion of a relativistic spinless charge in the same fields. When the electric and magnetic fields are weak relative to a critical field, we are able to separate the *Zitterbewegung* from the center of mass motion. The critical fields are so large that for all practical purposes this separation is valid.

Finally we should emphasize that the algebraic structure of the internal dynamics of the electron, derived and studied in Ref. 1 and here, transcend the particular four-dimensional representation. They are also valid for any relativistic system (e.g., hadrons) with internal degrees of freedom or extended structure when described in the Heisenberg representation.

II. HEISENBERG'S EQUATIONS OF MOTION FOR AN ELECTRON IN AN EXTERNAL ELECTROMAGNETIC FIELD

For an electron in an external electromagnetic field we use the minimal coupling rule $p_\mu \to \pi_\mu = p_\mu - eA_\mu$ and replace the free-particle proper-time Hamiltonian of Ref. 1 by

$$\mathscr{H} = -\pi_\mu \gamma^\mu \ . \tag{1}$$

The velocity of the electron is then

$$\dot{x}^\mu = \frac{i}{\hbar}[\mathscr{H}, x^\mu] = \gamma^\mu \ . \tag{2}$$

The Heisenberg equation for the motion of the velocity operator yields

$$\dot{\gamma}^\mu = \frac{i}{\hbar}[\mathscr{H}, \gamma^\mu] = \frac{2}{\hbar}\pi_\nu \sigma^{\mu\nu} \ . \tag{3}$$

Differentiating again we find

$$\begin{aligned}\ddot{\gamma}^\mu &= \frac{2i}{\hbar^2}[\mathscr{H}, \pi_\nu \sigma^{\mu\nu}] \\ &= -\frac{2i}{\hbar^2}(\pi_\alpha \pi_\nu [\gamma^\alpha, \sigma^{\mu\nu}] + [\pi_\alpha, \pi_\nu]\sigma^{\mu\nu}\gamma^\alpha) \\ &= \frac{4}{\hbar^2}\pi^\mu \pi\cdot\gamma - \frac{4\pi\cdot\pi}{\hbar}\gamma^\mu + \frac{2ie}{\hbar c}F^\mu{}_\nu \gamma^\nu + \frac{2e}{\hbar c}{}^*F^\mu{}_\nu \gamma^{5\nu} \ ,\end{aligned} \tag{4}$$

where we have used the commutation relation

$$[\pi_\mu, \pi_\nu] = -\frac{i\hbar e}{c}F_{\mu\nu} \tag{5}$$

and

$$^*F^{\mu\nu}=\tfrac{1}{2}\epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}$$

is the Hodge dual of the electromagnetic field tensor $F^{\mu\nu}$. The sum of the four terms on the right-hand side of (4) is γ^0 Hermitian and therefore any solution of this equation for $\gamma^\mu(s)$ must be γ^0 Hermitian. This result is to be expected since the γ^0-unitary proper-time evolution of $\gamma^\mu(0)$ will preserve its γ^0 Hermiticity.

Using the fact that

$$\dot{\gamma}^\mu\mathscr{H}=-2\pi_\nu\pi_\alpha\sigma^{\mu\nu}\gamma^\alpha$$
$$=\frac{2i}{\hbar}(\pi^\mu\pi\cdot\gamma-\pi\cdot\pi\gamma^\mu)-2\frac{e}{c}F^\mu{}_\nu\gamma^\nu+i\left[\frac{e}{c}\right]^* F^\mu{}_\nu\gamma^\nu$$

we can put (4) into the more tractable form

$$\ddot{\gamma}^\mu+\frac{2i}{\hbar}\dot{\gamma}^\mu\mathscr{H}+\frac{2ie}{\hbar c}F^\mu{}_\nu\gamma^\nu=0 \tag{6}$$

or

$$-\dot{\gamma}^\mu\mathscr{H}=\frac{e}{c}F^\mu{}_\nu\gamma^\nu+\frac{\hbar}{2i}\ddot{\gamma}^\mu \ . \tag{6'}$$

We note that since $-\mathscr{H}=mc$, when acting on positive-energy solutions of the Dirac equation, (6′) appears remarkably similar in form to the Lorentz-Dirac equation for a classical radiating point charge

$$mc\dot{v}^\mu=\frac{e}{c}F^\mu{}_\nu v^\nu+\frac{2}{3}\frac{e^2}{4\pi c}(\ddot{v}^\mu+v^\mu\dot{v}_\alpha\dot{v}^\alpha) \tag{7}$$

which can be written in the form

$$\frac{d}{ds}(m'cv^\mu)=\frac{e}{c}F^\mu{}_\nu v^\nu+\frac{2}{3}\frac{e^2}{4\pi c}\ddot{v}^\mu \ , \tag{7'}$$

where m' is the variable mass of the radiating charge

$$\frac{dm'}{ds}=-\frac{2}{3}\frac{e^2}{4\pi 6c}\dot{v}_\alpha\dot{v}^\alpha$$

and v^μ is the proper-time velocity. A crucial difference between (6′) and (7′) is the appearance of $i=\sqrt{-1}$ in front of the acceleration. A consequence of this factor of i is that while the classical Lorentz-Dirac equation has run-away solutions, the quantum-mechanical equation exhibits instead oscillatory *Zitterbewegung* solutions. Note also that when $\hbar$ is taken to zero in (6′) this equation reduces to an operator form of the classical Lorentz equation.

We shall now take $\hbar=c=1$. We can eliminate the first-order term in (6) by using the ansatz

$$\gamma^\mu(s)=y^\mu(s)e^{-i\mathscr{H}s} \ , \tag{8}$$

where $y^\mu(s)$, like $\gamma^\mu(s)$, is a four-by-four matrix operator. Substituting (8) into (6) we find that the terms containing $\dot{y}^\mu(s)$ cancel and we are left with

$$\ddot{y}^\mu(s)=-y^\nu[\mathscr{H}^2g_\nu{}^\mu-2ie\exp(-i\mathscr{H}s)$$
$$\times F_\nu{}^\mu(x(s))\exp(i\mathscr{H}s)] \ . \tag{9}$$

Now, the Heisenberg-picture operator $x^\mu(s)$ is given formally as

$$x^\mu(s)=e^{i\mathscr{H}s}x^\mu(0)e^{-i\mathscr{H}s}$$

and furthermore for any integer power of $x^\mu(s)$

$$[x^\mu(s)]^n=e^{i\mathscr{H}s}[x^\mu(0)]^n e^{-i\mathscr{H}s} \ .$$

Hence for any field $F_\nu{}^\mu(x)$, which can be expanded as a power series in x, we have

$$e^{-i\mathscr{H}s}F_\nu{}^\mu(x(s))e^{i\mathscr{H}s}=F_\nu{}^\mu(x(0)) \ .$$

Therefore Eq. (9) can be written in the form

$$\ddot{y}^\mu(s)=-y^\nu(s)M_\nu{}^\mu \ , \tag{10}$$

where the four-by-four matrix operator with elements

$$M_\nu{}^\mu=\mathscr{H}^2g_\nu{}^\mu-2ieF_\nu{}^\mu(x(0)) \tag{11}$$

is constant in proper time.

At first sight (10) may appear strangely nonlocal since y^μ, which is related to the velocity operator at proper time s, depends on the electromagnetic field at the position $x(0)$ at proper time zero. However, let us recall that in the proper-time Heisenberg picture developed in Ref. 1 the wave function is laid out for all space and time but is independent of proper time, while the dynamical variables, represented by Heisenberg operators, depend on s. Equations (10) and (11) mean simply that the external electromagnetic field should not be regarded as a dynamical variable depending on proper time, but like the wave function it is laid out in space-time.

Equation (10) yields the formal solution

$$y^\mu(s)=a^\nu[\exp(-iM^{1/2}s)]_\nu{}^\mu+b^\nu[\exp(iM^{1/2}s)]_\nu{}^\mu \ ,$$

where a^ν and b^ν are operators, constant in proper time, and $M^{1/2}$ is the square root of the matrix operator M. Hence

$$\gamma^\mu(s)=y^\mu(s)e^{-i\mathscr{H}s}$$
$$=a^\nu[\exp(-iM^{1/2}s)\exp(-i\mathscr{H}s)]_\nu{}^\mu$$
$$+b^\nu[\exp(iM^{1/2}s)\exp(-i\mathscr{H}s)]_\nu{}^\mu \ . \tag{12}$$

Since (6) is a second-order differential equation we require two initial conditions to specify its solution. The initial condition on $\gamma^\mu(s)$ is

$$\gamma^\mu(0)=a^\mu+b^\mu \tag{13}$$

while the initial condition on $\dot{\gamma}^\mu(s)$ gives

$$2\pi_\nu(0)\sigma^{\mu\nu}(0)=\gamma^\mu(0)$$
$$=-ia^\nu[M^{1/2}]_\nu{}^\mu-ia^\mu\mathscr{H}$$
$$+ib^\nu[M^{1/2}]_\nu{}^\mu-ib^\mu\mathscr{H} \ . \tag{14}$$

Henceforth all operators shall be taken at $s=0$ unless their dependence on s is given explicitly. Solving (13) simultaneously with (14), we find

$$a^\mu=\tfrac{1}{2}\gamma^\mu+\tfrac{1}{2}\pi^\nu[M^{-1/2}]_\nu{}^\mu-\frac{i}{2}\pi_\alpha\sigma^{\alpha\nu}[M^{-1/2}]_\nu{}^\mu \tag{15a}$$

and

$$b^\mu=\tfrac{1}{2}\gamma^\mu-\tfrac{1}{2}\pi^\nu[M^{-1/2}]_\nu{}^\mu+\frac{i}{2}\pi_\alpha\sigma^{\alpha\nu}[M^{-1/2}]_\nu{}^\mu \ . \tag{15b}$$

Thus if $M^{1/2}$, $M^{-1/2}$, and $\exp(\pm isM)$ can be found, the velocity operator for an electron in an arbitrary external electromagnetic field is given by (12) with (15).

III. METHOD OF SOLUTION FOR HEISENBERG EQUATIONS IN A HOMOGENEOUS EXTERNAL FIELD

When the external electromagnetic field is constant in space-time, the task of finding the square root of the matrix M becomes much simpler. We first diagonalize M by a similarity transformation

$$\begin{aligned} M' &= S^\dagger M S \\ &= \mathrm{diag}(\mathscr{H}^2+\lambda_j{}^2), \quad j=0,1,2,3 \end{aligned} \tag{16}$$

where S is a four-by-four unitary matrix, and $\lambda_j{}^2$ is the jth eigenvalue of the off-diagonal part $-2ie[F_\nu{}^\mu]$ of M. The similarity transformation which diagonalizes $[F_\nu{}^\mu]$ leaves the diagonal part $[\mathscr{H}^2 g_\nu{}^\mu]$ of M unchanged. This would not be the case if $F_\nu{}^\mu$ depended on x because then the matrix S would depend on x so that space-time derivatives in $\mathscr{H}$ would act on it and $S^\dagger[\mathscr{H}^2 g_\nu{}^\mu]S$ would not in general be diagonal.

