Memory-function approach to retardation effects in the relativistic two-body problem

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The memory-function approach is applied to the relativistic two-particle problem. We thus obtain integral equations similar to the Bethe-Salpeter equation, taking all retardation effects fully into account and eliminating the relative time variable. The formalism is studied for scalar Klein-Gordon fields as well as for spinorial Dirac fields. For both cases the memory function and the integral kernels are calculated for a given model explicitly. The scalar-scalar model of Cutkosky is treated in detail and numerical binding energies are obtained, very close to those of the original Bethe-Salpeter equation, thus showing the extreme importance of retardation effects. As an example for Dirac fields the memory function for a proton-antiproton pair interacting via exchange of a π^0 is established.

I. INTRODUCTION

In relativistic quantum field theory bound states or resonances are identified with poles appearing in the Green's functions. A simple relativistic generalization of the Schrödinger equation for bound states of two particles is unfortunately not available except for the limiting case of a particle in a static external potential (Dirac or Klein-Gordon equations).

The problem in a relativistic treatment is twofold: On the one hand we cannot disregard effects of retardation—a supplementary relative time appears—and on the other hand we have to take into account the quantum character of the field describing the interaction (e.g., the electromagnetic field in positronium). Thus one cannot avoid determining the two-particle Green's function. Bethe and Salpeter[1] proposed an equation in 1951 (the Bethe-Salpeter equation) which calculates this propagator by an iteration process as the sum of all Feynman diagrams. Because of this perturbation-theoretical method, in practice one is of course restricted to approximate calculations only. The advantage of the Bethe-Salpeter equation is, however, that the iteration process nevertheless takes into account an infinite sum of Feynman graphs when calculating the Green's function.

But even for the most simple forms of the Bethe-Salpeter equation its (numerical) solution gives rise to enormous difficulties. The problem comes from the relative time variable mentioned above which comes along with the relativistic retardation effects. Thus people soon searched for methods to get rid of this difficulty, and several approximations for the Bethe-Salpeter equation have been proposed. The first was due to Salpeter[2] himself and is the instantaneous approximation. Another group of approximations are the quasipotential approximations,[3–7] being similar in form to the more familiar Schrödinger, Klein-Gordon, or Dirac equations. In particular, the instantaneous approximation has often been used, e.g., to calculate the spectrum of positronium.[8] A complete list of references up to 1969 concerning the Bethe-Salpeter equation is given in the paper of Nakanishi.[9] Interesting more recent discussions may be found in the review papers of Stroscio[10] and Bodwin and Yennie.[11] The progress of the last few years in quark spectroscopy[12–14] renews interest in a relativistic description of bound states. Here the instantaneous approximation has been used, too.[15–18]

In the case of positronium and a coupling constant to the electromagnetic field of $\lambda=\frac{1}{137}$ the instantaneous approximation yields good results.[8] But it is not evident whether it is appropriate for calculating the meson spectrum, the coupling constant between quarks and gluons being of the order of unity. Indeed, for a second-order kernel, model calculations[19] indicate very important differences for $\lambda=1$ (more than a factor 2 for the binding energies) between the Bethe-Salpeter equation and the instantaneous approximation. The quasipotential approximation of Blankenbecler and Sugar[5] yields for two equal masses the same results as the instantaneous approximation and the one of Todorov[7] reveals even more important differences with the Bethe-Salpeter equation. Therefore it seems to be desirable to look for an approximation yielding results very close to the Bethe-Salpeter equation but avoiding the problem of the relative time. This will be done in this paper.

In Schrödinger-type many-body problems the so-called effective potentials are often introduced with the help of the Feshbach formalism.[20] These effective potentials depend only on the one energy which appears also in the Schrödinger equation as an eigenvalue of the Hamilton operator. For nonrelativistic many-body correlation functions this has been modeled in introducing integral equations whose kernel also only depends on one energy, i.e., there is no other relative energy corresponding to a relative time variable. This is generally known as the Mori or memory-function approach.[21] Here we want to generalize this formalism to the relativistic many-body problem.

In Sec. II we briefly recall the Bethe-Salpeter equation and the instantaneous approximation for the scalar-scalar model of Cutkosky.[22] This is the only model for which

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the Bethe-Salpeter equation with a second-order kernel (ladder graph) can be solved numerically quite easily without approximation. This is why this model will continue to serve as a reference also in Sec. IV.

In Sec. III we schematically outline the theory of the memory-function approach and in Sec. IV we make this explicit for the scalar-scalar model of Cutkosky. In Sec. IV A we neglect all vacuum correlations to make the calculations more transparent. In Sec. IV B, however, all vacuum correlations are taken into account and the relation with the Bethe-Salpeter equation is discussed. In Sec. IV C numerical results are presented and discussed, and compared with those of the Bethe-Salpeter equation and the instantaneous approximation.

In Sec. V we show how the formalism developed may be extended to interacting Dirac fields (strong interaction of protons, quantum electrodynamics, perturbative quantum chromodynamics, etc.). Section VI is devoted to the conclusions and in an appendix we give some details concerning the numerical method used in Sec. IV.

II. THE BETHE-SALPETER EQUATION AND THE INSTANTANEOUS APPROXIMATION, CUTKOSKY'S SCALAR-SCALAR MODEL

We define the two-particle propagator by

$$G(x_1,x_2,x_1',x_2') = -\langle 0 \,|\, T\phi_1(x_1)\phi_2(x_2)\phi_1^\dagger(x_1')\phi_2^\dagger(x_2') \,|\, 0\rangle \;, \qquad (2.1)$$

where $x_i=(t_i,\mathbf{x}_i)$. ϕ_i is the field of particle i, T is the chronological operator, and $|0\rangle$ is the physical (i.e., correlated) vacuum. The corresponding free propagator G_0 is obtained by replacing the fields and the vacuum by the corresponding quantities without any interaction. However, we suppose that all masses are already physical ones. Thus one can drop all graphs contributing only to mass renormalization.

In the Bethe-Salpeter equation

$$G=G_0+G_0KG \qquad (2.2a)$$

or explicitly

$$G(x_1,x_2,x_1',x_2')=G_0(x_1,x_2,x_1',x_2')+\int G_0(x_1,x_2,y_1,y_2)K(y_1,y_2,z_1,z_2)G(z_1,z_2,x_1',x_2')d^4y_1d^4y_2d^4z_1d^4z_2 \;, \qquad (2.2b)$$

the integration over y_1 and y_2 is trivial, G_0 containing corresponding δ distributions, but the integrations over z_1 and z_2 are real ones over the spatial coordinates $\mathbf{z}_1$ and $\mathbf{z}_2$ and over the two time variables z_1 and z_2.

One defines the Bethe-Salpeter amplitude for a bound state $|\chi\rangle$ as

$$\chi(x_1,x_2)=\langle 0 \,|\, T\phi_1(x_1)\phi_2(x_2) \,|\, \chi\rangle \qquad (2.3)$$

satisfying the homogeneous Bethe-Salpeter equation

$$\chi(x_1,x_2)=\int G_0(x_1,x_2,y_1,y_2)K(y_1,y_2,z_1,z_2)\,. \times\chi(z_1,z_2)d^4y_1d^4y_2d^4z_1d^4z_2 \;. \qquad (2.4)$$

For a given Lagrangian the kernel K may be computed by perturbation theory. We consider as a simple model the one of Cutkosky[22] which has been widely used for several applications.[9,23] It describes two complex scalar fields $\phi_1(x_1)$ and $\phi_2(x_2)$ of masses m_1 and m_2 interacting through a real scalar field $\Psi(x)$ of mass μ with an interaction Lagrangian

$$\mathscr{L}_{\rm int}[g_1{:}\phi_1^\dagger(x)\phi_1(x){:}+g_2{:}\phi_2^\dagger(x)\phi_2(x){:}]\Psi(x) \;. \qquad (2.5)$$

We define the dimensionless coupling parameter λ by

$$g_1g_2=16\pi m_1m_2\lambda \;. \qquad (2.6)$$

Generally it is more convenient to work in momentum space and to introduce total and relative variables by

$$X=\alpha x_1+(1-\alpha)x_2,\;\; x=x_1-x_2 \;,$$
$$P=p_1+p_2,\;\; p=(1-\alpha)p_1-\alpha p_2 \;, \qquad (2.7)$$
$$0\le\alpha\le 1 \;,$$

and to factorize out of all quantities a δ distribution according to

$$\chi(p,P)=\delta(P-K)\chi_K(p) \qquad (2.8)$$

because total momentum is conserved.

The sum of all irreducible Feynman diagrams constituting the kernel K being unknown one usually restricts oneself to the so-called ladder approximation for which only a single exchange of the particle of mass μ contributes to K (see also Fig. 1):

$$K(x_1,x_2,x_1',x_2')=\delta^4(x_1-x_1')\delta^4(x_2-x_2') \times i\Delta_F(x_1-x_2,\mu) \;. \qquad (2.9)$$

With this kernel, the Bethe-Salpeter equation reads

$$[(\alpha K+p)^2-m_1{}^2]\{[(1-\alpha)K-p]^2-m_2{}^2\}\chi_K(p) = \frac{16\pi i m_1m_2\lambda}{(2\pi)^4}\int\frac{\chi_K(p')}{(p-p')^2-\mu^2+i\epsilon}d^4p' \;. \qquad (2.10)$$

In the center-of-mass system ($\mathbf{K}=0$) the binding energy B is obtained after solving this equation as $K_0=m_1+m_2-B$.

It is only for $\mu=0$ where this equation is easily solvable due to the resulting O(4) symmetry. After a Wick rotation and a stereographic projection on a unit sphere, Cutkosky[22] was able to transform (2.10) into an ordinary dif-

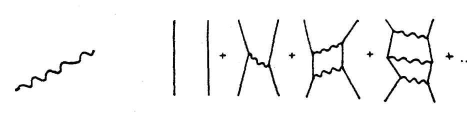

FIG. 1. The irreducible kernel in the ladder approximation (a) and resulting contributions to the propagator G (b).

ferential equation of second order (for details see Cutkosky's paper[22] or the book by Itzykson and Zuber[24]). This differential equation can be easily solved. The binding energies obtained by Silvestre-Brac *et al.*[19] will be used as a reference later on in this paper.

As soon as the problem becomes more complicated (e.g., positronium) the Bethe-Salpeter equation, even in the ladder approximation, cannot be solved exactly. The difficulty is most closely related to the presence of the relative time variable $t_r = t_1 - t_2$. Clearly, in the Bethe-Salpeter amplitude (2.3)

$$\chi(x_1,x_2) = \chi(t_1,\mathbf{x}_1,t_2,\mathbf{x}_2) \ . \tag{2.11}$$

$\mathbf{x}_1$ and $\mathbf{x}_2$ are the spatial coordinates of the two particles considered, but what is the meaning of t_1 and t_2?