Next we find the square root of M' by taking the square root of each of its diagonal elements. Using the fact that $\gamma^5=\gamma^0\gamma^1\gamma^2\gamma^3$ anticommutes with $\mathscr{H}=-\pi_\mu\gamma^\mu$ we can write

$$(\mathscr{H}^2+\lambda^2)^{1/2}=\pm(\mathscr{H}\pm i\lambda\gamma^5)\ . \tag{17}$$

Expression (17) shows an ambiguity in overall sign and an ambiguity in the relative sign between $\mathscr{H}$ and $(i\lambda_j\gamma^5)$. Changing the overall sign in (17) amounts to changing the sign of $M^{1/2}$ which amounts to simultaneously switching a^μ with b^μ and $\exp(iM^{1/2}s)$ with $\exp(-iM^{1/2}s)$ in (12), leaving the solution for $\gamma^\mu(s)$ unchanged. We shall also find that the remaining sign ambiguity in (17) does not present a problem.

The square root of M is then given by

$$M^{1/2}=S(M')^{1/2}S^\dagger\ . \tag{18}$$

Similarly we can exponentiate $M^{1/2}$ by exponentiating the diagonal matrix $M'^{\,1/2}$ and using

$$\exp(\pm iM^{1/2}s)=S\exp(\pm i(M')^{1/2}s)S^\dagger\ . \tag{19}$$

The calculation involved in the steps outlined above can be considerably simplified by taking the electric and magnetic fields in F to be parallel. So long as the field is non-null, i.e., does not satisfy

$$\mathbf{E}\cdot\mathbf{B}=0 \text{ and } |\mathbf{E}|^2-|\mathbf{B}|^2=0\ ,$$

this does not entail a loss of generality because one can always find a frame in which $\mathbf{E}$ and $\mathbf{B}$ lie in the same direction.[2]

In order to elucidate the separate effects of electric and magnetic fields on the motion of the electron, we shall obtain solutions for the velocity and position operators in an electric field and then in a magnetic field, both fields pointing in the third direction. Then, since the electric field affects only the zeroth and third components of the velocity and position nontrivially, while the magnetic field affects the one and two components, we can combine the two sets of solutions to give the motion of the electron in parallel electric and magnetic fields. The velocity and position operators for an electron in an otherwise arbitrary non-null homogeneous electromagnetic field can then be obtained by Lorentz transforming these solutions. For null fields another method has already been developed elsewhere.[3]

IV. RELATIVISTIC MOTION OF A SPINLESS CHARGE

We shall need, in order to separate out the center-of-mass motion of the electron, the motion in proper time of a relativistic spinless charge acted upon by the Lorentz force in a constant electric and constant magnetic field. We recall from classical relativistic dynamics that a particle with charge e has the proper-time velocity

$$v^\mu=\frac{\pi^\mu}{m}=\frac{p^\mu-eA^\mu}{m} \tag{20}$$

and obeys the equation of motion

$$m\dot v^\mu=eF^\mu{}_\nu v^\nu\ . \tag{21}$$

When $\mathbf{B}=(0,0,B)$ and $\mathbf{E}=(0,0,E)$, the solutions of (21) are

$$\begin{aligned} v^1(s)&=v^1(0)\cos\left[\frac{\mathscr{B}}{m}s\right]+v^2(0)\sin\left[\frac{\mathscr{B}}{m}s\right],\\ v^2(s)&=v^2(0)\cos\left[\frac{\mathscr{B}}{m}s\right]-v^1(0)\sin\left[\frac{\mathscr{B}}{m}s\right],\\ v^0(s)&=v^0(0)\cosh\left[\frac{\epsilon}{m}s\right]+v^3(0)\sinh\left[\frac{\epsilon}{m}s\right],\\ v^3(s)&=v^3(0)\cosh\left[\frac{\epsilon}{m}s\right]+v^0(0)\sinh\left[\frac{\epsilon}{m}s\right]. \end{aligned} \tag{22}$$

Integrating, we obtain

$$\begin{aligned} x^0(s)&=c^0+\frac{m}{\epsilon}\left[v^0(0)\sinh\left[\frac{\epsilon}{m}s\right]+v^3(0)\cosh\left[\frac{\epsilon}{m}s\right]\right],\\ x^1(s)&=c^1+\frac{m}{\mathscr{B}}\left[v^1(0)\sin\left[\frac{\mathscr{B}}{m}s\right]-v^2(0)\cos\left[\frac{\mathscr{B}}{m}s\right]\right],\\ x^2(s)&=c^2+\frac{m}{\mathscr{B}}\left[v^2(0)\sin\left[\frac{\mathscr{B}}{m}s\right]+v^1(0)\cos\left[\frac{\mathscr{B}}{m}s\right]\right],\\ x^3(s)&=c^3+\frac{m}{\epsilon}\left[v^3(0)\sinh\left[\frac{\epsilon}{m}s\right]+v^0(0)\cosh\left[\frac{\epsilon}{m}s\right]\right]. \end{aligned} \tag{23}$$

The particle moves with angular frequency B/m along a circle in the first and second directions and accelerates with hyperbolic frequency ϵ/m in the 0-3 plane along a hyperboloid:

$$(x^1-c^1)^2+(x^2-c^2)^2=\frac{m^2}{\mathscr{B}^2}[(v^1(0))^2+(v^2(0))^2]\ , \tag{24}$$

$$(x^0-c^0)^2-(x^3-c^3)^2=\frac{m^2}{\epsilon^2}[(v^0(0))^2-(v^3(0))^2]\ . \tag{25}$$

Results similar to (20), (22), and (23) can be obtained for a Klein-Gordon particle in the Heisenberg picture in proper time.

V. MOTION IN A HOMOGENEOUS ELECTRIC FIELD

For the case that $B=0$ and $\mathbf{E}=(0,0,E)$, the matrix M is given by

$$M=\begin{bmatrix} \mathscr{H}^2 & 0 & 0 & 2i\epsilon \\ 0 & \mathscr{H}^2 & 0 & 0 \\ 0 & 0 & \mathscr{H}^2 & 0 \\ 2i\epsilon & 0 & 0 & \mathscr{H}^2 \end{bmatrix} . \tag{26}$$

The square root of the submatrix $\begin{pmatrix} M_1^1 & M_1^2 \\ M_2^1 & M_2^2 \end{pmatrix}$ is $\begin{pmatrix} \mathscr{H} & 0 \\ 0 & \mathscr{H} \end{pmatrix}$. Using (12) and (15) we can immediately write solutions for the one and two components of the velocity

$$\gamma^j(s)=\tfrac{1}{2}(\gamma^j+\pi^j\mathscr{H}^{-1}-i\pi_\alpha\sigma^{\alpha j}\mathscr{H}^{-1})e^{-2i\mathscr{H}s} + \tfrac{1}{2}(\gamma^j-\pi^j\mathscr{H}^{-1}+i\pi_\alpha\sigma^{\alpha j}\mathscr{H}^{-1}),\quad j=1,2 . \tag{27}$$

In these solutions the condition that all operators act on states for which $\mathscr{H}^2=m^2$ has not been applied yet.

Let us now consider the submatrix

$$m=\begin{bmatrix} M_0{}^0 & M_0{}^3 \\ M_3{}^0 & M_3{}^3 \end{bmatrix}=\begin{bmatrix} \mathscr{H}^2 & 2i\epsilon \\ 2i\epsilon & \mathscr{H}^2 \end{bmatrix} .$$

By inspection we see that the matrix which diagonalizes m is

$$S=\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}=S^\dagger \tag{28}$$

and

$$m'=\frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} \mathscr{H}^2 & 2i\epsilon \\ 2i\epsilon & \mathscr{H}^2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} \mathscr{H}^2+2i\epsilon & 0 \\ 0 & \mathscr{H}^2-2i\epsilon \end{bmatrix} .$$

Equation (17) gives four possibilities for the square root of each diagonal element in m'. From these possibilities we choose

$$(m')^{1/2}=\begin{bmatrix} \mathscr{H}+\sqrt{\epsilon}(1-i)\gamma^5 & 0 \\ 0 & \mathscr{H}+\sqrt{\epsilon}(1+i)\gamma^5 \end{bmatrix}$$

and using (18) we find

$$m^{1/2}=\begin{bmatrix} \mathscr{H}+\sqrt{\epsilon}\gamma^5 & -i\sqrt{\epsilon}\gamma^5 \\ -i\sqrt{\epsilon}\gamma^5 & \mathscr{H}+\sqrt{\epsilon}\gamma^5 \end{bmatrix} . \tag{29}$$

To find the inverse of $m^{1/2}$ we let

$$m^{-1/2}=\begin{bmatrix} A & B \\ B & A \end{bmatrix}$$

and write

$$\begin{bmatrix} \mathscr{H}+\sqrt{\epsilon}\gamma^5 & -i\sqrt{\epsilon}\gamma^5 \\ -i\sqrt{\epsilon}\gamma^5 & \mathscr{H}\sqrt{\epsilon}\gamma^5 \end{bmatrix}\begin{bmatrix} A & B \\ B & A \end{bmatrix}=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} ,$$

giving two simultaneous equations for A and B which yield the solutions

$$B=-i(\mathscr{H}^4+4\epsilon^2)^{-1}(2\epsilon\mathscr{H}+\sqrt{\epsilon}(\mathscr{H}^2+2\epsilon)\gamma^5) = -i(2\epsilon\mathscr{H}+\sqrt{\epsilon}(\mathscr{H}^2+2\epsilon)\gamma^5)(\mathscr{H}^4+4\epsilon^2)^{-1}$$

and

$$A=(\mathscr{H}^3+\sqrt{\epsilon}(\mathscr{H}^2-2\epsilon)\gamma^5)(\mathscr{H}^4+4\epsilon^2)^{-1} .$$

We have used the fact that even powers of $\mathscr{H}$ commute with $\mathscr{H}$ and γ^5 and therefore every term in the power series expansion for the exponential in

$$(\mathscr{H}^4+4\epsilon^2)^{-1}=-i\int_0^\infty d\tau \exp[i\tau(\mathscr{H}^4+4\epsilon^2+i\delta)] ,$$

where δ is an infinitesimal, can be commuted to the right. Thus we have

$$m^{-1/2}=\begin{bmatrix} (\mathscr{H}^3+\sqrt{\epsilon}(\mathscr{H}^2-2\epsilon)\gamma^5)(\mathscr{H}^4+4\epsilon^2)^{-1} & -i(2\epsilon\mathscr{H}+\sqrt{\epsilon}(\mathscr{H}^2+2\epsilon)\gamma^5)(\mathscr{H}^4+4\epsilon^2)^{-1} \\ -i(2\epsilon\mathscr{H}+\sqrt{\epsilon}(\mathscr{H}^2+2\epsilon)\gamma^5)(\mathscr{H}^4+4\epsilon^2)^{-1} & (\mathscr{H}^3+\sqrt{\epsilon}(\mathscr{H}^2-2\epsilon)\gamma^5)(\mathscr{H}^4+4\epsilon^2)^{-1} \end{bmatrix} . \tag{30}$$

To find $\exp(iM^{1/2}s)$ and $\exp(-iM^{1/2}s)$ we use (19) and obtain

$$\begin{aligned}
\exp(iM^{1/2}s)^0{}_0&=\tfrac{1}{2}(e^{is(\mathscr{H}+\sqrt{\epsilon}(1-i)\gamma^5)}+e^{is(\mathscr{H}+\sqrt{\epsilon}(1+i)\gamma^5)})=\exp(iM^{1/2}s)^3{}_3 ,\\
\exp(iM^{1/2}s)^0{}_3&=\tfrac{1}{2}(e^{is(\mathscr{H}+\sqrt{\epsilon}(1-i)\gamma^5)}-e^{is(\mathscr{H}+(1+i)\gamma^5)})=\exp(iM^{1/2}s)^3{}_0 ,\\
\exp(-iM^{1/2}s)^0{}_0&=\tfrac{1}{2}(e^{-is(\mathscr{H}+\sqrt{\epsilon}(1-i)\gamma^5)}+e^{-is(\mathscr{H}+\sqrt{\epsilon}(1+i)\gamma^5)})=\exp(-iM^{1/2}s)^3{}_3 ,\\
\exp(-iM^{1/2}s)^0{}_3&=\tfrac{1}{2}(e^{-is(\mathscr{H}+\sqrt{\epsilon}(1-i)\gamma^5)}-e^{-is(\mathscr{H}+\sqrt{\epsilon}(1+i)\gamma^5)})=\exp(-iM^{1/2}s)^3{}_0 .
\end{aligned} \tag{31}$$

Combining (31), (30), and (15) in (12) we find the following operator solutions for $\gamma^0(s)$ and $\gamma^3(s)$:

$$
\begin{aligned}
\gamma^0(s)=&\ \tfrac{1}{2}\gamma^0[\cos(\mathscr{H}+\sqrt{\epsilon}(1-i)\gamma^5)s+\cos(\mathscr{H}+\sqrt{\epsilon}(1+i)\gamma^5)s]e^{-i\mathscr{H}s}\\
&\times\tfrac{1}{2}\gamma^3[\cos(\mathscr{H}+\sqrt{\epsilon}(1-i)\gamma^5)s-\cos(\mathscr{H}+\sqrt{\epsilon}(1+i)\gamma^5)s]e^{-i\mathscr{H}s}\\
&-\frac{i}{2}\pi^0\frac{(\mathscr{H}^3+\sqrt{\epsilon}(\mathscr{H}^2-2\epsilon)\gamma^5)}{\mathscr{H}^4+4\epsilon^2}[\sin(\mathscr{H}+\sqrt{\epsilon}(1-i)\gamma^5)s+\sin(\mathscr{H}+\sqrt{\epsilon}(1+i)\gamma^5)s]e^{-i\mathscr{H}s}\\
&-\frac{1}{2}\pi^3\frac{(2\epsilon\mathscr{H}+\sqrt{\epsilon}(\mathscr{H}^2+2\epsilon)\gamma^5)}{\mathscr{H}^4+4\epsilon^2}[\sin(\mathscr{H}+\sqrt{\epsilon}(1-i)\gamma^5)s+\sin(\mathscr{H}+\sqrt{\epsilon}(1+i)\gamma^5)s]e^{-i\mathscr{H}s}\\
&-\frac{1}{2}\pi_\alpha\sigma^{\alpha 0}\frac{(\mathscr{H}^3+\sqrt{\epsilon}(\mathscr{H}^2-2\epsilon)\gamma^5)}{\mathscr{H}^2+4\epsilon^2}[\sin(\mathscr{H}+\sqrt{\epsilon}(1-i)\gamma^5)s+\sin(\mathscr{H}+\sqrt{\epsilon}(1+i)\gamma^5)s]e^{-i\mathscr{H}s}\\
&+\frac{i}{2}\pi_\alpha\sigma^{\alpha 3}\frac{(2\epsilon\mathscr{H}+\sqrt{\epsilon}(\mathscr{H}^2+2\epsilon)\gamma^5)}{\mathscr{H}^4+4\epsilon^2}[\sin(\mathscr{H}+\sqrt{\epsilon}(1-i)\gamma^5)s+\sin(\mathscr{H}+\sqrt{\epsilon}(1+i)\gamma^5)s]e^{-i\mathscr{H}s}\\
&-\frac{i}{2}\pi^3\frac{(\mathscr{H}^3+\sqrt{\epsilon}(\mathscr{H}^2-2\epsilon)\gamma^5)}{\mathscr{H}^4+4\epsilon^2}[\sin(\mathscr{H}+\sqrt{\epsilon}(1-i)\gamma^5)s-\sin(\mathscr{H}+\sqrt{\epsilon}(1+i)\gamma^5)s]e^{-i\mathscr{H}s}\\
&-\frac{1}{2}\pi^0\frac{(2\epsilon\mathscr{H}+\sqrt{\epsilon}(\mathscr{H}^2+2\epsilon)\gamma^5)}{\mathscr{H}^4+4\epsilon^2}[\sin(\mathscr{H}+\sqrt{\epsilon}(1-i)\gamma^5)s-\sin(\mathscr{H}+\sqrt{\epsilon}(1+i)\gamma^5)s]e^{-i\mathscr{H}s}\\
&-\frac{1}{2}\pi_\alpha\sigma^{\alpha 3}\frac{(\mathscr{H}^3+\sqrt{\epsilon}(\mathscr{H}^2-2\epsilon)\gamma^5)}{\mathscr{H}^4+4\epsilon^2}[\sin(\mathscr{H}+\sqrt{\epsilon}(1-i)\gamma^5)s-\sin(\mathscr{H}+\sqrt{\epsilon}(1+i)\gamma^5)s]e^{-i\mathscr{H}s}\\
&+\frac{i}{2}\pi_\alpha\sigma^{\alpha 0}\frac{(2\epsilon\mathscr{H}+\sqrt{\epsilon}(\mathscr{H}^2+2\epsilon)\gamma^5)}{\mathscr{H}^4+4\epsilon^2}[\sin(\mathscr{H}+\sqrt{\epsilon}(1-i)\gamma^5)s\ ,\\
\gamma^3(s)=&\ (\text{same with } 0\leftrightarrow 3)\ ,
\end{aligned}
\tag{32}
$$

where the sines, cosines, and exponentials are defined by their power-series expansions.

In order to interpret the solutions (27) and (32) physically we need to impose the condition that they act upon states for which

$$\mathscr{H}^2\psi=m^2\psi\ , \tag{33}$$

i.e., positive- and negative-frequency solutions of the Dirac equation. With this restriction we have

$$
\begin{aligned}
&\cos[\mathscr{H}+\sqrt{\epsilon}(1+i)\gamma^5]s=\cos\Omega_2 s\ ,\\
&\cos[\mathscr{H}+\sqrt{\epsilon}(1+i)\gamma^5]s=[\mathscr{H}+\sqrt{\epsilon}(1+i)\gamma^5]\frac{\sin\Omega_2 s}{\Omega_2}\ ,\\
&\cos[\mathscr{H}+\sqrt{\epsilon}(1-i)\gamma^5]s=\cos\Omega_1 s\ ,\\
&\sin[\mathscr{H}+\sqrt{\epsilon}(1-i)\gamma^5]s=[\mathscr{H}+\sqrt{\epsilon}(1-i)\gamma^5]\frac{\sin\Omega_1 s}{\Omega_1}\ ,
\end{aligned}
\tag{34}
$$

where

$$
\begin{aligned}
&\Omega_2=(m^2-2i\epsilon)^{1/2}=\omega-i\mu\ ,\\
&\Omega_1=(m^2+2i\epsilon)^{1/2}=\omega+i\mu\ ,\\
&\omega=\mathrm{Re}\Omega_1=\frac{1}{\sqrt{2}}\{[(m^4+4\epsilon^2)^{1/2}+m^2]^{1/2}\}\ ,\\
&\mu=\mathrm{Im}\Omega_1=\frac{1}{\sqrt{2}}\{[(m^4+4\epsilon^2)^{1/2}-m^2]^{1/2}\}\ ,
\end{aligned}
\tag{35}
$$

as can be seen from the power series of the sines and cosines. It is possible to set $\mathscr{H}^2=m^2$ in (32) because, as we have mentioned before, even powers of $\mathscr{H}$ commute with γ^5. We also use $\mathscr{H}^{-1}=\mathscr{H}/m^2$ in (27).

We also encounter the term $\pi_\alpha\sigma^{\alpha\mu}$. Since this term acts on states for which $\mathscr{H}^2=m^2$ we can write

$$
\begin{aligned}
\pi_\alpha\sigma^{\alpha\mu}\mathscr{H}&=-\frac{i}{2}(\pi\cdot\gamma\gamma^\mu-\gamma^\mu\pi\cdot\gamma)\pi\cdot\gamma\\
&=-\frac{i}{2}(\pi^\mu-i\pi_\alpha\sigma^{\alpha\mu})\pi\cdot\gamma+\frac{i}{2}m^2\gamma^\mu\\
&=\tfrac{1}{2}\pi_\alpha\sigma^{\alpha\mu}\mathscr{H}+\frac{i}{2}(m^2\gamma^\mu-\pi^\mu\pi\cdot\gamma)
\end{aligned}
$$

and obtain

$$\pi_\alpha\sigma^{\alpha\mu}\mathscr{H}=i(m^2\gamma^\mu-\pi^\mu\pi\cdot\gamma)\ . \tag{36}$$

Inserting these relations into (32) we then obtain the following expressions for the components of the velocity of an electron in the constant electric field $\mathbf{E}=(0,0,E)$:

$$\gamma^j(s)=\frac{\pi^j\pi\cdot\gamma}{m^2}+\left[\gamma^j-\frac{\pi^j\pi\cdot\gamma}{m^2}\right]\cos 2ms-\frac{\pi_\alpha}{m}\sigma^{\alpha j}\sin 2ms \tag{37a}$$

for $j=1,2$, and

$$
\begin{aligned}
\gamma^0(s)=&\ \cos((\omega+m)s)\cosh(\mu s)\left[\frac{1}{2}\gamma^0\frac{\omega^2+\mu^2+m\omega}{\omega^2+\mu^2}-\frac{i}{2}\gamma^3\frac{m\mu}{\omega^2+\mu^2}-\frac{\pi^0\pi\cdot\gamma\omega}{m(\omega^2+\mu^2)}+\frac{i\pi^3\pi\cdot\gamma\mu}{m(\omega^2+\mu^2)}\right]\\
&+\sin((\omega+m)s)\sinh(\mu s)\left[-\frac{1}{2}\gamma^0\frac{\mu m}{\omega^2+\mu^2}-\frac{i}{2}\gamma^3\frac{\omega^2+\mu^2+m\omega}{\omega^2+\mu^2}+\frac{\pi^0\pi\cdot\gamma\mu}{m(\omega^2+\mu^2)}+\frac{i\pi^3\pi\cdot\gamma\omega}{m(\omega^2+\mu^2)}\right]
\end{aligned}
$$

$$+\sin((\omega+m)s)\cosh(\mu s)\left[\frac{i}{2}\pi^0\frac{\omega^2+\mu^2-m\omega}{m(\omega^2+\mu^2)}-\frac{1}{2}\pi^3\frac{\mu}{\omega^2+\mu^2}-\frac{1}{2}\pi_\alpha\sigma^{\alpha 0}\frac{\omega^2+\mu^2+m\omega}{m(\omega^2+\mu^2)}+\frac{i}{2}\pi_\alpha\sigma^{\alpha 3}\frac{\mu}{\omega^2+\mu^2}\right]$$