Physically one is interested to know the probability amplitude at one time to find particle 1 at x_1 and particle 2 at x_2, i.e., one wants to know χ only for $t = t_1 = t_2$, $t_r = 0$.

The instantaneous approximation consists in putting $p_0 = p_0'$ in $K(p,P,p',P') = K(p_0,\mathbf{p},P,p_0',\mathbf{p}',P')$ (the exchange of the particle of mass μ becomes instantaneous) and to integrate χ over p_0 (corresponding to set $t_r = 0$):

$$\Phi_K(\mathbf{p}) = \int \chi_K(p_0,\mathbf{p})dp_0 \ . \tag{2.12}$$

Thus, (2.10) becomes (i.e., in the ladder approximation)

$$\frac{\omega_1(\mathbf{p})\omega_2(\mathbf{p})}{\pi[\omega_1(\mathbf{p})+\omega_2(\mathbf{p})]}\{K_0{}^2 - [\omega_1(\mathbf{p})+\omega_2(\mathbf{p})]^2\}\Phi_K(\mathbf{p}) = -\frac{16\pi m_1 m_2}{(2\pi)^4}\int d^3p' \frac{\Phi_K(\mathbf{p}')}{(\mathbf{p}-\mathbf{p}')^2+\mu^2 - i\epsilon}$$

with

$$\omega_i(\mathbf{p}) = (m_i{}^2 + \mathbf{p}^2)^{1/2} \tag{2.13}$$

(where a remaining p_0 integration has been performed in the upper or lower complex half plane).

Determining the values of K_0 that admit a solution yields again the binding energies. The nonrelativistic limit of (2.13) is the Schrödinger equation with a Yukawa potential. Equation (2.13) can be solved numerically; after angular decomposition and going to a discrete limit one is left with a diagonalization of a nonsymmetric matrix which is easily performed. More details and the numerical results are found again in Silvestre-Brac *et al.*[19]

III. OUTLINE OF THE GENERAL THEORY

Our claim here will be that instead of Eq. (2.2) which are in fact three coupled equations for three different time orderings in

$$G(t_1,\mathbf{x}_1,t_2,\mathbf{x}_2,t_1',\mathbf{x}_1',t_2',\mathbf{x}_2') = -\langle 0 \,|\, T\phi(t_1,\mathbf{x}_1)\phi(t_2,\mathbf{x}_2)\phi^\dagger(t_1',\mathbf{x}_1')\phi^\dagger(t_2',\mathbf{x}_2') \,|\, 0\rangle \tag{3.1}$$

(corresponding to the u, s, and t channels in the relativistic case and to two possible particle-hole channels and one particle-particle/hole-hole channel in the nonrelativistic case) we can establish integral equations for Green's functions characterizing each of the three channels separately. This can be achieved in setting the different times in (3.1) equal pairwise. For example, $t_1 = t_2$ and $t_1' = t_2'$ corresponds to the particle-particle (pp)/hole-hole (hh) channel or $t_1 = t_2'$ and $t_2 = t_1'$ to one of the two possible particle-hole (ph) channels (we do not make the distinction any more, unless necessary, between the relativistic and nonrelativistic case, assuming that the notation pp and ph is self-evident in both cases.)

To be specific let us consider the pp/hh channel, everything being completely analogous in the ph channels (in Sec. V a ph channel is treated explicitly). We therefore will study the function

$$G(t-t',\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_1',\mathbf{x}_2') = -\langle 0 \,|\, T\phi(t,\mathbf{x}_1)\phi(t,\mathbf{x}_2)\phi^\dagger(t',\mathbf{x}_1')\phi^\dagger(t',\mathbf{x}_2') \,|\, 0\rangle \ . \tag{3.2}$$

We do not introduce a new symbol but think that the Green's functions are sufficiently characterized in specifying their arguments. [Often we will write simply $G(t-t')$ for G according to (3.2) and $G(t_1,t_2,t_1',t_2')$ for G of (2.1).] The spectral representation of $G(t-t')$ yields the bound and scattering states of the pp/hh (particle-particle/antiparticle-antiparticle) pair and we will establish an integral equation for this function analogous to the Bethe-Salpeter equation. We only want to outline the principle here. More detailed derivations can be found for the nonrelativistic case in the paper of Mori[21] (condensed matter) and the papers of Werner,[25] Schuck,[26,27] Schuck and Ethofer,[28] or in a textbook by Ring and Schuck.[29] The relativistic case is treated in detail in the next two sections. The basic idea is also similar to those developed by Logunov and Tavkhelidze[3] and by Fishbane and Namyslowski.[30]

Let us thus derive an equation of motion for the spatial Fourier transform of G of (3.2), $G(t-t',\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3',\mathbf{k}_4')$. First we have to find the "inverse" operator of the free part G_0 of G. $G_0{}^{-1}(t-t')$ contains time derivation operators $\partial/\partial t$. Then we calculate $G_0{}^{-1}G$ with the help of the Heisenberg equations of motion for the field operators in G:

$$\int G_0{}^{-1}(t-t'',\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_1'',\mathbf{k}_2'')G(t''-t',\mathbf{k}_1'',\mathbf{k}_2'',\mathbf{k}_1',\mathbf{k}_2')dt''d^3k_1''d^3k_2'' = N(t-t',\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_1',\mathbf{k}_2') + R(t-t',\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_1',\mathbf{k}_2') \tag{3.3a}$$

or in a symbolic "matrix" notation

$$G_0{}^{-1}G = N + R \ . \tag{3.3b}$$

N is simply $G_0{}^{-1}G_0$ (in contrast to the usual definition) and R contains all other terms. In principle R is completely determined by (3.3): $R = G_0{}^{-1}(G-G_0)$, and its explicit form can easily be found using the Heisenberg equations for ϕ and $\phi^\dagger$. If no interaction were present, $G = G_0$ and $R = 0$. Indeed, R explicitly contains the coupling constant to first order. In the next section we will show that it is in fact of second order in the coupling constants.

Because we restricted the propagators to a specific channel, $N=G_0^{-1}G_0$ is not the identity in the Hilbert space of all possible states, but the projector onto the subspace that corresponds to the channel under consideration. It is only in this sense that G_0^{-1} is the inverse of G_0 (more exactly, G_0^{-1} is the pseudoinverse of G_0). Thus, in general, the inverse of N does not exist, but one can apply the usual procedure of diagonalization and elimination of eigenvalues zero, which means nothing else than the restriction to the subspace N is projecting on.

Now it is easy to establish an integral equation for G. Defining the pseudoinverse G^{-1} as the inverse of G in the corresponding subspace and taking into account that $G_0G_0^{-1}G=G$ and $G_0N=G_0$ Eq. (3.3) yields upon left multiplication with G_0

$$G=G_0+G_0MG\ , \tag{3.4a}$$

with

$$M=RG^{-1}\ . \tag{3.4b}$$

M is usually called the memory function or mass operator. If one does not want to go beyond a second-order kernel one can approximate G^{-1} by G_0^{-1} to obtain $M=RG_0^{-1}$. In general we obtain from (3.4a)

$$G^{-1}=G_0^{-1}-M \tag{3.5}$$

and combining this with (3.4b) yields an integral equation to determine the complete M:

$$M=RG_0^{-1}-RM\ . \tag{3.6}$$

This equation may either be solved exactly or M is obtained as a perturbation series

$$M=RG_0^{-1}-R(RG_0^{-1})+RR(RG_0^{-1})-\cdots\ . \tag{3.7}$$

This situation is of course the same as for the kernel K of Eq. (2.2): because one cannot calculate all irreducible graphs, K is never given exactly (in the last section we choose the ladder approximation). Indeed, we will see in the following section that we can establish a one to one correspondence between the perturbation series of K and M (one-boson exchange for example).

Equation (3.4a) has the decisive advantage over Eq. (2.2) that—after Fourier transformation in the time variable—it depends only on one energy: the total energy in the channel considered. Every relative energy or relative time is eliminated, but without having introduced (as does the instantaneous approximation) any further approximation. Thus we get

$$G(E)=G_0(E)+G_0(E)M(E)G(E)\ , \tag{3.8a}$$

where only spatial variables are integrated. For bound states it reduces to an "eigenvalue problem" of the usual type with an energy-dependent potential $M(E)$: If χ is the wave function of a bound state of energy E it obeys the homogenous equation corresponding to (3.8a):

$$\chi(E)=G_0(E)M(E)\chi(E)\ . \tag{3.8b}$$

In the choice of approximations for M we have to be guided by our physical intuition (unless we have a smallness parameter) as is always the case for approximate forms of the Bethe-Salpeter equation. In the nonrelativistic many-body theory a well known and often very useful approximation in the ph channel to Eq. (3.8) is the random-phase approximation[29] (RPA) or equivalently the summation of the infinite series of ph bubbles. This approximation implies replacing $M(E)$ by an energy-independent (static) potential.

In the next section we will describe a relativistic model that gives an energy-dependent M due to a one-boson exchange. This corresponds to developing $M(E)$ up to second order in the coupling constants.

IV. THE MEMORY FUNCTION IN CUTKOSKY'S MODEL

A. Neglecting vacuum correlations

In this section we want to calculate the memory function $M(E)$ for Cutkosky's model, i.e., for a theory of two complex scalar fields ϕ_1 and ϕ_2 interacting through a real scalar field ψ as we described in Sec. II and with a Lagrangian given by (2.5). For the expansion in momentum space of the field operators we adopt the conventions of Bjorken and Drell:[31]

$$\phi_i(t,x)=\int\frac{d^3k}{[(2\pi)^3 2\omega_k^i]^{1/2}}[a_{i+}(t,k)e^{ikx}+a_{i-}^\dagger(t,k)e^{-ikx}]\ ,\quad i=1,2 \text{ and } \omega_k^i=(m_i^2+k^2)^{1/2}\ , \tag{4.1}$$

$$\psi(t,x)=\psi^\dagger(t,x)=\int\frac{d^3k}{[(2\pi)^3 2\omega_k^\mu]^{1/2}}[b(t,k)e^{ikx}+b^\dagger(t,k)e^{-ikx}] \text{ with } \omega_k^\mu=(\mu^2+k^2)^{1/2}\ . \tag{4.2}$$

Here and in what follows we no longer distinguish three-vectors because we always write time and spatial variables separately, so that no confusion is possible. For example, e^{ikx} means $e^{i\mathbf{k}\cdot\mathbf{x}}$.