$$+\cos((\omega+m)s)\sinh(\mu s)\left[-\frac{i}{2}\pi^0\frac{\mu}{\omega^2+\mu^2}-\frac{1}{2}\pi^3\frac{\omega^2+\mu^2-m\omega}{m(\omega^2+\mu^2)}-\frac{1}{2}\pi_\alpha\sigma^{\alpha 0}\frac{\mu}{\omega^2+\mu^2}-\frac{i}{2}\pi_\alpha\sigma^{\alpha 3}\frac{\omega^2+\mu^2+m\omega}{m(\omega^2+\mu^2)}\right]$$

$$+\cos((\omega-m)s)\cosh(\mu s)\left[\frac{1}{2}\gamma^0\frac{\omega^2+\mu^2-m\omega}{\omega^2+\mu^2}+\frac{i}{2}\gamma^3\frac{m\mu}{\omega^2+\mu^2}+\frac{\pi^0\pi\cdot\gamma\omega}{m(\omega^2+\mu^2)}-\frac{i\pi^3\pi\cdot\gamma\mu}{m(\omega^2+\mu^2)}\right]$$

$$+\sin((\omega-m)s)\sinh(\mu s)\left[\frac{1}{2}\gamma^0\frac{m\mu}{\omega^2+\mu^2}-\frac{i}{2}\gamma^3\frac{\omega^2+\mu^2-m\omega}{\omega^2+\mu^2}-\frac{\pi^0\pi\cdot\gamma\mu}{m(\omega^2+\mu^2)}-\frac{i\pi^3\pi\cdot\gamma\omega}{m(\omega^2+\mu^2)}\right]$$

$$+\sin((\omega-m)s)\cosh(\mu s)\left[-\frac{i}{2}\pi^0\frac{\omega^2+\mu^2+m\omega}{m(\omega^2+\mu^2)}-\frac{1}{2}\frac{\pi^3\mu}{\omega^2+\mu^2}+\frac{1}{2}\pi_\alpha\sigma^{\alpha 0}\frac{\omega^2+\mu^2-m\omega}{m(\omega^2+\mu^2)}+\frac{i}{2}\pi_\alpha\sigma^{\alpha 3}\frac{\mu}{\omega^2+\mu^2}\right]$$

$$+\cos((\omega-m)s)\sinh(\mu s)\left[-\frac{i}{2}\pi^0\frac{\mu}{\omega^2+\mu^2}+\frac{1}{2}\pi^3\frac{\omega^2+\mu^2+m\omega}{m(\omega^2+\mu^2)}-\frac{1}{2}\pi_\alpha\sigma^{\alpha 0}\frac{\mu}{\omega^2+\mu^2}+\frac{i}{2}\pi_\alpha\sigma^{\alpha 3}\frac{\omega^2+\mu^2-m\omega}{m(\omega^2+\mu^2)}\right],$$

$$\gamma^3(s)=(\text{same with }0\leftrightarrow 3)\ . \tag{37c}$$

Integrating the velocity operator over proper time we obtain the position operator

$$x^j(s)=c^j+\frac{\pi^j\pi\cdot\gamma}{m^2}s+\frac{\pi_\alpha\sigma^{\alpha j}}{2m^2}\cos 2ms+\left[\frac{1}{2m}\right]\left[\gamma^j-\frac{\pi^j\pi\cdot\gamma}{m^2}\right]\sin 2ms,\quad j=1,2 \tag{38a}$$

$$x^0(s)=c^0+\sin((\omega+m)s)\cosh(\mu s)\left[\tfrac{1}{2}\gamma^0\frac{\omega}{\omega^2+\mu^2}-\frac{i}{2}\gamma^3\frac{\mu}{\omega^2+\mu^2}+\frac{(\pi^0\pi\cdot\gamma(\mu^2-\omega^2-m\omega)+i\pi^3\pi\cdot\gamma(2\omega+m)\mu)}{m(\omega^2+\mu^2)[(\omega+m)^2+\mu^2]}\right]$$

$$+\cos((\omega+m)s)\sinh(\mu s)\left[\frac{1}{2}\gamma^0\frac{\mu}{\omega^2+\mu^2}+\frac{i}{2}\gamma^3\frac{\omega}{\omega^2+\mu^2}+\frac{(-\pi^0\pi\cdot\gamma(2\omega+m)\mu+i\pi^3\pi\cdot\gamma(\mu^2-\omega^2-m\omega))}{m(\omega^2+\mu^2)[(\omega+m)^2+\mu^2]}\right]$$

$$+\cos((\omega-m)s)\cosh(\mu s)\left[\frac{1}{2}\frac{[i\pi^0(\omega(\omega^2+\mu^2-m^2)-2m\mu^2)+\pi^3\mu(\omega^2+2m\omega+\mu^2-m^2)]}{m(\omega^2+\mu^2)((\omega-m)^2+\mu^2)}\right.$$
$$\left.-\frac{\pi_\alpha\sigma^{\alpha 0}\omega}{2m(\omega^2+\mu^2)}+i\frac{\pi_\alpha\sigma^{\alpha 3}\mu}{2m(\omega^2+\mu^2)}\right]$$

$$+\sin((\omega-m)s)\cosh(\mu s)\left[\frac{[-i\pi^0\mu(\omega^2+\mu^2+2m\omega-m^2)+\pi^3(\omega(\omega^2+\mu^2-m^2)-2m\mu^2)]}{m(\omega^2+\mu^2)((\omega-m)^2+\mu^2)}\right.$$
$$\left.+\frac{\pi_\alpha\sigma^{\alpha 0}\mu}{2m(\omega^2+\mu^2)}+\frac{i\pi_\alpha\sigma^{\alpha 3}\omega}{2m(\omega^2+\mu^2)}\right]$$

$$+\cos((\omega+m)s)\cosh(\mu s)\left[\frac{1}{2}\frac{[-i\pi^0(\omega(\omega^2+\mu^2-m^2)+2m\mu^2)+\pi^3\mu(m^2+2m\omega-\omega^2-\mu^2)]}{m(\omega^2+\mu^2)((\omega+m)^2+\mu^2)}\right.$$
$$\left.+\frac{1}{2}\frac{\pi_\alpha\sigma^{\alpha 0}\omega}{m(\omega^2+\mu^2)}-\frac{i}{2}\frac{\pi_\alpha\sigma^{\alpha 3}\mu}{m(\omega^2+\mu^2)}\right]$$

$$+\sin((\omega+m)s)\sinh(\mu s)\left[\frac{1}{2}\frac{[i\pi^0\mu(\omega^2+\mu^2-m^2-2m\omega)-\pi^3(\omega(\omega^2+\mu^2-m^2)+2m\mu^2)]}{m(\omega^2+\mu^2)((\omega+m)^2+\mu^2)}\right.$$
$$\left.-\frac{\pi_\alpha\sigma^{\alpha 0}\mu}{2m(\omega^2+\mu^2)}-\frac{i\pi_\alpha\sigma^{\alpha 3}\omega}{2m(\omega^2+\mu^2)}\right]$$

$$+\sin((\omega-m)s)\cosh(\mu s)\left[\frac{1}{2}\gamma^0\frac{\omega}{\omega^2+\mu^2}-\frac{i}{2}\gamma^3\frac{\mu}{\omega^2+\mu^2}+\frac{(\pi^0\pi\cdot\gamma(\omega^2-m\omega-\mu^2)+i\pi^3\pi\cdot\gamma\mu(m-2\omega))}{m(\omega^2+\mu^2)((\omega-m)^2+\mu^2)}\right]$$

$$+\cos((\omega-m)s)\sinh(\mu s)\left[\frac{1}{2}\gamma^0\frac{\mu}{\omega^2+\mu^2}+\frac{i}{2}\gamma^3\frac{\omega}{\omega^2+\mu^2}+\frac{[\pi^0\pi\cdot\gamma(\mu(2\omega-m))+i\pi^3\pi\cdot\gamma(\omega^2\mu^2-m\omega)]}{m(\omega^2+\mu^2)((\omega-m)^2+\mu^2)}\right], \tag{38b}$$

$$x^3(s)=(\text{same with }0\leftrightarrow 3)\ , \tag{38c}$$

where c^μ is a constant operator determined by the initial condition.

Expressions (37a) and (38a) agree with the corresponding free particle expressions with p_μ replaced by $\pi_\mu = p_\mu - eA_\mu$. This result is expected since the electric field exerts no force in the first or second directions so that motion in these directions should correspond as closely as possible to that of the free particle, while at the same time p must be replaced by π in order that gauge invariance be preserved.

It is remarkable that although γ^5 appeared in the solution (32), we had to introduce it in order to find the square root (29), all terms containing γ^5 have disappeared from (37) and (38). The only Dirac matrices appearing in (37) and (38) are γ^μ and $\sigma^{\mu\nu}$, with μ and ν ranging from zero to three. These are the same Dirac matrices which appear in the free particle solutions of Ref. 1. This means that the SO(3,2) dynamical system described in Ref. 1 has not been enlarged to the SO(4,2) dynamical system as a result of the interaction with an electric field.

Equations (37) and (38) also reveal that there is no simple separation between *Zitterbewegung* and center-of-mass motion since all of the terms in these solutions are oscillatory. We see from these solutions that the position and velocity operators oscillate with frequencies $\omega \pm m$ while at the same time accelerating hyperbolically with hyperbolic frequency μ.

VI. MOTION IN A HOMOGENEOUS MAGNETIC FIELD

Now let $\mathbf{E}=0$ and $\mathbf{B}=(0,0,B)$. In this case M is given by

$$M = \begin{bmatrix} \mathscr{H}^2 & 0 & 0 & 0 \\ 0 & \mathscr{H}^2 & -2i\mathscr{B} & 0 \\ 0 & 2i\mathscr{B} & \mathscr{H}^2 & 0 \\ 0 & 0 & 0 & \mathscr{H}^2 \end{bmatrix}, \tag{39}$$

where $\mathscr{B} = eB$. Again we can immediately write down solutions for two components of the velocity, in this case the zero and three components

$$\gamma^\mu(s) = \tfrac{1}{2}(\gamma^\mu + \pi^\mu \mathscr{H}^{-1} - i\pi_\alpha \sigma^{\alpha\mu}\mathscr{H}^{-1})e^{-2i\mathscr{H}s} + \tfrac{1}{2}(\gamma^\mu - \pi^\mu\mathscr{H}^{-1} + i\pi_\alpha\sigma^{\alpha\mu}\mathscr{H}^{-1}), \quad \mu = 0,3 \,. \tag{40}$$

To solve for the remaining two components we consider the submatrix

$$m = \begin{bmatrix} M^1{}_1 & M^1{}_2 \\ M^1{}_1 & M^2{}_2 \end{bmatrix} = \begin{bmatrix} \mathscr{H}^2 & -2i\mathscr{B} \\ 2i\mathscr{B} & \mathscr{H}^2 \end{bmatrix}. \tag{41}$$

This submatrix is diagonalized via a similarity transformation by

$$S = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}, \quad S^\dagger = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix} \tag{42}$$

giving

$$m' = \begin{bmatrix} \mathscr{H}^2 + 2i\mathscr{B} & 0 \\ 0 & \mathscr{H}^2 - 2i\mathscr{B} \end{bmatrix}.$$

Taking the square root of m' and inverting the similarity transformation that diagonalized m, we obtain

$$m^{1/2} = \begin{bmatrix} \mathscr{H} + \left[\dfrac{1+i}{\sqrt{2}}\right]\gamma^5\sqrt{\mathscr{B}} & \left[\dfrac{1+i}{\sqrt{2}}\right]\gamma^5\sqrt{\mathscr{B}} \\ -\left[\dfrac{1+i}{\sqrt{2}}\right]\gamma^5\sqrt{\mathscr{B}} & \mathscr{H} + \left[\dfrac{1+i}{\sqrt{2}}\right]\gamma^5\sqrt{\mathscr{B}} \end{bmatrix}. \tag{43}$$

This matrix has the following inverse:

$$m^{-1/2} = \begin{bmatrix} (\mathscr{H}^3 + \gamma^5\sqrt{i\mathscr{B}}(\mathscr{H}^2 - 2i\mathscr{B}))(\mathscr{H}^4 - 4\mathscr{B}^2)^{-1} & (2i\mathscr{B}\mathscr{H} + \gamma^5\sqrt{i\mathscr{B}}(\mathscr{H}^2 + 2i\mathscr{B}))(\mathscr{H}^4 - 4\mathscr{B}^2)^{-1} \\ (-2i\mathscr{B}\mathscr{H} - \gamma^5\sqrt{i\mathscr{B}}(\mathscr{H}^2 + 2i\mathscr{B}))(\mathscr{H}^4 - 4\mathscr{B}^2)^{-1} & (\mathscr{H}^3 + \gamma^5\sqrt{i\mathscr{B}}(\mathscr{H}^2 - 2i\mathscr{B}))(\mathscr{H}^4 - 4\mathscr{B}^2)^{-1} \end{bmatrix}. \tag{44}$$

Exponentiating $m^{1/2}$ we find

$$\exp(im^{1/2}s)^1{}_1 = \tfrac{1}{2}(e^{i(\mathscr{H}+i\sqrt{2\mathscr{B}}\gamma^5)s} + e^{i(\mathscr{H}+\sqrt{2\mathscr{B}}\gamma^5)s}) = \exp(im^{1/2}s)^2{}_2 \,,$$

$$\exp(im^{1/2}s)^1{}_2 = \frac{1}{2i}(e^{i(\mathscr{H}+i\sqrt{2\mathscr{B}}\gamma^5)s} - e^{i(\mathscr{H}+\sqrt{2\mathscr{B}}\gamma^5)s}) = -\exp(im^{1/2}s)^2{}_1 \,,$$

and (45)

$$\exp(-im^{1/2}s)^1{}_1 = \tfrac{1}{2}(e^{-i(\mathscr{H}+i\sqrt{2\mathscr{B}}\gamma^5)s} + e^{-i(\mathscr{H}+\sqrt{2\mathscr{B}}\gamma^5)s}) = -\exp(-im^{1/2}s)^2{}_2 \,,$$

$$\exp(-im^{1/2}s)^1{}_2 = \frac{1}{2i}(e^{-i(\mathscr{H}+i\sqrt{2\mathscr{B}}\gamma^5)s} - e^{-i(\mathscr{H}+\sqrt{2\mathscr{B}}\gamma^5)s}) = -\exp(-im^{1/2}s)^2{}_1 \,,$$

Finally, combining terms, we obtain the solutions

$$\gamma^1(s) = \tfrac{1}{2}\gamma^1[\cos(\mathscr{H}+i\gamma^5\sqrt{2\mathscr{B}})s + \cos(\mathscr{H}+\gamma^5\sqrt{2\mathscr{B}})s]e^{-i\mathscr{H}s}$$

$$+ \frac{i}{2}\gamma^2[\cos(\mathscr{H}+i\gamma^5\sqrt{2\mathscr{B}})s - \cos(\mathscr{H}+\gamma^5\sqrt{2\mathscr{B}})s]e^{-i\mathscr{H}s}$$

$$
\begin{aligned}
&-\frac{i}{2}\pi^1\frac{(\mathscr{H}^3+\sqrt{i\mathscr{B}}(\mathscr{H}^2-2i\mathscr{B})\gamma^5)}{\mathscr{H}^4-4\mathscr{B}^2}[\sin(\mathscr{H}+i\gamma^5\sqrt{2\mathscr{B}})s+\sin(\mathscr{H}+\gamma^5\sqrt{2\mathscr{B}})s]e^{-i\mathscr{H}s}\\
&+\frac{i}{2}\pi^2\frac{(2i\mathscr{B}\mathscr{H}+\sqrt{i\mathscr{B}}(\mathscr{H}^2+2i\mathscr{B})\gamma^5)}{\mathscr{H}^4-4\mathscr{B}^2}[\sin(\mathscr{H}+i\gamma^5\sqrt{2\mathscr{B}})s+\sin(\mathscr{H}+\gamma^5\sqrt{2\mathscr{B}})s]e^{-i\mathscr{H}s}\\
&-\frac{1}{2}\pi_\alpha\sigma^{\alpha 1}\frac{(\mathscr{H}+\sqrt{i\mathscr{B}}(\mathscr{H}^2-2i\mathscr{B})\gamma^5)}{\mathscr{H}-4\mathscr{B}^2}[\sin(\mathscr{H}+i\gamma^5\sqrt{2\mathscr{B}})s+\sin(\mathscr{H}+\gamma^5\sqrt{2\mathscr{B}})s]e^{-i\mathscr{H}s}\\
&+\frac{1}{2}\pi_\alpha\sigma^{\alpha 2}\frac{(2i\mathscr{B}\mathscr{H}+\sqrt{i\mathscr{B}}(\mathscr{H}^2+2i\mathscr{B})\gamma^5)}{\mathscr{H}^4-4\mathscr{B}^2}[\sin(\mathscr{H}+i\gamma^5\sqrt{2\mathscr{B}})s+\sin(\mathscr{H}+\gamma^5\sqrt{2\mathscr{B}})s]e^{-i\mathscr{H}s}\\
&+\frac{1}{2}\pi^1\frac{(2i\mathscr{B}\mathscr{H}+\sqrt{i\mathscr{B}}(\mathscr{H}^2+2i\mathscr{B})\gamma^5)}{\mathscr{H}^4-4\mathscr{B}^2}[\sin(\mathscr{H}+i\gamma^5\sqrt{2\mathscr{B}})s-\sin(\mathscr{H}+\gamma^5\sqrt{2\mathscr{B}})s]e^{-i\mathscr{H}s}\\
&+\frac{1}{2}\pi^2\frac{(\mathscr{H}^3+\sqrt{i\mathscr{B}}(\mathscr{H}^2-2i\mathscr{B})\gamma^5)}{\mathscr{H}^4-4\mathscr{B}^2}[\sin(\mathscr{H}+i\gamma^5\sqrt{2\mathscr{B}})s-\sin(\mathscr{H}+\gamma^5\sqrt{2\mathscr{B}})s]e^{-i\mathscr{H}s}\\
&-\frac{i}{2}\pi_\alpha\sigma^{\alpha 1}\frac{(2i\mathscr{B}\mathscr{H}+\sqrt{i\mathscr{B}}(\mathscr{H}^2+2i\mathscr{B})\gamma^5)}{\mathscr{H}^4-4\mathscr{B}^2}[\sin(\mathscr{H}+i\gamma^5\sqrt{2\mathscr{B}})s-\sin(\mathscr{H}+\gamma^5\sqrt{2\mathscr{B}})s]e^{-i\mathscr{H}s}\\
&-\frac{i}{2}\pi_\alpha\sigma^{\alpha 2}\frac{(\mathscr{H}^3+\sqrt{i\mathscr{B}}(\mathscr{H}^2-2i\mathscr{B})\gamma^5)}{\mathscr{H}^4-4\mathscr{B}^2}[\sin(\mathscr{H}+i\gamma^5\sqrt{2\mathscr{B}})s-\sin(\mathscr{H}+\gamma^5\sqrt{2\mathscr{B}})s]e^{-i\mathscr{H}s}\,,
\end{aligned}
\tag{46}
$$

$\gamma^2(s)=(\text{same with } \pi^1\to\pi^2, \pi^2\to-\pi^1, \text{etc.})$.

We again use the condition that, acting to the right, $\mathscr{H}^2=m^2$ and find

$$
\begin{aligned}
&\cos(\mathscr{H}+i\gamma^5\sqrt{2\mathscr{B}})s=\cos(\omega_1 s),\quad \sin(\mathscr{H}+i\gamma^5\sqrt{2\mathscr{B}})s=(\mathscr{H}+i\gamma^5\sqrt{2\mathscr{B}})\frac{\sin(\omega_1 s)}{\omega_1}\,,\\
&\cos(\mathscr{H}+\gamma^5\sqrt{2\mathscr{B}})s=\cos(\omega_2 s),\quad \sin(\mathscr{H}+\gamma^5\sqrt{2\mathscr{B}})s=(\mathscr{H}+\gamma^5\sqrt{2\mathscr{B}})\frac{\sin(\omega_2 s)}{\omega_2}\,,
\end{aligned}
\tag{47}
$$

with

$$\omega_1=(m^2+2\mathscr{B})^{1/2} \text{ and } \omega_2=(m^2-2\mathscr{B})^{1/2} \tag{48}$$

leading to the following results for the velocity operator:

$$
\begin{aligned}
\gamma^1(s)=&\tfrac{1}{2}\cos(\omega_1+m)s\left[\tfrac{1}{2}(\gamma^1+i\gamma^2)\left[\frac{\omega_1+m}{\omega_1}\right]-\frac{(\pi^1+i\pi^2)\pi\cdot\gamma}{m\omega_1}\right]\\
&+\tfrac{1}{2}\sin(\omega_1+m)s\left[\frac{i}{2}(\pi^1+i\pi^2)\left[\frac{\omega_1-m}{m\omega_1}\right]-\frac{1}{2}\frac{\pi_\alpha}{m}(\sigma^{\alpha 1}+i\sigma^{\alpha 2})\frac{\omega_1+m}{\omega_1}\right]\\
&+\tfrac{1}{2}\cos(\omega_1-m)s\left[\tfrac{1}{2}(\gamma^1+i\gamma^2)\left[\frac{\omega_1-m}{\omega_1}\right]+\frac{(\pi^1+i\pi^2)\pi\cdot\gamma}{m\omega_1}\right]\\
&+\tfrac{1}{2}\sin(\omega_1-m)s\left[-\frac{i}{2}\frac{(\pi^1+i\pi^2)(\omega_1+m)}{m\omega_1}+\frac{1}{2}\frac{\pi_\alpha}{m}(\sigma^{\alpha 1}+i\sigma^{\alpha 2})\frac{\omega_1-m}{\omega_1}\right]\\
&+\tfrac{1}{2}\cos(\omega_2+m)s\left[\tfrac{1}{2}(\gamma^1-i\gamma^2)\left[\frac{\omega_2+m}{\omega_2}\right]-\frac{(\pi^1-i\pi^2)\pi\cdot\gamma}{m\omega_2}\right]\\
&+\tfrac{1}{2}\sin(\omega_2+m)s\left[\frac{i}{2}(\pi^1-i\pi^2)\left[\frac{\omega_2-m}{m\omega_2}\right]-\frac{1}{2}\frac{\pi_\alpha}{m}(\sigma^{\alpha 1}-i\sigma^{\alpha 2})\frac{\omega_2+m}{\omega_2}\right]\\
&+\tfrac{1}{2}\cos(\omega_2-m)s\left[(\gamma^1-i\gamma^2)\left[\frac{\omega_2-m}{\omega_2}\right]+\frac{(\pi^1-i\pi^2)\pi\cdot\gamma}{m\omega_2}\right]\\
&+\tfrac{1}{2}\sin(\omega_2-m)s\left[-\frac{i}{2}(\pi^1-i\pi^2)\left[\frac{\omega_2+m}{m\omega_2}\right]+\frac{1}{2}\pi_\alpha(\sigma^{\alpha_1}-i\sigma^{\alpha 2})\frac{\omega_2-m}{m\omega_2}\right]\,,
\end{aligned}
\tag{49a}
$$