We have the following equal-time commutation relations:

$$[a_{i+}(t,k),a_{j+}^\dagger(t,k')]=[a_{i-}(t,k),a_{j-}^\dagger(t,k')]=\delta_{ij}\delta^3(k-k')\ . \tag{4.3}$$

All + operators commute with all − operators at equal times as well as all operators of particle 1 with those of particle 2.

The uncorrelated vacuum (and in this section we will just say "the vacuum") is defined by

$$a_{i+}(t,k)\,|\,0\rangle=a_{i-}(t,k)\,|\,0\rangle=b(t,k)\,|\,0\rangle=0 \tag{4.4}$$

for all t and k. Of course, the Hermitian conjugate relations hold, too. The Heisenberg equations of motion for any operator $A(t)$ are

$$A(t)=e^{iH(t-t')}A(t')e^{-iH(t-t')},\quad i\frac{d}{dt}A(t)=[A(t),H]\ ,\quad H=H_0+H_{\rm int},\quad H_{\rm int}=-\int \mathscr{L}_{\rm int}(t,x)d^3x\ . \tag{4.5}$$

Inserting the expansions in momentum space of the fields we obtain

$$H_0=\sum_i \int d^3k\,\omega_k^i[a_{i+}^\dagger(k)a_{i+}(k)+a_{i-}^\dagger(k)a_{i-}(k)]+\int d^3k\,\omega_k^\mu b^\dagger(k)b(k) \tag{4.6}$$

and

$$\begin{aligned}H_{\rm int}=\sum_i g_i\int \frac{d^3k_1d^3k_2}{[(2\pi)^3 8\omega_1\omega_2]^{1/2}}\Bigg[&\frac{a_{i+}^\dagger(k_1)a_{i+}(k_2)b(k_1-k_2)}{(\omega_{1-2}^\mu)^{1/2}}+\frac{a_{i+}^\dagger(k_1)a_{i+}(k_2)b^\dagger(k_2-k_1)}{(\omega_{1-2}^\mu)^{1/2}}\\&+\frac{a_{i-}^\dagger(k_2)a_{i-}(k_1)b(k_2-k_1)}{(\omega_{1-2}^\mu)^{1/2}}+\frac{a_{i-}^\dagger(k_2)a_{i-}(k_1)b^\dagger(k_1-k_2)}{(\omega_{1-2}^\mu)^{1/2}}\\&+\frac{a_{i+}^\dagger(k_1)a_{i-}^\dagger(k_2)b(k_1+k_2)}{(\omega_{1+2}^\mu)^{1/2}}+\frac{a_{i-}(k_1)a_{i+}(k_2)b^\dagger(k_1+k_2)}{(\omega_{1+2}^\mu)^{1/2}}\\&+\frac{a_{i+}^\dagger(k_1)a_{i-}^\dagger(k_2)b^\dagger(-k_1-k_2)}{(\omega_{1+2}^\mu)^{1/2}}+\frac{a_{i-}(k_1)a_{i+}(k_2)b(-k_1-k_2)}{(\omega_{1+2}^\mu)^{1/2}}\Bigg]\ .\end{aligned} \tag{4.7}$$

The first four expressions correspond to vertices where a + or − quantum of the field ϕ_1 or ϕ_2 is scattered by a quantum of the field Ψ, the fifth corresponds to pair creation, the sixth to pair annihilation, the seventh and eighth to vertices of vacuum bubbles (vacuum correlations). In formulas (4.6) and (4.7) we dropped the time argument for simplicity and also wrote ω_1 instead of $\omega_{k_1}^1$. It is evident that H is completely symmetric under permutation of + and − operators, or equivalently, the theory is invariant under charge conjugation. With the help of (4.3), (4.5), (4.6), and (4.7) we want to establish the equations of motion for the a and b operators because we will need them later on:

$$i\frac{\partial}{\partial t}a_{i+}(k)=\omega_k^i a_{i+}(k)+c_{k,k_3}^i[a_{i+}(k_3)+a_{i-}^\dagger(-k_3)][b(k-k_3)+b^\dagger(k_3-k)]\ , \tag{4.8}$$

$$c_{k,k_3}^i f(k,k_3)=g_i\int \frac{d^3k_3 f(k,k_3)}{[(2\pi)^3 8\omega_k^i\omega_{k_3}^i\omega_{k-k_3}^\mu]^{1/2}}\ . \tag{4.9}$$

Due to the invariance of the Hamilton operator under charge conjugation we obtain the corresponding equation for a_{i-} from (4.8) by exchanging all + and − operators. It is easy to establish some rules for expressions of the form (4.9) which will be needed in what follows:

$$\begin{aligned}&({\rm i})\ c_{k,k_3}^i f(k_3)=c_{k,-k_3}^i f(-k_3)\ ,\\&({\rm ii})\ c_{-k-k_3,-k_3}^i f(k,k_3)=c_{k+k_3,k_3}^i f(k,k_3)\ ,\\&({\rm iii})\ c_{-k+k_3,k_3}^i f(k,k_3)=c_{k-k_3,-k_3}^i f(k,k_3)\ ,\\&({\rm iv})\ c_{k,k_3}^i f(k,k_3)=c_{-k,-k_3}^i f(k,k_3)\ ,\\&({\rm v})\ c_{-k+k_3,k_3}^i f(-k+k_3,k_3)=c_{k+k_3,k_3}^i f(k_3,k+k_3)\ .\end{aligned} \tag{4.10}$$

To find $(\partial^2/\partial t^2)a_{i\pm}$ we have to calculate $i(\partial/\partial t)[b(k-k_3)+b^\dagger(k_3-k)]$. One gets

$$\begin{aligned}i\frac{\partial}{\partial t}b(k_4)=[b(k_4),H]=\omega_{k_4}^\mu b(k_4)+\sum_i\{&c_{k_4-k_3,-k_3}^i[a_{i+}^\dagger(k_3-k_4)a_{i+}(k_3)+a_{i-}(k_4-k_3)a_{i+}(k_3)]\\&+c_{k_4+k_3,k_3}^i[a_{i-}^\dagger(k_3)a_{i-}(k_3+k_4)+a_{i+}^\dagger(-k_3-k_4)a_{i-}^\dagger(k_3)]\}\ .\end{aligned} \tag{4.11}$$

With the rules (4.10) it is easy to show that this is symmetric in + and − operators as it should be because of the corresponding symmetry of the Hamilton operator. Taking the Hermitian conjugate and substituting $-k_4$ for k_4 we get an analogous equation for $b^\dagger(-k_4)$. Using the rules (4.10) one shows that all terms containing a_i operators are exactly the same as in (4.11), but with a reversed sign, thus they cancel and we obtain simply

$$i\frac{\partial}{\partial t}[b(k-k_3)+b^\dagger(k_3-k)]=\omega_{k-k_3}^\mu[b(k-k_3)-b^\dagger(k_3-k)]\ . \tag{4.12}$$

Using (4.8), the corresponding equation for $a_{i-}^\dagger$, (4.12) and the rules (4.10) $(\partial^2/\partial t^2)a_{i+}$ is given by

$$-\frac{\partial^2}{\partial t^2}a_{i+}(k)=(\omega_k^i)^2a_{i+}(k)+c_{k,k_3}^i[(\omega_k^i+\omega_{k_3}^i)a_{i+}(k_3)+(\omega_k^i-\omega_{k_3}^i)a_{i-}^\dagger(-k_3)][b(k-k_3)+b^\dagger(k_3-k)]$$
$$+c_{k,k_3}^i\omega_{k-k_3}^\mu[a_{i+}(k_3)+a_{i-}^\dagger(-k_3)][b(k-k_3)-b^\dagger(k_3-k)]\ . \tag{4.13}$$

The corresponding equations for a_{i-}, $a_{i+}^\dagger$, and $a_{i-}^\dagger$ are again obtained by interchanging all + and − operators or by taking the Hermitian conjugate. We abbreviate the inhomogeneities in the differential equations (4.8) and (4.13) by $p_{i+}(k)$ and $j_{i+}(k)$. $j_{i-}(k)$ is obtained from j_{i+} by interchanging + and − operators, $p_{i+}^\dagger(k)$ is the Hermitian conjugate of $p_{i+}(k)$, etc. Thus we can summarize the equations of motion as

$$i\frac{\partial}{\partial t}a_{i+}(k)=\omega_k^i a_{i+}(k)+p_{i+}(k),\quad i\frac{\partial}{\partial t}a_{i-}(k)=\omega_k^i a_{i-}(k)+p_{i-}(k)\ ,$$
$$i\frac{\partial}{\partial t}a_{i+}^\dagger(k)=-\omega_k^i a_{i+}^\dagger(k)-p_{i+}^\dagger(k),\quad i\frac{\partial}{\partial t}a_{i-}^\dagger(k)=-\omega_k^i a_{i-}^\dagger(k)-p_{i-}^\dagger(k)\ , \tag{4.14}$$

with

$$p_{i+}(k)=c_{k,k_3}^i[a_{i+}(k_3)+a_{i-}^\dagger(-k_3)][b(k-k_3)+b^\dagger(k_3-k)] \tag{4.15}$$

and

$$-\frac{\partial^2}{\partial t^2}a_{i+}(k)=(\omega_k^i)^2a_{i+}(k)+j_{i+}(k),\quad -\frac{\partial^2}{\partial t^2}a_{i-}(k)=(\omega_k^i)^2a_{i-}(k)+j_{i-}(k)\ ,$$
$$-\frac{\partial^2}{\partial t^2}a_{i+}^\dagger(k)=(\omega_k^i)^2a_{i+}^\dagger(k)+j_{i+}^\dagger(k),\quad -\frac{\partial^2}{\partial t^2}a_{i-}^\dagger(k)=(\omega_k^i)^2a_{i-}^\dagger(k)+j_{i-}^\dagger(k)\ , \tag{4.16}$$

with

$$j_{i+}(k)=c_{k,k_3}^i[(\omega_k^i+\omega_{k_3}^i)a_{i+}(k_3)+(\omega_k^i-\omega_{k_3}^i)a_{i-}^\dagger(-k_3)][b(k-k_3)+b^\dagger(k_3-k)]$$
$$+c_{k,k_3}^i\omega_{k-k_3}^\mu[a_{i+}(k_3)+a_{i-}(-k_3)][b(k-k_3)-b^\dagger(k_3-k)]\ , \tag{4.17}$$

where c^i is given by (4.9). The p_i and j_i contain one coupling constant explicitly because c^i does so. Using the rules (4.10) it is easy to see that

$$p_{i-}^\dagger(-k)=p_{i+}(k),\quad p_{i+}^\dagger(-k)=p_{i-}(k) \tag{4.18}$$

and

$$j_{i+}(k)+j_{i-}^\dagger(-k)=2\omega_k p_{i+}(k)\ . \tag{4.19}$$

Now we can define the equal-time propagator (or two-time Green's function) in the particle-particle/hole-hole channel as in (3.2). Inserting the momentum-space expansions of the fields $\phi_1,\phi_2,\phi_1^\dagger,\phi_2^\dagger$ and taking into account (4.4) (i.e., in this paragraph we define the propagator with the uncorrelated vacuum) yields

$$G(t-t',x_1,x_2,x_1',x_2')=-\int\frac{d^3k_1d^3k_2d^3k_1'd^3k_2'}{[(2\pi)^{12}16\omega_1\omega_2\omega_{1'}\omega_{2'}]^{1/2}}$$
$$\times\{\theta(t-t')\langle 0\,|\,a_{1+}(t,k_1)a_{2+}(t,k_2)a_{1+}^\dagger(t',k_1')a_{2+}^\dagger(t',k_2')\,|\,0\rangle$$
$$\times\exp[i(k_1x_1+k_2x_2-k_1'x_1'-k_2'x_2')]$$
$$+\theta(t'-t)\langle 0\,|\,a_{1-}(t',k_1')a_{2-}(t',k_2')a_{1-}^\dagger(t,k_1)a_{2-}^\dagger(t,k_2)\,|\,0\rangle$$
$$\times\exp[i(k_1'x_1'+k_2'x_2'-k_1x_1-k_2x_2)]\}\ . \tag{4.20}$$