$$\begin{aligned}
\gamma^2(s)= {} & \tfrac{1}{2}\cos(\omega_1+m)s\left[\tfrac{1}{2}(\gamma^2-i\gamma^1)\left[\frac{\omega_1+m}{\omega_1}\right]-\frac{(\pi^2-i\pi^1)\pi\cdot\gamma}{m\omega_1}\right]\\
&+\tfrac{1}{2}\sin(\omega_1+m)s\left[\frac{i}{2}(\pi^2-i\pi^1)\left[\frac{\omega_1-m}{\omega_1}\right]-\frac{1}{2}\frac{\pi_\alpha}{m}(\sigma^{\alpha 2}-i\sigma^{\alpha 1})\frac{\omega_1+m}{\omega_1}\right]\\
&+\tfrac{1}{2}\cos(\omega_1-m)s\left[\tfrac{1}{2}(\gamma^2-i\gamma^1)\left[\frac{\omega_1-m}{\omega_1}\right]+\frac{(\pi^2-i\pi^1)\pi\cdot\gamma}{m\omega_1}\right]\\
&+\tfrac{1}{2}\sin(\omega_1-m)s\left[-\frac{i}{2}(\pi^2-i\pi^1)\left[\frac{\omega_1+m}{m\omega_1}\right]+\frac{1}{2}\frac{\pi_\alpha}{m}(\sigma^{\alpha 2}-i\sigma^{\alpha 1})\frac{\omega_1-m}{\omega_1}\right]\\
&+\tfrac{1}{2}\cos(\omega_2+m)s\left[\tfrac{1}{2}(\gamma^2+i\gamma^1)\left[\frac{\omega_2+m}{\omega_2}\right]-\frac{(\pi^2+i\pi^1)\pi\cdot\gamma}{m\omega_2}\right]\\
&+\tfrac{1}{2}\sin(\omega_2+m)s\left[\frac{i}{2}(\pi^2+i\pi^1)\left[\frac{\omega_2-m}{m\omega_2}\right]-\frac{1}{2}\frac{\pi_\alpha}{m}(\sigma^{\alpha 2}+i\sigma^{\alpha 1})\frac{\omega_2+m}{\omega_2}\right]\\
&+\tfrac{1}{2}\cos(\omega_2-m)s\left[\tfrac{1}{2}(\gamma^2+i\gamma^1)\left[\frac{\omega_2-m}{\omega_2}\right]+\frac{(\pi^2+i\pi^1)\pi\cdot\gamma}{m\omega_2}\right]\\
&+\tfrac{1}{2}\sin(\omega_2-m)s\left[-\frac{i}{2}(\pi^2+i\pi^1)\left[\frac{\omega_2+m}{m\omega_2}\right]+\frac{1}{2}\frac{\pi_\alpha}{m}(\sigma^{\alpha 2}+i\sigma^{\alpha 1})\frac{\omega_2-m}{\omega_2}\right]. \qquad (49b)
\end{aligned}$$

Integrating, we find the position operator

$$\begin{aligned}
x^1(s)= {} & c^1+\frac{\sin(\omega_1+m)s}{2(\omega_1+m)}\left[\tfrac{1}{2}(\gamma^1+i\gamma^2)\left[\frac{\omega_1+m}{\omega_1}\right]-\frac{(\pi^1+i\pi^2)\pi\cdot\gamma}{m\omega_1}\right]\\
&-\frac{\cos(\omega_1+m)s}{2(\omega_1+m)}\left[\frac{i}{2}(\pi^1+i\pi^2)\left[\frac{\omega_1-m}{m\omega_1}\right]-\tfrac{1}{2}\pi_\alpha(\sigma^{\alpha 1}+i\sigma^{\alpha 2})\frac{\omega_1+m}{m\omega_1}\right]\\
&+\frac{\sin(\omega_1-m)s}{2(\omega_1-m)}\left[\tfrac{1}{2}(\gamma^1+i\gamma^2)\left[\frac{\omega_1-m}{\omega_1}\right]+\frac{(\pi^1+i\pi^2)\pi\cdot\gamma}{m\omega_1}\right]\\
&-\frac{\cos(\omega_1-m)s}{2(\omega_1-m)}\left[-\frac{i}{2}(\pi^1+i\pi^2)\left[\frac{\omega_1+m}{m\omega_1}\right]+\tfrac{1}{2}\pi_\alpha(\sigma^{\alpha 1}+i\sigma^{\alpha 2})\frac{\omega_1-m}{m\omega_1}\right]\\
&+\frac{\sin(\omega_2+m)s}{2(\omega_2+m)}\left[\tfrac{1}{2}(\gamma^1-1\gamma^2)\left[\frac{\omega_2+m}{\omega_2}\right]-\frac{(\pi^1-i\pi^2)\pi\cdot\gamma}{m\omega_2}\right]\\
&-\frac{\cos(\omega_2+m)s}{2(\omega_2+m)}\left[\frac{i}{2}(\pi^1-i\pi^2)\left[\frac{\omega_2-m}{\omega_2}\right]-\tfrac{1}{2}\pi_\alpha(\sigma^{\alpha 1}-i\sigma^{\alpha 2})\frac{\omega_2+m}{m\omega_2}\right]\\
&+\frac{\sin(\omega_2-m)s}{2(\omega_2-m)}\left[\tfrac{1}{2}(\gamma^1-i\gamma^2)\left[\frac{\omega_2-m}{\omega_2}\right]+\frac{(\pi^1-i\pi^2)\pi\cdot\gamma}{m\omega_2}\right]\\
&-\frac{\cos(\omega_2-m)s}{2(\omega_2-m)}\left[-\frac{i}{2}(\pi^1-i\pi^2)\left[\frac{\omega_2+m}{m\omega_2}\right]+\tfrac{1}{2}\pi_\alpha(\sigma^{\alpha 1}-i\sigma^{\alpha 2})\frac{\omega_2-m}{m\omega_2}\right], \qquad (50a)
\end{aligned}$$

$$\begin{aligned}
x^2(s)= {} & c^2+\frac{\sin(\omega_1+m)s}{2(\omega_1+m)}\left[\tfrac{1}{2}(\gamma^2-i\gamma^1)\left[\frac{\omega_1+m}{\omega_1}\right]-\frac{(\pi^2-i\pi^1)\pi\cdot\gamma}{m\omega_1}\right]\\
&-\frac{\cos(\omega_1+m)s}{2(\omega_1+m)}\left[\frac{i}{2}(\pi^2-i\pi^1)\left[\frac{\omega_1-m}{\omega_1}\right]-\tfrac{1}{2}\pi_\alpha(\sigma^{\alpha 2}-i\sigma^{\alpha 1})\frac{\omega_1+m}{m\omega_1}\right]\\
&+\frac{\sin(\omega_1-m)s}{2(\omega_1-m)}\left[\tfrac{1}{2}(\gamma^2-i\gamma^1)\left[\frac{\omega_1-m}{\omega_1}\right]+\frac{(\pi^2-i\pi^1)\pi\cdot\gamma}{m\omega_1}\right]\\
&-\frac{\cos(\omega_1-m)s}{2(\omega_1-m)}\left[-\frac{i}{2}(\pi^2-i\pi^1)\left[\frac{\omega_1+m}{\omega_1}\right]+\tfrac{1}{2}\pi_\alpha(\sigma^{\alpha 2}-i\sigma^{\alpha 1})\frac{\omega_1-m}{m\omega_1}\right]
\end{aligned}$$

$$+\frac{\sin(\omega_2+m)s}{2(\omega_2+m)}\left[\tfrac{1}{2}(\gamma^2+i\gamma^1)\left[\frac{\omega_2+m}{\omega_2}\right]-\frac{(\pi^2+i\pi^1)\pi\cdot\gamma}{m\omega_2}\right]$$
$$-\frac{\cos(\omega_2+m)s}{2(\omega_2+m)}\left[\frac{i}{2}(\pi^2+i\pi^1)\left[\frac{\omega_2-m}{\omega_2}\right]-\tfrac{1}{2}\pi_\alpha(\sigma^{\alpha 2}+i\sigma^{\alpha 1})\frac{\omega_2+m}{m\omega_2}\right]$$
$$+\frac{\sin(\omega_2-m)s}{2(\omega_2-m)}\left[\tfrac{1}{2}(\gamma^2+i\gamma^1)\left[\frac{\omega_2-m}{\omega_2}\right]+\frac{(\pi^2+i\pi^1)\pi\cdot\gamma}{m\omega_2}\right]$$
$$-\frac{\cos(\omega_2-m)s}{2(\omega_2-m)}\left[-\frac{i}{2}(\pi^2+i\pi^1)\left[\frac{\omega_2+m}{\omega_2}\right]+\tfrac{1}{2}\pi_\alpha(\sigma^{\alpha 2}+i\sigma^{\alpha 1})\frac{\omega_2-m}{m\omega_2}\right]. \tag{50b}$$

The motion in the zeroth and third directions corresponds to the free particle solution (37a) and (38a), since no force acts in the 0-3 plane.