Other vacuum expectation values do not appear because all operators of the field 1 commute with all operators of the field 2 at the same time and $a_\pm$ annihilates the uncorrelated vacuum. We define Fourier transformation by

$$\mathscr{F}(f(t-t'))=f(E)=(2\pi)^{-1/2}\int f(t-t')e^{iE(t-t')}dt\ ,$$
$$\mathscr{F}^{-1}(f(E))=f(t-t')=(2\pi)^{-1/2}\int f(E)e^{-iE(t-t')}dE\ , \tag{4.21}$$

and

$$f(x_1,x_2,x_1',x_2')=(2\pi)^{-6}\int d^3k_1d^3k_2d^3k_1'd^3k_2'f(k_1,k_2,k_1',k_2')\exp[i(k_1x_1+k_2x_2-k_1'x_1'-k_2'x_2')]\ ,$$
$$f(k_1,k_2,k_1',k_2')=(2\pi)^{-6}\int d^3k_1d^3k_2d^3k_1'd^3k_2'f(x_1,x_2,x_1',x_2')\exp[-i(k_1x_1+k_2x_2-k_1'x_1'-k_2'x_2')]\ . \tag{4.22}$$

However it is advantageous to adopt the convention that for a Green's function like (4.18) we do not define $G(t-t',k_1,k_2,k_1',k_2')$ as the Fourier transform of (4.18) in the sense of (4.22), but apply this definition only for the first term, proportional to $\theta(t-t')$, while for the second one, proportional to $\theta(t'-t)$, the sign of the exponent in (4.22) will be reversed. If we further introduce the abbreviation

$$d_{11'22'}=(16\omega_1\omega_{1'}\omega_2\omega_{2'})^{-1/2}\ , \tag{4.23}$$

we obtain

$$G(t-t',k_1,k_2,k_1',k_2')=-d_{11'22'}[\theta(t-t')\langle 0|a_{1+}(t,k_1)a_{2+}(t,k_2)a_{1+}^\dagger(t',k_1')a_{2+}^\dagger(t',k_2')|0\rangle$$
$$+\theta(t'-t)\langle 0|a_{1-}(t',k_1')a_{2-}(t',k_2')a_{1-}^\dagger(t,k_1)a_{2-}^\dagger(t,k_2)|0\rangle]\ . \tag{4.24}$$

We see that $G(t-t',k_1,k_2,k_1',k_2')$ describes for $t>t'$ the creation of two particles at t' with momenta k_1' and k_2' and their destruction at t with momenta k_1 and k_2, and for $t'>t$ the creation of two antiparticles at t with momenta k_1 and k_2 and their destruction at t' with momenta k_1' and k_2'.

Because at a given time we always write first the a_1 and then the a_2 operator we will drop the indices 1 and 2 in what follows unless this may cause confusion. The two operators on the left have unprimed arguments and the two on the right have primed ones if this is so for the θ function [i.e., when the factor is $\theta(t-t')$] and the other way round. Thus it is clear that $\theta(t''-t')\langle a_-a_-a_-^\dagger a_-^\dagger\rangle$ means

$$\theta(t''-t')\langle 0|a_{1-}(t'',k_1'')a_{2-}(t'',k_2'')a_{1-}^\dagger(t',k_1')a_{2-}^\dagger(t',k_2')|0\rangle\ .$$

And finally we simply write $\langle\cdots\rangle$ instead of $\langle 0|\cdots|0\rangle$.

Equation (4.24) then reads

$$G(t-t',k_1,k_2,k_1',k_2')=-d_{11'22'}[\theta(t-t')\langle a_+a_+a_+^\dagger a_+^\dagger\rangle+\theta(t'-t)\langle a_-a_-a_-^\dagger a_-^\dagger\rangle]\ . \tag{4.25}$$

To obtain the free propagator $G_0(t-t')$ from (4.25) we only have to observe that without any interaction ($g_1=g_2=0$) the temporal evolution of the operators is simply given by

$$a_{i\pm}(t,k)=e^{-i\omega_k^i t}a_{i\pm}(0,k),\quad a_{i\pm}^\dagger(t,k)=e^{i\omega_k^i t}a_{i\pm}^\dagger(0,k)\ , \tag{4.26}$$

and that for the equal-time vacuum expectation values (with the uncorrelated vacuum) we have

$$\langle a_{1\pm}a_{2\pm}a_{1\pm}^\dagger a_{2\pm}^\dagger\rangle_{t=t'}=\delta^3(k_1-k_1')\delta^3(k_2-k_2')\ , \tag{4.27}$$

and thus for $G_0(t-t')$,

$$G_0(t-t',k_1,k_2,k_1',k_2')=-(4\omega_1\omega_2)^{-1}\delta^3(k_1-k_1')\delta^3(k_2-k_2')[\theta(t-t')e^{-i(\omega_1+\omega_2)(t-t')}+\theta(t'-t)e^{i(\omega_1+\omega_2)(t-t')}]\ . \tag{4.28}$$

If fields 1 and 2 are identical we have to replace $\delta^3(k_1-k_1')\delta^3(k_2-k_2')$ by $2Id_{11'22'}$, where Id is a symmetrized δ distribution:

$$Id_{11'22'}=\tfrac{1}{2}[\delta^3(k_1-k_1')\delta^3(k_2-k_2')+\delta^3(k_1-k_2')\delta^3(k_2-k_1')]\ .$$

However, to keep our formulas as short as possible we will assume that fields 1 and 2 are not identical. If they were one obtains the corresponding formulas by replacing always $\delta^3(k_1-k_1')\delta^3(k_2-k_2')$ by $Id_{11'22'}$ (up to a factor 2 or $\frac{1}{2}$) in what follows.

We want to calculate the Fourier transform of (4.28) with respect to the time variable: $G_0(E)$. According to (4.21) we obtain

$$G_0(E,k_1,k_2,k_1',k_2')=-\frac{i}{(2\pi)^{1/2}}\frac{\omega_1+\omega_2}{2\omega_1\omega_2}\frac{1}{E^2-(\omega_1+\omega_2)^2}\delta^3(k_1-k_1')\delta^3(k_2-k_2')\ . \tag{4.29}$$

The inverses (pseudoinverses) $G_0{}^{-1}(E)$ and $G_0{}^{-1}(t-t')$ are defined by

$$\int G_0{}^{-1}(E,k_1,k_2,k_1'',k_2'')G_0(E,k_1'',k_2'',k_1',k_2')d^3k_1''d^3k_2''=\delta^3(k_1-k_1')\delta^3(k_2-k_2')\ , \tag{4.30}$$

$$\int G_0{}^{-1}(t-t'',k_1,k_2,k_1'',k_2'')G_0(t''-t',k_1'',k_2'',k_1',k_2')dt''d^3k_1''d^3k_2''=\delta(t-t')\delta^3(k_1-k_1')\delta^3(k_2-k_2')\ . \tag{4.31}$$

It is easy to obtain $G_0{}^{-1}(E)$,

$$G_0{}^{-1}(E,k_1,k_2,k_1',k_2')=i(2\pi)^{1/2}\frac{2\omega_1\omega_2}{\omega_1+\omega_2}[E^2-(\omega_1+\omega_2)^2]\delta^3(k_1-k_1')\delta^3(k_2-k_2') . \quad (4.32)$$

$G_0{}^{-1}(t-t')$ is then given by $(2\pi)^{-1}$ times the Fourier transform of $G_0{}^{-1}(E)$,

$$G_0{}^{-1}(t-t',k_1,k_2,k_1',k_2')=i\delta(t-t')\delta^3(k_1-k_1')\delta^3(k_2-k_2')\frac{2\omega_1\omega_2}{\omega_1+\omega_2}\left[-\frac{\partial^2}{\partial t'^2}-(\omega_1+\omega_2)^2\right] . \quad (4.33)$$

An equivalent form is also

$$G_0{}^{-1}(t-t',k_1,k_2,k_1',k_2')=\left[-\frac{\overleftarrow{\partial}^2}{\partial t^2}-(\omega_1+\omega_2)^2\right]\frac{2\omega_1\omega_2}{\omega_1+\omega_2}i\delta(t-t')\delta^3(k_1-k_1')\delta^3(k_2-k_2') . \quad (4.34)$$

Of course, one can check directly that $G_0{}^{-1}(t-t')$ from (4.33) and $G_0(t-t')$ from (4.28) fulfill Eq. (4.31).