We see again that all terms containing γ^5 have canceled out of the results (49) and (50). The SO(3,2) dynamical system is not enlarged to the SO(4,2) system as a result of interaction with a magnetic field. These solutions also reveal that the motion of the electron in a constant magnetic field is oscillatory with four distinct frequencies, $\omega_1\pm m$ and $\omega_2\pm m$. It is interesting to note that the combination $\gamma^1(s)\pm i\gamma^2(s)$ oscillates only with frequencies $\omega_1\pm m$, respectively,

$$\gamma^1(s)+i\gamma^2(s)=\frac{1}{2}(\gamma^1+i\gamma^2)\left[\left[\frac{\omega_1+m}{\omega_1}\right]\cos((\omega_1+m)s)+\left[\frac{\omega_1-m}{\omega_1}\right]\cos((\omega_1-m)s)\right]$$
$$+\frac{(\pi^1+i\pi^2)\pi\cdot\gamma}{m\omega_1}[\cos((\omega_1-m)s)-\cos((\omega_1+m)s)]$$
$$+\frac{i}{2}(\pi^1+i\pi^2)\left[\left[\frac{\omega_1-m}{m\omega_1}\right]\sin((\omega_1+m)s)-\left[\frac{\omega_1+m}{m\omega_1}\right]\sin((\omega_1-m)s)\right]$$
$$+\frac{1}{2}\pi_\alpha(\sigma^{\alpha 1}+i\sigma^{\alpha 2})\left[\left[\frac{\omega_1-m}{m\omega_1}\right]\sin((\omega_1-m)s)-\left[\frac{\omega_1+m}{m\omega_1}\right]\sin((\omega_1+m)s)\right], \tag{51a}$$

while $\gamma^1(s)-i\gamma^2(s)$ oscillates with frequencies $\omega_2\pm m$

$$\gamma^1(s)-i\gamma^2(s)=\tfrac{1}{2}(\gamma^1-i\gamma^2)\left[\left[\frac{\omega_2+m}{\omega_2}\right]\cos((\omega_2+m)s)+\left[\frac{\omega_2-m}{\omega_2}\right]\cos((\omega_2-m)s)\right]$$
$$+\frac{(\pi^1-i\pi^2)\pi\cdot\gamma}{m\omega_2}[\cos((\omega_2-m)s)-\cos((\omega_2+m)s)]$$
$$+\frac{i}{2}(\pi^1-i\pi^2)\left[\left[\frac{\omega_2-m}{m\omega_2}\right]\sin((\omega_2+m)s)-\left[\frac{\omega_2+m}{m\omega_2}\right]\sin((\omega_2-m)s)\right]$$
$$+\frac{1}{2}\pi_\alpha(\sigma^{\alpha 1}-i\sigma^{\alpha 2})\left[\left[\frac{\omega_2-m}{m\omega_2}\right]\sin((\omega_2-m)s)-\left[\frac{\omega_2+m}{m\omega_2}\right]\sin((\omega_2+m)s)\right]. \tag{51b}$$

When $\mathscr{B}>m/2$, ω_2 becomes imaginary. This means that sines and cosines of $(\omega_2 s)$ become hyperbolic sines and cosines of $(\,|\omega_2|\,s)$, and the motion acquires a hyperbolic acceleration in addition to the oscillation. We expect, on physical grounds, such a threshold (or phase transition) to occur when the cyclotron frequency $(\mathscr{B}/m)$ becomes equal to the *Zitterbewegung* frequency. However, for fields of this strength pair creation effects become important and the single-particle theory must be extended or interpreted differently.

VII. CENTER OF MASS AND INTERNAL MOTIONS IN WEAK ELECTROMAGNETIC FIELDS

We have found that the exact solutions (37), (38), (49), and (50) for the velocity and position in constant electric and magnetic fields do not allow a straightforward separation into center-of-mass motion plus *Zitterbewegung;* they are intertwined in a complex way. However, we do know that when no external fields are present, this separation is possible. Therefore we expect that when the external fields are sufficiently weak, we can separate the internal and center-of-mass motions. In this section we investigate the solutions (37b), (37c), (38b), (38c), (49), and (50) which together give the velocity and position of an electron in constant electric and magnetic fields both pointing in the third direction.

In the limit that $\epsilon \ll m^2$ and $\mathscr{B}\ll m^2$, we can approximate

$$\omega\approx m+\frac{\epsilon^2}{m^3}\approx m,\quad \mu\approx\frac{\epsilon}{m}, \tag{52}$$

$$\omega_1 \approx m + \frac{\mathscr{B}}{m}, \quad \omega_2 \approx m - \frac{\mathscr{B}}{m} .$$

These limits correspond to

$$E_{\rm crit} = 1.3 \times 10^{16}\ {\rm V/cm}\ ,$$

$$B_{\rm crit} = 4.4 \times 10^{13}\ {\rm G}\ .$$

For comparison, at the Compton wavelength, the fields produced by an electron are $E \approx 10^{13}$ V/cm, $B \approx 2.5 \times 10^{10}$ G. Dropping terms of order ϵ/m^2 and $\mathscr{B}/m^2$, unless they appear in the arguments of the (circular or hyperbolic) sines and cosines, we find that the velocity operator has the form

$$\gamma^\mu(s) = v^\mu_{\rm c.m.}(s) + \eta^\mu(s)\ , \tag{53}$$

where the components of the center-of-mass velocity are

$$\begin{aligned} v_{\rm c.m.}{}^0(s) &= \frac{\pi^0 \pi\cdot\gamma}{m^2}\cosh\left[\frac{\epsilon}{m}s\right] + \frac{\pi^3}{m}\sinh\left[\frac{\epsilon}{m}s\right]\ ,\\ v_{\rm c.m.}{}^3(s) &= \frac{\pi^3 \pi\cdot\gamma}{m^2}\cosh\left[\frac{\epsilon}{m}s\right] + \frac{\pi^0}{m}\sinh\left[\frac{\epsilon}{m}s\right]\ ,\\ v_{\rm c.m.}{}^1(s) &= \frac{\pi^1 \pi\cdot\gamma}{m^2}\cos\left[\frac{\mathscr{B}}{m}s\right] + \frac{\pi^2}{m}\sin\left[\frac{\mathscr{B}}{m}s\right]\ ,\\ v_{\rm c.m.}{}^2(s) &= \frac{\pi^2 \pi\cdot\gamma}{m^2}\cos\left[\frac{\mathscr{B}}{m}s\right] - \frac{\pi^1}{m}\sin\left[\frac{\mathscr{B}}{m}s\right]\ , \end{aligned} \tag{54}$$

and the components of internal velocity are

$$\begin{aligned} \eta^0(s) =& \cosh\left[\frac{\epsilon}{m}s\right]\left[\left[\gamma^0 - \frac{\pi^0\pi\cdot\gamma}{m^2}\right]\cos(2ms) - \frac{\pi_\alpha}{m}\sigma^{\alpha 0}\sin(2ms)\right]\\ &+ i\sinh\left[\frac{\epsilon}{m}s\right]\left[-\frac{\pi_\alpha}{m}\sigma^{\alpha 3}\cos(2ms) - \left[\gamma^3 - \frac{\pi^3\pi\cdot\gamma}{m^2}\right]\sin(2ms)\right]\ ,\\ \eta^3(s) =& \cosh\left[\frac{\epsilon}{m}s\right]\left[\left[\gamma^3 - \frac{\pi^3\pi\cdot\gamma}{m^2}\right]\cos(2ms) - \frac{\pi_\alpha}{m}\sigma^{\alpha 3}\sin(2ms)\right]\\ &+ i\sinh\left[\frac{\epsilon}{m}s\right]\left[-\frac{\pi_\alpha}{m}\sigma^{\alpha 0}\cos(2ms) - \left[\gamma^0 - \frac{\pi^0\pi\cdot\gamma}{m^2}\right]\sin(2ms)\right]\ ,\\ \eta^1(s) =& \cosh\left[\frac{\mathscr{B}}{m}s\right]\left[\left[\gamma^1 - \frac{\pi^3\pi\cdot\gamma}{m^2}\right]\cos(2ms) - \frac{\pi_\alpha}{m}\sigma^{\alpha 1}\sin(2ms)\right]\\ &+ i\sinh\left[\frac{\mathscr{B}}{m}s\right]\left[-\frac{\pi_\alpha}{m}\sigma^{\alpha 2}\cos(2ms) - \left[\gamma^2 - \frac{\pi^2\pi\cdot\gamma}{m^2}\right]\sin(2ms)\right]\ ,\\ \eta^2(s) =& \cosh\left[\frac{\mathscr{B}}{m}s\right]\left[\left[\gamma^2 - \frac{\pi^2\pi\cdot\gamma}{m^2}\right]\cos(2ms) - \frac{\pi_\alpha}{m}\sigma^{\alpha 2}\sin(2ms)\right]\ . \end{aligned} \tag{55}$$

Keeping only the terms of lowest order in ϵ/m^2 and $\mathscr{B}/m^2$ we find that the position operator $x^\mu(s)$ of the charge also separates into a center of mass $X^\mu(s)$ plus an oscillatory internal coordinate $Q^\mu(s)$ where

$$\begin{aligned} &X^0(s) = \frac{m}{\epsilon}\frac{\pi^0\pi\cdot\gamma}{m^2}\sinh\left[\frac{\epsilon}{m}s\right] + \frac{m}{\epsilon}\frac{\pi^3}{m}\cosh\left[\frac{\epsilon}{m}s\right] + c^0,\quad X^3(s) = \frac{m}{\epsilon}\frac{\pi^3\pi\cdot\gamma}{m^2}\sinh\left[\frac{\epsilon}{m}s\right] + \frac{m}{\epsilon}\frac{\pi^0}{m}\cosh\left[\frac{\epsilon}{m}s\right] + c^3\ ,\\ &X^1(s) = \frac{m}{\mathscr{B}}\frac{\pi^1\pi\cdot\gamma}{m^2}\sin\left[\frac{\mathscr{B}}{m}s\right] - \frac{m}{\mathscr{B}}\frac{\pi^3}{m}\cos\left[\frac{\mathscr{B}}{m}s\right] + c^1,\quad X^2(s) = \frac{m}{\mathscr{B}}\frac{\pi^2\pi\cdot\gamma}{m^2}\sin\left[\frac{\mathscr{B}}{m}s\right] + \frac{m}{\mathscr{B}}\frac{\pi^1}{m}\cos\left[\frac{\mathscr{B}}{m}s\right] + c^2\ , \end{aligned} \tag{56}$$

and

$$\begin{aligned} Q^0(s) =& \frac{i}{2m}\sinh\left[\frac{\epsilon}{m}s\right]\left[\left[\gamma^3 - \frac{\pi^3\pi\cdot\gamma}{m^2}\right]\cos(2ms) - \frac{\pi_\alpha}{m}\sigma^{\alpha 3}\sin(2ms)\right]\\ &+ \frac{1}{2m}\cosh\left[\frac{\epsilon}{m}s\right]\left[\frac{\pi_\alpha}{m}\sigma^{\alpha 0}\cos(2ms) + \left[\gamma^0 - \frac{\pi^0\pi\cdot\gamma}{m^2}\right]\sin(2ms)\right]\ ,\\ Q^3(s) =& \frac{1}{2m}\cosh\left[\frac{\epsilon}{m}s\right]\left[\frac{\pi_\alpha}{m}\sigma^{\alpha 3}\cos(2ms) + \left[\gamma^3 - \frac{\pi^3\pi\cdot\gamma}{m^2}\right]\sin(2ms)\right]\\ &+ \frac{i}{2m}\sinh\left[\frac{\epsilon}{m}s\right]\left[\left[\gamma^0 - \frac{\pi^0\pi\cdot\gamma}{m^2}\right]\cos(2ms) - \frac{\pi_\alpha}{m}\sigma^{\alpha 0}\sin(2ms)\right]\ , \end{aligned} \tag{57}$$

$$Q^1(s)=\frac{1}{2m}\cos\left[\frac{\mathcal{B}}{m}s\right]\left[\frac{\pi_\alpha}{m}\sigma^{\alpha 1}\cos(2ms)+\left[\gamma^1-\frac{\pi^1\pi\cdot\gamma}{m^2}\right]\sin(2ms)\right]$$
$$+\frac{i}{2m}\sin\left[\frac{\mathcal{B}}{m}s\right]\left[\left[\gamma^2-\frac{\pi^2\pi\cdot\gamma}{m^2}\right]\cos(2ms)-\frac{\pi_\alpha}{m}\sigma^{\alpha 2}\sin(2ms)\right],$$

$$Q^2(s)=\frac{1}{2m}\cos\left[\frac{\mathcal{B}}{m}s\right]\left[\frac{\pi_\alpha}{m}\sigma^{\alpha 2}\cos(2ms)+\left[\gamma^2-\frac{\pi^2\pi\cdot\gamma}{m^2}\right]\sin(2ms)\right].$$

It is interesting to compare (54) and (56) with the corresponding expressions (22) and (23) for the velocity and position of a relativistic spinless charge in the same electric and magnetic fields: on positive-energy solutions of the Dirac equation $\pi\cdot\gamma=m$ and Eqs. (54) and (56) reduce to (22) and (23). The center of mass of the electron behaves like a relativistic spinless charge. In fact (54) is a solution of the equation

$$\dot{V}^\mu_{\text{c.m.}}\pi\cdot\gamma=eF^\mu{}_\nu V^\nu_{\text{c.m.}} \tag{58}$$

with the initial condition

$$V^\mu_{\text{c.m.}}(0)=\frac{\pi^\mu(0)\pi\cdot\gamma}{m^2}. \tag{59}$$

Equation (58) can be thought of as a semiclassical approximation to (6′) in which the term explicitly containing $\hbar$ has been dropped. The initial velocity (59) corresponds, with minimal coupling replacement $p\to\pi$, to the center-of-mass velocity of a free electron.

Let us now turn our attention to the influence of "weak" electric and magnetic fields on the *Zitterbewegung*. We recall from Ref. 1 that the free electron *Zitterbewegung* is a harmonic oscillator which can be represented as a

$$\begin{bmatrix} Q^\mu(s) \\ \frac{1}{2m}\eta^\mu(s)\end{bmatrix}=\begin{bmatrix}\cos(2ms) & \sin(2ms)\\ -\sin(2ms) & \cos(2ms)\end{bmatrix}\begin{bmatrix} Q^\mu(0) \\ \frac{1}{2m}\eta^\mu(0)\end{bmatrix}.$$

Motivated by the free particle case, we rewrite (55) and (57), together in terms of a phase space rotation as follows:

$$\begin{bmatrix} Q^0(s) \\ \frac{1}{2m}\eta^0(s) \\ Q^3(s) \\ \frac{1}{2m}\eta^3(s)\end{bmatrix}=\begin{bmatrix} R(2ms)\cosh\left[\frac{\epsilon}{m}s\right] & iR\left[2ms+\frac{\pi}{2}\right]\sinh\left[\frac{\epsilon}{m}s\right] \\ iR\left[2ms+\frac{\pi}{2}\right]\sinh\left[\frac{\epsilon}{m}s\right] & R(2ms)\cosh\left[\frac{\epsilon}{m}s\right]\end{bmatrix}\begin{bmatrix} Q^0(0) \\ \frac{1}{2m}\eta^0(0) \\ Q^3(0) \\ \frac{1}{2m}\eta^3(0)\end{bmatrix},$$
$$\begin{bmatrix} Q^1(s) \\ \frac{1}{2m}\eta^1(s) \\ Q^2(s) \\ \frac{1}{2m}\eta^2(s)\end{bmatrix}=\begin{bmatrix} R(2ms)\cos\left[\frac{\mathcal{B}}{m}s\right] & iR\left[2ms+\frac{\pi}{2}\right]\sinh\left[\frac{\mathcal{B}}{m}s\right] \\ -iR\left[2ms+\frac{\pi}{2}\right]\sinh\left[\frac{\mathcal{B}}{m}s\right] & R(2ms)\cosh\left[\frac{\mathcal{B}}{m}s\right]\end{bmatrix}\begin{bmatrix} Q^1(0) \\ \frac{1}{2m}\eta^1(0) \\ Q^2(0) \\ \frac{1}{2m}\eta^2(0)\end{bmatrix}, \tag{60}$$

where

$$R(\alpha)=\begin{bmatrix}\cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha\end{bmatrix} \tag{61}$$

and the initial internal position and velocity operators are

$$Q^\mu(0)=\frac{1}{2m}\frac{\pi_\alpha}{m}\sigma^{\alpha\mu},$$
$$\eta^\mu(0)=\left[\gamma^\mu(0)-\frac{\pi^\mu(0)\pi\cdot\gamma}{m^2}\right]. \tag{62}$$

We see in the result (60) a "tensor product" structure (with $\pi/2$ phase shift and multiplication by $i=\sqrt{-1}$ in the off-diagonal elements) between a circular (or hyperbolic) rotation, induced by the magnetic (or electric) field, on the one hand, and a rotation in the position-velocity phase space, coming from *Zitterbewegung*, on the other hand. An expression similar to (60) can be derived, by differentiation, for motion in the velocity-acceleration phase space.

VIII. HEISENBERG EQUATIONS FOR THE SPIN TENSOR

For the sake of completeness we also record the Heisenberg equation of motion for the spin tensor $S_{\mu\nu}=(i/4)[\gamma_\mu,\gamma_\nu]$:

$$-\dot{S}^{\mu\nu}\mathscr{H}=\frac{\hbar}{2i}\ddot{S}^{\mu\nu}+\frac{e}{c}(S^{\nu\alpha}F_{\alpha}{}^{\mu}-S^{\mu\alpha}F_{\alpha}{}^{\nu}) . \tag{62}$$

Using the fact that $S^{0j}=(i/2)\alpha^j$ and $S^{jk}=\epsilon^{jkl}S^l$, we can write down separately the vector and axial-vector components of (62):

$$-\left[\frac{i}{2}\dot{\boldsymbol{\alpha}}\right]\mathscr{H}=\frac{\hbar}{2i}\left[\frac{i}{2}\ddot{\boldsymbol{\alpha}}\right]+\frac{e}{c}\left[\frac{i}{2}\boldsymbol{\alpha}\times\mathbf{B}+\mathbf{S}\times\mathbf{E}\right] , \tag{63a}$$

$$-\dot{\mathbf{S}}\mathscr{H}=\frac{\hbar}{2i}\ddot{\mathbf{S}}+\frac{e}{c}\left[\mathbf{S}\times\mathbf{B}-\frac{i}{2}\boldsymbol{\alpha}\times\mathbf{E}\right] . \tag{63b}$$

We note that when $\mathbf{E}=0$, $\mathbf{S}$ and $i\boldsymbol{\alpha}$ decouple and Eqs. (63a) and (63b) are each equivalent to (6′) for the velocity operator. In this case we can repeat the procedure in Sec. V and obtain solutions for the proper time evolution of the operators $\mathbf{S}$ and $i\boldsymbol{\alpha}$ in the constant magnetic field $\mathbf{B}=(0,0,B)$. In terms of the "raising and lowering" operators

$$S^+=S^1+iS^2,\quad S^-=S^1-iS^2$$

we obtain

$$\begin{aligned}
S^+(s)=&\ \tfrac{1}{2}\cos((\omega_1+m)s)\left[S^+(0)\frac{\omega_1-m}{\omega_1}+i\dot{S}^+(0)\frac{\mathscr{H}}{m\omega_1}\right]+\tfrac{1}{2}\sin((\omega_1+m)s)\left[\dot{S}^+(0)\frac{1}{\omega_1}-iS^+(0)\mathscr{H}\frac{\omega_1-m}{m\omega_1}\right]\\
&+\tfrac{1}{2}\cos((\omega_1-m)s)\left[S^+(0)\frac{\omega_1+m}{\omega_1}-i\dot{S}^+(0)\frac{\mathscr{H}}{m\omega_1}\right]+\tfrac{1}{2}\sin((\omega_1-m)s)\left[\dot{S}^+(0)\frac{1}{\omega_1}+iS^+(0)\mathscr{H}\frac{\omega_1+m}{m\omega_1}\right] ,\\
S^-(s)=&\ \tfrac{1}{2}\cos((\omega_2+m)s)\left[S^-(0)\frac{\omega_2-m}{\omega_2}+i\dot{S}^-(0)\frac{\mathscr{H}}{m\omega_2}\right]+\tfrac{1}{2}\sin((\omega_2+m)s)\left[\dot{S}^-(0)\frac{1}{\omega_2}-iS^-(0)\mathscr{H}\frac{\omega_2-m}{m\omega_2}\right]\\
&+\tfrac{1}{2}\cos((\omega_2-m)s)\left[S^-(0)\frac{\omega_2+m}{\omega_2}-i\dot{S}^-(0)\frac{\mathscr{H}}{m\omega_2}\right]+\tfrac{1}{2}\sin((\omega_2-m)s)\left[\dot{S}^-(0)\frac{1}{\omega_2}+iS^-(0)\mathscr{H}\frac{\omega_2+m}{m\omega_2}\right] ,\\
S^3(s)=&\ S^3(0)-\frac{i}{2}\dot{S}^3(0)\mathscr{H}^{-1}+\frac{i}{2}\dot{S}^3(0)\mathscr{H}^{-1}\exp(-2i\mathscr{H}s) ,
\end{aligned} \tag{64}$$

with similar expressions for $i(\alpha^+,\alpha^-,\alpha^3)$ where again $\omega_2=(m^2+2\mathscr{B})^{1/2}$ and $\omega_2=(m^2-2\mathscr{B})^{1/2}$.

For weak fields one can write Eqs. (64) in the form of Eqs. (53), (54), and (55) for the velocities.

IX. CONCLUSION

In this work we have obtained exact solutions of the Heisenberg equations in proper time for the motion of an electron in constant electric and magnetic fields. Barut and Bracken[4] have previously solved the Heisenberg equations in ordinary time in a constant magnetic field. The equations for an electron in a constant electric field are, however, quite intractable in ordinary time, since the electric potential appears as an extra term in the Hamiltonian, in contrast to the magnetic vector potential which is minimally coupled to the momentum. In the covariant proper time formalism used in this paper both potentials couple minimally to the four-momentum, and the equations for both the electric and magnetic fields can be solved in a unified manner.

Our solutions reveal that in an electric field the velocity and position operator of the electron accelerate hyperbolically, like the relativistic spinless charge, while at the same time oscillating at two frequencies depending on the strength of the field and the mass. In a magnetic field the velocity and position, as well as spin, of the electron oscillate at four distinct frequencies. When the electric and magnetic fields are much weaker than the critical values (10^{16} V/cm, 10^{13} G), the oscillatory *Zitterbewegung* and the center-of-mass motion can be separated. The center of mass of the electron then behaves like a relativistic spinless charge, while the internal motion exhibits a structure similar to a tensor product between a hyperbolic (or circular) motion, induced by the electric (or magnetic) field, on the one hand, and the much faster oscillation at frequency $2m$ characteristic of the *Zitterbewegung* on the other hand.

Besides providing insight into the influence of the electromagnetic fields on the internal and center-of-mass motions of the electron, and more generally on relativistic extended particles, the results of this work may be used to calculate the Green's function of the Dirac field using the proper time method of Schwinger.[5] The solutions of the Heisenberg equations are also important in radiation problems. For example, the form of the acceleration, $\ddot{x}^2$, leads immediately to a derivation of the spontaneous emission formula.[6]

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