According to our general discussion in Sec. III we now have to calculate $G_0{}^{-1}G$. $G_0{}^{-1}(t-t')$ contains the operator $\partial^2/\partial t'^2$ and $G(t-t')$ is given by (4.24) or (4.25). We thus have to know first and second derivations of the a operators with respect to time. They have already been given in formulas (4.14) to (4.17). We obtain

$$\begin{aligned}
-\frac{\partial^2}{\partial t^2}G(t-t',k_1,k_2,k_1',k_2') &= -d_{11'22'}i\frac{\partial}{\partial t}i\frac{\partial}{\partial t}[\theta(t-t')\langle a_+a_+a_+^\dagger a_+^\dagger\rangle+\theta(t'-t)\langle a_-a_-a_-^\dagger a_-^\dagger\rangle]\\
&= -d_{11'22'}i\frac{\partial}{\partial t}\left[i\delta(t-t')(\langle a_+a_+a_+^\dagger a_+^\dagger\rangle-\langle a_-a_-a_-^\dagger a_-^\dagger\rangle)\right.\\
&\qquad \left.+\theta(t-t')i\frac{\partial}{\partial t}\langle a_+a_+a_+^\dagger a_+^\dagger\rangle+\theta(t'-t)i\frac{\partial}{\partial t}\langle a_-a_-a_-^\dagger a_-^\dagger\rangle\right]\\
&= -d_{11'22'}\left[0+i\delta(t-t')\left[i\frac{\partial}{\partial t}\langle a_+a_+a_+^\dagger a_+^\dagger\rangle-i\frac{\partial}{\partial t}\langle a_-a_-a_-^\dagger a_-^\dagger\rangle\right]\right.\\
&\qquad \left.+\theta(t-t')\left[-\frac{\partial^2}{\partial t^2}\langle a_+a_+a_+^\dagger a_+^\dagger\rangle\right]+\theta(t'-t)\left[-\frac{\partial^2}{\partial t^2}\langle a_-a_-a_-^\dagger a_-^\dagger\rangle\right]\right] .
\end{aligned} \quad (4.35)$$

We have

$$\delta(t-t')(\langle a_+a_+a_+^\dagger a_+^\dagger\rangle-\langle a_-a_-a_-^\dagger a_-^\dagger\rangle)=\delta(t-t')[\delta^3(k_1-k_1')\delta^3(k_2-k_2')-\delta^3(k_1-k_1')\delta^3(k_2-k_2')]=0 .$$

And furthermore

$$\begin{aligned}
i\frac{\partial}{\partial t}\langle a_+(t)a_+(t)a_+^\dagger(t')a_+^\dagger(t')\rangle-i\frac{\partial}{\partial t}\langle a_-(t')a_-(t')a_-^\dagger(t)a_-^\dagger(t)\rangle\\
=(\omega_1+\omega_2)\langle a_+a_+a_+^\dagger a_+^\dagger\rangle+\langle p_+a_+a_+^\dagger a_+^\dagger\rangle+\langle a_+p_+a_+^\dagger a_+^\dagger\rangle\\
+(\omega_1+\omega_2)\langle a_-a_-a_-^\dagger a_-^\dagger\rangle+\langle a_-a_-p_-^\dagger a_-^\dagger\rangle+\langle a_-a_-a_-^\dagger p_-^\dagger\rangle .
\end{aligned} \quad (4.36)$$

Because of the factor $\delta(t-t')$ all times are equal in this expression and thus $\langle a_+a_+a_+^\dagger a_+^\dagger\rangle$ and $\langle a_-a_-a_-^\dagger a_-^\dagger\rangle$ reduce to $\delta^3(k_1-k_1')\delta^3(k_2-k_2')$ each, and the terms containing one p are zero because they contain a b and a $b^\dagger$ operator which commutes with all a operators at the same time t, and therefore we end up with $\langle 0|b^\dagger$ or $b|0\rangle$ which are both zero (of course, this is true only with an uncorrelated vacuum as dealt with in this subsection). The other terms yield

$$\begin{aligned}
-\frac{\partial^2}{\partial t^2}\langle a_+(t)a_+(t)a_+^\dagger(t')a_+^\dagger(t')\rangle &= \left\langle\left[-\frac{\partial^2}{\partial t^2}a_+\right]a_+a_+^\dagger a_+^\dagger\right\rangle+\left\langle a_+\left[\frac{\partial^2}{\partial t^2}a_+\right]a_+^\dagger a_+^\dagger\right\rangle+2\left\langle\left[i\frac{\partial}{\partial t}a_+\right]\left[i\frac{\partial}{\partial t}a_+\right]a_+^\dagger a_+^\dagger\right\rangle\\
&= (\omega_1)^2\langle a_+a_+a_+^\dagger a_+^\dagger\rangle+\langle j_+a_+a_+^\dagger a_+^\dagger\rangle+(\omega_2)^2\langle a_+a_+a_+^\dagger a_+^\dagger\rangle\\
&\quad +\langle a_+j_+a_+^\dagger a_+^\dagger\rangle+2\omega_1\omega_2\langle a_+a_+a_+^\dagger a_+^\dagger\rangle+2\omega_1\langle a_+p_+a_+^\dagger a_+^\dagger\rangle\\
&\quad +2\omega_2\langle p_+a_+a_+^\dagger a_+^\dagger\rangle+2\langle p_+p_+a_+^\dagger a_+^\dagger\rangle\\
&= (\omega_1+\omega_2)^2\langle a_+a_+a_+^\dagger a_+^\dagger\rangle+\langle j_+a_+a_+^\dagger a_+^\dagger\rangle+\langle a_+j_+a_+^\dagger a_+^\dagger\rangle\\
&\quad +2\omega_1\langle a_+p_+a_+^\dagger a_+^\dagger\rangle+2\omega_2\langle p_+a_+a_+^\dagger a_+^\dagger\rangle+2\langle p_+p_+a_+^\dagger a_+^\dagger\rangle ,
\end{aligned} \quad (4.37)$$

and in a completely analogous way,

$$-\frac{\partial^2}{\partial t^2}\langle a_-a_-a_-^\dagger a_-^\dagger\rangle=(\omega_1+\omega_2)^2\langle a_-a_-a_-^\dagger a_-^\dagger\rangle+\langle a_-a_-j_-^\dagger a_-^\dagger\rangle$$

$$+\langle a_-a_-a_-^\dagger j_-^\dagger\rangle+2\omega_1\langle a_-a_-a_-^\dagger p_-^\dagger\rangle+2\omega_2\langle a_-a_-p_-^\dagger a_-^\dagger\rangle+2\langle a_-a_-p_-^\dagger p_-^\dagger\rangle\ . \tag{4.38}$$

We define the Green's functions

$$G(t-t',j(k_1),k_2,k_1',k_2')=-d_{11'22'}[\theta(t-t')\langle j_+a_+a_+^\dagger a_+^\dagger\rangle+\theta(t'-t)\langle a_-a_-j_-^\dagger a_-^\dagger\rangle] \tag{4.39}$$

and $G(t-t',k_1,p(k_2),k_1',k_2')$, etc., in the same manner and taking into account (4.33) and (4.34) we obtain

$$\int G_0^{-1}(t-t'',k_1,k_2,k_1'',k_2'')G(t''-t',k_1'',k_2'',k_1',k_2')dt''d^3k_1''d^3k_2''=\delta(t-t')\delta^3(k_1-k_1')\delta^3(k_2-k_2')$$
$$+R(t-t',k_1,k_2,k_1',k_2')$$

with

$$R(t-t',k_1,k_2,k_1',k_2')=i\frac{2\omega_1\omega_2}{\omega_1+\omega_2}[G(t-t',j(k_1),k_2,k_1',k_2')+G(t-t',k_1,j(k_2),k_1',k_2')$$
$$+2\omega_1G(t-t',k_1,p(k_2),k_1',k_2')+2\omega_2G(t-t',p(k_1),k_2,k_1',k_2')$$
$$+2G(t-t',p(k_1),p(k_2),k_1',k_2')]\ . \tag{4.40}$$

The function R contains with the j and p operators explicitly one coupling constant g_i. In the last section we already anticipated that R is in fact of second order in the coupling constants. A typical term from R is up to factors independent of g_i:

$$\theta(t-t')\langle p_+a_+a_+^\dagger a_+^\dagger\rangle\sim g_1\langle[a_+(t)+a_-^\dagger(t)][b(t)+b^\dagger(t)]a_+(t)a_+^\dagger(t')a_+^\dagger(t')\rangle$$
$$=g_1\langle a_+(t)b(t)a_+(t)a_+^\dagger(t')a_+^\dagger(t')\rangle \tag{4.41}$$

because of (4.4) and the fact that b and $b^\dagger$ commute with a_+ at equal times. But in (4.41) the b operator cannot be simply brought to the right to get zero [because of $b(t)\,|\,0\rangle=0$], since $b(t)$ does not commute with the a operators at another time t'. Equation (4.41) already explicitly contains one g_i. To prove that R is of second order we have to show that the commutator between $b(t)$ and $a_+^\dagger(t')$ is in lowest order proportional to g.

If there were no interaction, $b(t)$ and $a(t')$ would commute [$a(t')$ stands for $a_\pm(t')$ or $a_\pm^\dagger(t')$]. Thus, a development of the commutator in powers of g cannot contain a constant term. To show that the coefficient of the linear term does not vanish, we define a function

$$f_{t'}(t)=[a_+(t,k),b(t',k')]\ , \tag{4.42}$$

which satisfies the following differential equation:

$$i\frac{d}{dt}f_{t'}(t)=\omega_kf_{t'}(t)+g_{t'}(t)$$

with

$$g_{t'}(t)=[p_+(t,k),b(t',k')]\ , \tag{4.43}$$

with solution

$$f_{t'}(t)=e^{-i\omega_k(t-t')}f_{t'}(t')-i\int_{t'}^{t}e^{-i\omega_k(t-s)}g_{t'}(s)ds\ . \tag{4.44}$$

Since a and b commute at equal times we have $f_{t'}(t')=0$. The function $g_{t'}(s)$ already explicitly contains one g_i. Thus, we can calculate all operators in lowest order. The time evolution is then given by (4.26) and we obtain

$$g_{t'}(s)=c_{k,k_3}^i[e^{-i\omega_{k_3}(s-t')}a_+(t',k_3)+e^{i\omega_{k_3}(s-t')}a_-^\dagger(t',-k_3)]e^{i\omega_{k-k_3}^\mu(s-t')}[b^\dagger(t',k_3-k),b(t',k')]$$
$$=g_i\frac{[e^{-i\omega_{k+k'}(s-t')}a_+(t',k+k')+e^{i\omega_{k+k'}(s-t')}a_-^\dagger(t',-k-k')]}{[(4\pi)^3\omega_k\omega_{k+k'}\omega_{k'}^\mu]^{1/2}}e^{i\omega_{k'}^\mu(s-t')}\ . \tag{4.45}$$

It is trivial to insert this into (4.44) and to perform the integration to get the commutator (4.42) to lowest order. For us, it is only important that $g_{t'}(s)$ and thus also $f_{t'}(t)$ is proportional to g_i in lowest order. Therefore R is of second (and higher) order in the coupling constants.

According to the general equations (3.6) and (3.7), M may now be computed in any order. In this work we will restrict ourselves to a calculation up to second order: $M=RG_0^{-1}$. The contributions in this order are either proportional to g_1g_2 or g_1^2 or g_2^2. A g_i being always connected with a vertex of particle i, the terms proportional to g_1^2 or g_2^2 describe graphs containing two vertices of the same particle. In second order these can only be self-energy diagrams contributing to a mass renormalization. But we assume all masses already renormalized and thus drop all terms proportional to g_1^2 or g_2^2 and retain only those of order g_1g_2. To compute M as RG_0^{-1} we have to calculate essentially $[-\partial^2/\partial t'^2-(\omega_{1'}+\omega_{2'})^2]R(t-t')$. This is very similar to the calculations (4.35) to (4.40).

For example one obtains

$$\left[-\frac{\partial^2}{\partial t'^2}-(\omega_{1'}+\omega_{2'})^2\right]G(t-t',j(k_1),k_2,k_1',k_2')$$
$$=-d_{11'22'}\Bigg[i\frac{\partial}{\partial t'}[-i\delta(t-t')(\langle j_+a_+a_+^\dagger a_+^\dagger\rangle-\langle a_-a_-j_-^\dagger a_-^\dagger\rangle)]-i\delta(t-t')\left[i\frac{\partial}{\partial t'}\langle j_+a_+a_+^\dagger a_+^\dagger\rangle-i\frac{\partial}{\partial t'}\langle a_-a_-j_-^\dagger a_-^\dagger\rangle\right]$$
$$+\theta(t-t')(\langle j_+a_+j_+^\dagger a_+^\dagger\rangle+\langle j_+a_+a_+^\dagger j_+^\dagger\rangle+2\omega_{1'}\langle j_+a_+a_+^\dagger p_+^\dagger\rangle+2\omega_{2'}\langle j_+a_+p_+^\dagger a_+^\dagger\rangle+2\langle j_+a_+p_+^\dagger p_+^\dagger\rangle)$$
$$+\theta(t'-t)(\langle j_-a_-j_-^\dagger a_-^\dagger\rangle+\langle a_-j_-j_-^\dagger a_-^\dagger\rangle+2\omega_{1'}\langle a_-p_-j_-^\dagger a_-^\dagger\rangle+2\omega_{2'}\langle p_-a_-j_-^\dagger a_-^\dagger\rangle+\langle p_-p_-j_-^\dagger a_-^\dagger\rangle)\Bigg]\ . \tag{4.46}$$

The first term in the large square bracket vanishes because the factor $\delta(t-t')$ puts all times equal and then the b and $b^\dagger$ contained in j commute with all a operators and due to (4.4) one gets zero. The second term in the large square bracket yields

$$-i\delta(t-t')[(\omega_{1'}-\omega_{2'})\langle j_+a_+a_+^\dagger a_+^\dagger\rangle-(\omega_{1'}+\omega_{2'})\langle a_-a_-j_-^\dagger a_-^\dagger\rangle$$
$$-\langle j_+a_+p_+^\dagger a_+^\dagger\rangle-\langle j_+a_+a_+^\dagger p_+^\dagger\rangle-\langle p_-a_-j_-^\dagger a_-^\dagger\rangle-\langle a_-p_-j_-^\dagger a_-^\dagger\rangle]\ . \tag{4.47}$$

The first two vacuum expectation values vanish again and the other four equal-time vacuum expectation values are easily calculated using the definitions (4.15) and (4.17) of p and j and the equal-time commutation relations for the a and b operators: $\langle j_+a_+p_+^\dagger a_+^\dagger\rangle$ and $\langle p_-a_-j_-^\dagger a_-^\dagger\rangle$ give divergent integrals, which is not astonishing because they are proportional to g_1^2 and thus contribute to the mass renormalization. As already mentioned we will not retain these terms. Thus we are left with $\langle j_+a_+a_+^\dagger p_+^\dagger\rangle$ and $\langle a_-p_-j_-^\dagger a_-^\dagger\rangle$. Each of them yields

$$g_1g_2\frac{(\omega_1+\omega_{1'}+\omega^\mu)\delta^3(k_1+k_2-k_1'-k_2')}{(4\pi)^3(\omega_1\omega_2\omega_{1'}\omega_{2'})^{1/2}\omega^\mu}=g_1g_2\frac{(\omega_1+\omega_{1'}+\omega^\mu)}{(4\pi)^3\omega^\mu}4d_{11'22'}\delta^3(K-K')\ , \tag{4.48}$$

where we abbreviated $\omega^\mu_{k_1-k_2}$ by ω^μ. The terms containing the θ functions give (in the order g_1g_2) with an evident generalization of the definition (4.39)

$$G(t-t',j(k_1),k_2,k_1',j(k_2'))+2\omega_{1'}G(t-t',j(k_1),k_2,k_1',p(k_2'))$$

which we will also write as

$$G(t-t',j(k_1),k_2,k_1',[j(k_2')+2\omega_{1'}p(k_2')])\ .$$

Equation (4.46) then yields using (4.48)

$$\left[-\frac{\partial^2}{\partial t'^2}-(\omega_{1'}+\omega_{2'})^2\right]G(t-t',j(k_1),k_2,k_1',k_2')=-i\delta(t-t')(d_{11'22'})^2g_1g_2\frac{8(\omega_1+\omega_2+\omega^\mu)}{(4\pi)^3\omega^\mu}\delta^3(K-K')$$
$$+G(t-t',j(k_1),k_2,k_1',[j(k_2')+2\omega_{1'}p(k_2')])\ . \tag{4.49}$$

The corresponding calculations for the other terms in R are completely analogous, only $G(t-t',p(k_1),p(k_2),k_1',k_2')$ is somewhat different. For the term corresponding to the first one in the large square brackets in (4.46) it is not possible to conclude in the same way that it is vanishing because it now contains a product of two b and four a operators, but using the rules (4.10) it is easy to show that $\langle p_+p_+a_+^\dagger a_+^\dagger\rangle_{t=t'}=\langle a_-a_-p_-^\dagger p_-^\dagger\rangle_{t=t'}$, and their difference vanishes. By the same reason the term

$$-d_{11'22'}(-i)\delta(t-t')[i(\partial/\partial t')\langle p_+p_+a_+^\dagger a_+^\dagger\rangle - i(\partial/\partial t')\langle a_-a_-p_-^\dagger p_-^\dagger\rangle]$$
$$= -d_{11'22'}(-i)\delta(t-t')[(-\omega_{1'}-\omega_{2'})\langle p_+p_+a_+^\dagger a_+^\dagger\rangle - (\omega_{1'}+\omega_{2'})\langle a_-a_-p_-^\dagger p_-^\dagger\rangle]$$

yields simply

$$-id_{11'22'}\delta(t-t')2(\omega_{1'}+\omega_{2'})\langle p_+p_+a_+^\dagger a_+^\dagger\rangle = -i\delta(t-t')(d_{11'22'})^2 g_1 g_2 \frac{8(\omega_{1'}+\omega_{2'})}{(4\pi)^3\omega^\mu}\delta^3(K-K') \; . \tag{4.50}$$

Collecting all terms, we finally obtain for $M=RG_0^{-1}$ in the order g_1g_2

$$M(t-t',k_1,k_2,k_1',k_2') = i\frac{2\omega_1\omega_2 2\omega_{1'}\omega_{2'}}{(\omega_1+\omega_2)(\omega_{1'}+\omega_{2'})} i\{-i\delta(t-t')(d_{11'22'})^2 g_1 g_2[(4\pi)^3\omega^\mu]^{-1}\delta^3(K-K')$$
$$\times 8(\omega_1+\omega_{1'}+\omega^\mu+\omega_2+\omega_{2'}+\omega^\mu+2\omega_1+2\omega_2+2\omega_{1'}+2\omega_{2'})$$
$$+G(t-t',[j(k_1)+2\omega_2 p(k_1)],k_2,k_1',[j(k_2')+2\omega_{1'}p(k_2')])$$
$$+G(t-t',k_1,[j(k_2)+2\omega_1 p(k_2)],[j(k_1')+2\omega_{2'}p(k_1')],k_2')\} \; . \tag{4.51}$$

The two Green's functions in (4.51) explicitly contain a factor g_1g_2 and we can replace the operators a and b by the corresponding operators of free fields. Since time evolution is known [Eq. (4.26) and a corresponding one for b and $b^\dagger$], we obtain for M the following explicit expression:

$$M(t-t',k_1,k_2,k_1',k_2') = \frac{g_1g_2\delta^3(K-K')}{(4\pi)^3(\omega_1+\omega_2)(\omega_{1'}+\omega_{2'})\omega^\mu}\{2i\delta(t-t')[3(\omega_1+\omega_2+\omega_{1'}+\omega_{2'})+2\omega^\mu]$$
$$+(\omega_1+\omega_{1'}+\omega^\mu+2\omega_2)(\omega_2+\omega_{2'}+\omega^\mu+\omega_{1'})$$
$$\times[\theta(t-t')e^{-i(\omega_2+\omega_{1'}+\omega^\mu)(t-t')}+\theta(t'-t)e^{i(\omega_2+\omega_{1'}+\omega^\mu)(t-t')}]$$
$$+(\omega_2+\omega_{2'}+\omega^\mu+2\omega_1)(\omega_1+\omega_{1'}+\omega^\mu+2\omega_{2'})$$
$$\times[\theta(t-t')e^{-i(\omega_1+\omega_{2'}+\omega^\mu)(t-t')}+\theta(t'-t)e^{i(\omega_1+\omega_{2'}+\omega^\mu)(t-t')}]\} \; . \tag{4.52}$$

We want to interpret a typical term of M. Since $\theta(t'-t)\theta(t'-t)$ is zero, we can write

$$\theta(t-t')e^{-i(\omega_2+\omega_{1'}+\omega^\mu)(t-t')}+\theta(t'-t)e^{i(\omega_2+\omega_{1'}+\omega^\mu)(t-t')} = [\theta(t-t')e^{-i\omega_2(t-t')}+\theta(t'-t)e^{i\omega_2(t-t')}]$$
$$\times[\theta(t-t')e^{-i\omega_{1'}(t-t')}+\theta(t'-t)e^{i\omega_{1'}(t-t')}]$$
$$\times[\theta(t-t')e^{-i\omega^\mu(t-t')}+\theta(t'-t)e^{i\omega^\mu(t-t')}] \; . \tag{4.53}$$

This is, up to factors, just the product of three Feynman propagators from t' to t for particle 1 with momentum k_1', for particle 2 with momentum k_2 and for the exchange particle with momentum $k_1-k_1'=k_2-k_2'$. In Fig. 2(a) this is shown graphically for $t>t'$. Figure 2(b) shows the corresponding graph for $t'>t$ and 2(c) the other term with the θ functions in (4.53) for $t>t'$ while 2(d) illustrates the meaning of the static term proportional to $\delta(t-t')$. The fact that the four times of the propagator are not allowed to vary independently (because $t_1=t_2, t_1'=t_2'$) is compensated because M not only contains the exchange particle propagator (as does the kernel K of the Bethe-Salpeter equation in the ladder approximation) but also the two one-particle propagators of the fields 1 and 2. The relation of our approach to the Bethe-Salpeter equation will be studied in Sec. IV C, because it is clear that this question can only be treated when we include all vacuum correlations.

Expression (4.53) for $M(t-t')$ is completely symmetric under permutation of all primed and unprimed quantities as well as under $1\leftrightarrow 2$. Using (4.21) the Fourier transform $M(E)$ is easily calculated:

$$M(E,k_1,k_2,k_1',k_2') = \frac{g_1g_2\delta^3(K-K')i}{(4\pi)^3(\omega_1+\omega_2)(\omega_{1'}+\omega_{2'})\omega^\mu(2\pi)^{1/2}}\left[6(\omega_2+\omega_{1'})+2\omega^\mu+(\omega_1+\omega_{1'}+\omega^\mu+2\omega_2)(\omega_2+\omega_{2'}+\omega^\mu+2\omega_{1'})\right.$$
$$\left.\times\left[\frac{1}{E-\omega_2-\omega_{1'}-\omega^\mu}-\frac{1}{E+\omega_2+\omega_{1'}+\omega^\mu}\right]\right]+(1\leftrightarrow 2) \; , \tag{4.54}$$

where E is the total energy of the two-particle system. Note that we have symmetry $E\leftrightarrow -E$ corresponding to a particle-antiparticle symmetry. For the numerical solution it is advantageous to write $M(E)$ in another form obtained by decomposition into partial fractions with respect to ω^μ:

$$M(E,k_1,k_2,k_1',k_2')=-\frac{g_1g_2\delta^3(K-K')i}{(4\pi)^3(\omega_1+\omega_2)(\omega_{1'}+\omega_{2'})\omega^\mu(2\pi)^{1/2}}\times\left[\frac{(\omega_1+\omega_2-E)(\omega_{1'}+\omega_{2'}-E)}{\omega^\mu+\omega_2+\omega_{1'}+E}+\frac{(\omega_1+\omega_2+E)(\omega_{1'}+\omega_{2'}+E)}{\omega^\mu+\omega_2+\omega_{1'}-E}\right]+(1\leftrightarrow 2)\ . \tag{4.55}$$

When taking the Fourier transform of the equation

$$G(t-t')=G_0(t-t')+\int G_0(t-t'')M(t''-t''')G(t'''-t')dt''dt'''\ , \tag{4.56a}$$

a factor $(2\pi)^{1/2}$ appears two times:

$$G(E)=G_0(E)+2\pi G_0(E)M(E)G(E)\ . \tag{4.56b}$$

We either include this factor 2π in the definition of $M(E)$ (as we did in Sec. III without mentioning it explicitly) or we define the integral kernel $(G_0M)(E)$ by

$$(G_0M)(E)=2\pi G_0(E)M(E)\ . \tag{4.57}$$

Using (4.29) and (4.55) we obtain

$$(G_0M)(E,k_1,k_2,k_1',k_2')=-\frac{g_1g_2\delta^3(K-K')}{(4\pi)^3 2\omega_1\omega_2(\omega_{1'}+\omega_{2'})(E^2-(\omega_1+\omega_2)^2)\omega^\mu}\times\left[\frac{(\omega_1+\omega_2-E)(\omega_{1'}+\omega_{2'}-E)}{\omega^\mu+\omega_2+\omega_{1'}+E}+\frac{(\omega_1+\omega_2+E)(\omega_{1'}+\omega_{2'}+E)}{\omega^\mu+\omega_2+\omega_{1'}-E}\right]+(1\leftrightarrow 2)\ . \tag{4.58}$$

We want to examine the two limiting cases $m_1=m_2$ and $m_2\to\infty$ in some more detail.

(a) $m_1=m_2=2m$. We introduce total and relative coordinates by (2.7) (e.g., with $\alpha=\frac{1}{2}$). They are inverted by

$$k_1=\tfrac{1}{2}K+k,\quad k_2=\tfrac{1}{2}K-k \tag{4.59}$$

and in the center-of-mass system ($\mathbf{K}=0$) we have $\mathbf{k}_1=\mathbf{k}$ and $\mathbf{k}_2=-\mathbf{k}$ and thus $\omega_1=\omega_2=(4m^2+k^2)^{1/2}\equiv\omega$ and $\omega_{1'}=\omega_{2'}=\omega'$. We factorize from all quantities a $\delta^3(K-K')$ distribution characterizing the conservation of total momentum, in particular

$$(G_0M)(E,K,k,K',k')=\delta^3(K-K')(G_0M)(E,K,k,k') \tag{4.60}$$

and obtain in the center-of-mass system when introducing λ by (2.6)

$$(G_0M)(E,K=0,k,k')=-\frac{2m^2\lambda}{\pi^2\omega[E^2-4(\omega)^2)]\omega^\mu}\left[\frac{[1-E/(2\omega)][1-E/(2\omega')]}{\omega^\mu+\omega+\omega'+E}+\frac{[1+E/(2\omega)][1+E/(2\omega')]}{\omega^\mu+\omega+\omega'-E}\right]\ . \tag{4.61}$$

(b) $m_2\to\infty$. In this case we have $m=m_1$, where we again introduced total and relative variables. With the binding energy B we put $E=m_2+m-B$ and thus $E^2-(\omega_1+\omega_2)^2\to 2m_2(m-B-\omega)$. Equations (4.58) and (4.60) then yield

$$(G_0M)(E,K=0,k,k')=-\frac{m\lambda}{4\pi^2\omega(m-B-\omega)\omega^\mu}\left[\frac{1}{\omega^\mu+\omega-m+B}+\frac{1}{\omega^\mu+\omega'-m+B}\right]\ . \tag{4.62}$$

In any case, a bound state χ is a solution of

$$\chi_E(k)=\int(G_0M)(E,K=0,k,k')\chi_E(k')dk'\ . \tag{4.63}$$

We have solved this integral equation with the kernels (4.61) and (4.62) by going to a discrete limit and determining the energies making the resulting nonsymmetric matrix singular. We calculated the binding energies for several levels and with different λ. Details concerning the numerical procedure can be found in the Appendix. Before discussing the energies we found, we will show in Sec. IV B how to calculate M including all vacuum correlations. In Sec. IV C we will discuss the binding energies with and without taking into account vacuum correlations and compare them to the corresponding energies given by the Bethe-Salpeter equation and the instantaneous approximation.

B. Taking vacuum correlations into account

In this section we will calculate M to order g_1g_2 without any further approximation. We keep the same model of Cutkosky with the momentum expansions (4.1) and (4.2) of the fields and the Hamilton operator (4.6) and (4.7). Thus the equations of motion (4.14) to (4.17) do not change. The only difference is that now the Green's function G is defined with the physical, i.e., the correlated vacuum. G_0 remains the same because it is defined as the limit of G for $g_i\to 0$, and in this limit the correlated vacuum becomes the uncorrelated one.

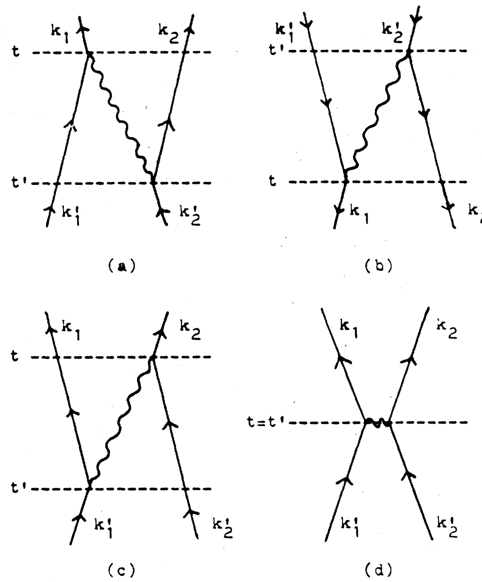

FIG. 2. Graphs described by $M(t-t')$. $M(t-t')$ describes the evolution between the dashed lines at t and t'.

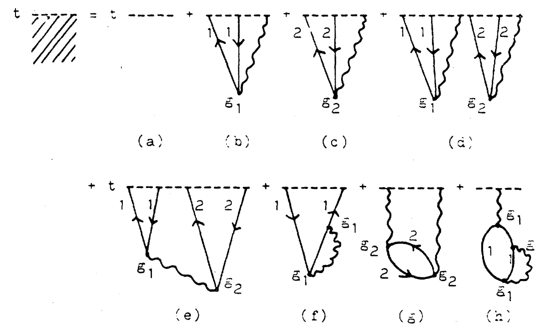

FIG. 3. Graphs contributing to the correlated vacuum.

We will call the uncorrelated vacuum $|0\rangle$ and the correlated one $|\tilde{0}\rangle$. The correlated vacuum can be considered as emerging from the uncorrelated one by pair creation:

$$|\tilde{0}\rangle = c_0|0\rangle + c_1 a^\dagger_{1+}a^\dagger_{1-}b^\dagger|0\rangle + c_2 a^\dagger_{2+}a^\dagger_{2-}b^\dagger|0\rangle + c_3 a^\dagger_{1+}a^\dagger_{1-}b^\dagger a^\dagger_{2+}a^\dagger_{2-}b^\dagger|0\rangle + c_4 a^\dagger_{1+}a^\dagger_{1-}b\, a^\dagger_{2+}a^\dagger_{2-}b^\dagger|0\rangle + \cdots , \quad (4.64)$$

where c_j symbolizes multiplication with a time- and momentum-dependent factor and integration over all possible momenta and times from $-\infty$ to the time the correlated vacuum is taken. In Fig. 3 we give a graphical interpretation of the terms in (4.64) [3(a)–3(e)] and show also some other diagrams contributing to the physical vacuum. As shown in the figure, every vertex corresponds to a coupling constant g_i. Of course, the relation (4.4) does not hold any longer and thus G is not simply given by (4.18) or (4.24). *A priori* there are 16 possible combinations of a operators when inserting the momentum space expansions of the fields in (3.2). (We will write $\langle \cdots \rangle$ as an abbreviation of $\langle\tilde{0}|\cdots|\tilde{0}\rangle$.) Among the terms proportional to $\theta(t-t')$ there are, for example,

$$\langle a_{1+}(t,k_1)a_{2+}(t,k_2)a^\dagger_{1+}(t',k'_1)a^\dagger_{2+}(t',k'_2)\rangle ,$$

$$\langle a_{1+}(t,k_1)a^\dagger_{2-}(t,-k_2)a^\dagger_{1+}(t',k'_1)a_{2-}(t',-k'_2)\rangle ,$$

and among those proportional to $\theta(t'-t)$ is, e.g.,

$$\langle a_{1-}(t',k'_1)a_{2-}(t',k'_2)a_{1+}(t,-k_1)a^\dagger_{2-}(t,k_2)\rangle .$$

In these vacuum expectation values an operator will be said to be on a "right" position if it is a creation operator and occupies one of the two positions on the right in the vacuum expectation value or if it is a destruction operator and occupies one of the two positions on the left. Otherwise, it will be said to be in a "wrong" position. The whole operator sequence is called a "right" or "correct" one if all operators are in "right" positions. In $a_{1+}a^\dagger_{2-}a^\dagger_{1+}a_{2-}$, for example, the operators a_{1+} and $a^\dagger_{1+}$ are in correct position, while $a^\dagger_{2-}$ and a_{2-} are in a wrong one. It is now easy to see that all operators in correct positions have an argument $+k$ whereas all those in a wrong one have an argument $-k$. Furthermore we have unprimed arguments for the two operators on the left if they are a_+ or $a^\dagger_-$ and primed ones for those on the right if they are $a^\dagger_+$ or a_-, and the other way round. Thus we can drop all arguments again. For example, $\langle a^\dagger_- a^\dagger_- a^\dagger_+ a_-\rangle$ means

$$\langle a^\dagger_{1-}(t,-k_1)a^\dagger_{2-}(t,-k_2)a^\dagger_{1+}(t',k'_1)a_{2-}(t',-k'_2)\rangle .$$

Then the Green's function G is

$$\begin{aligned} G(t-t',k_1,k_2,k'_1,k'_2) = -d_{11'22'}\{\theta(t-t')[\, & \langle a_+a_+a^\dagger_+a^\dagger_+\rangle + \langle a_+a_+a^\dagger_+a_-\rangle + (1\leftrightarrow 2) + \langle a_+a_+a_-a_-\rangle + \langle a_+a^\dagger_-a^\dagger_+a^\dagger_+\rangle + (1\leftrightarrow 2) \\ & + \langle a_+a^\dagger_-a^\dagger_+a_-\rangle + (1\leftrightarrow 2) + \langle a_+a^\dagger_-a_-a^\dagger_+\rangle + (1\leftrightarrow 2) + \langle a_+a^\dagger_-a_-a_-\rangle + (1\leftrightarrow 2) \\ & + \langle a^\dagger_-a^\dagger_-a^\dagger_+a^\dagger_+\rangle + \langle a^\dagger_-a^\dagger_-a^\dagger_+a_-\rangle + (1\leftrightarrow 2) + \langle a^\dagger_-a^\dagger_-a_-a_-\rangle] \\ + \theta(t'-t)[\, & \langle a_-a_-a^\dagger_-a^\dagger_-\rangle + \langle a_-a_-a^\dagger_-a_+\rangle + (1\leftrightarrow 2) + \langle a_-a_-a_+a_+\rangle + \langle a_-a^\dagger_+a^\dagger_-a^\dagger_-\rangle + (1\leftrightarrow 2) \\ & + \langle a_-a^\dagger_+a^\dagger_-a_+\rangle + (1\leftrightarrow 2) + \langle a_-a^\dagger_+a_+a^\dagger_-\rangle + (1\leftrightarrow 2) + \langle a_-a^\dagger_+a_+a_+\rangle + (1\leftrightarrow 2) \\ & + \langle a^\dagger_+a^\dagger_+a^\dagger_-a^\dagger_-\rangle + \langle a^\dagger_+a^\dagger_+a^\dagger_-a_+\rangle + (1\leftrightarrow 2) + \langle a^\dagger_+a^\dagger_+a_+a_+\rangle]\} \end{aligned} \quad (4.65)$$

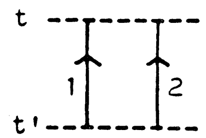

FIG. 4. Graphical representation of $\langle a_+ a_+ a_+^\dagger a_+^\dagger \rangle$.

[where $\langle a_+ a_+ a_+^\dagger a_- \rangle + (1 \leftrightarrow 2)$ means $\langle a_+ a_+ a_+^\dagger a_- \rangle + \langle a_+ a_+ a_- a_+^\dagger \rangle$)]. It is easy to check that to each vacuum expectation value for $t > t'$ corresponds one for $t' > t$ obtained from the first one by taking the complex conjugate and interchange + and −.

We now have to discuss the 16 vacuum expectation values appearing in (4.65): in which order do they contribute? In zeroth order only $\langle a_+ a_+ a_+^\dagger a_+^\dagger \rangle$ (and $\langle a_- a_- a_-^\dagger a_-^\dagger \rangle$) give a contribution. We represent this graphically as shown in Fig. 4. Of course, the diagram is only a symbolic one, because all kinds of interactions are possible between t' and t. The $a_+^\dagger$ operators describe the creation of two particles at the time t' and the a_+ their destruction at t. Let us now consider $\langle a_+ a_+ a_+^\dagger a_- \rangle$: Particle 1 is created at t' and destroyed at t. Particle 2 is destroyed by a_{2+} at t, at t' however it is now created but a_{2-} destroys an antiparticle 2. This is only possible if at time t' a particle-antiparticle pair was already present. But this is possible due to the virtual pair creation in the correlated vacuum. However, an exchange particle is created with the pair and has to be destroyed at some later time. This is shown in Fig. 5. We see that the term $\langle a_+ a_+ a_+^\dagger a_- \rangle$ yields contributions of order $g_2 g_2$ and $g_1 g_2$ (also of order g^4). Of course, we could have obtained the same results in an analytical way, using (4.64),

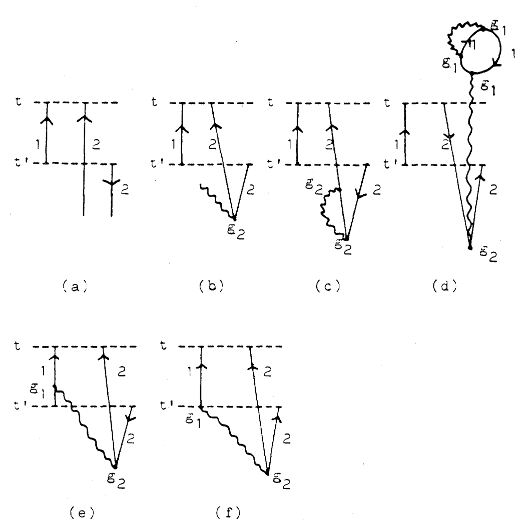

FIG. 5. Graphical representation of $\langle a_+ a_+ a_+^\dagger a_- \rangle$ (a); with possible contribution from the correlated vacuum (b). The possible couplings of the line of the exchange particle and the resulting orders in the coupling constants are given in (c) to (f).

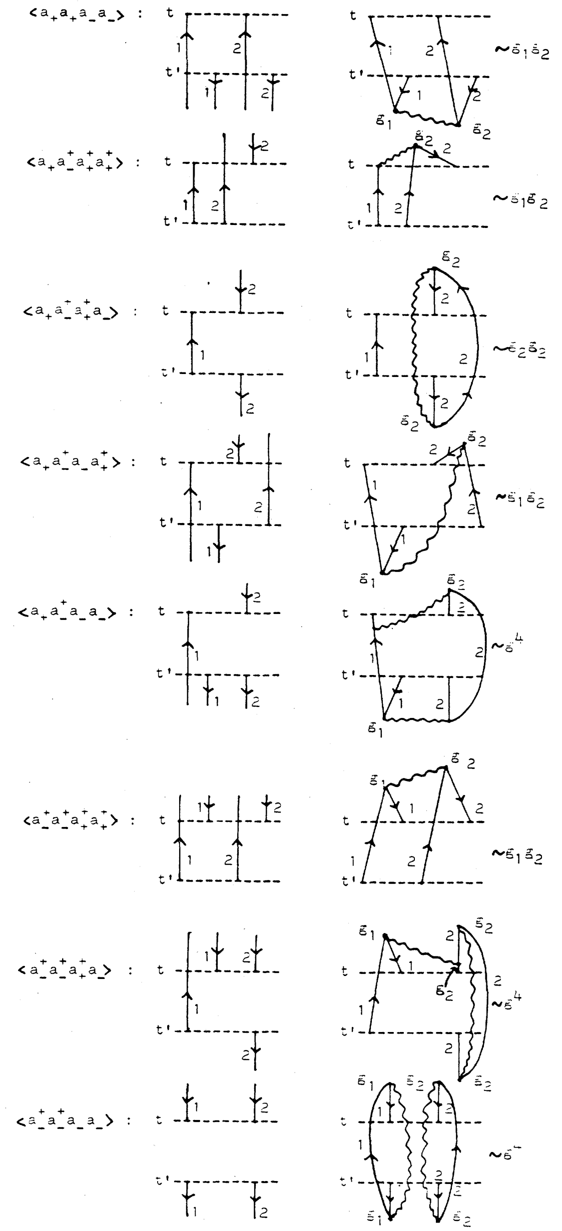

FIG. 6. Contributions of the different vacuum expectation values.

but we prefer the more intuitive graphical discussion. In a similar way, one can discuss the other vacuum expectation values in (4.65) (see Fig. 6). From this figure, we see that any operator in a wrong position yields one g: an operator a_1 in a wrong position yields a g_1, an a_2 yields a g_2. In $\langle a_+ a_-^\dagger a_+^\dagger a_- \rangle$, e.g., $a_{2-}^\dagger$ and a_{2-} are in a wrong position, this term is proportional (in lowest order) to g_2^2. If only one a is in a wrong position, e.g., a_1, the term gives contributions of order $g_1 g_2$ and g_1^2. If more than two a operators are in wrong positions, the term is of order g^4.

Again, we only keep terms giving a contribution to order $g_1 g_2$. Expression (4.65) then reduces to

$$
\begin{aligned}
G(t-t',k_1,k_2,k_1',k_2') = -d_{11'22'}\{\theta(t-t')[\,&\langle a_+a_+a_+^\dagger a_+^\dagger\rangle+\langle a_+a_+a_-a_-\rangle+\langle a_-^\dagger a_-^\dagger a_+^\dagger a_+^\dagger\rangle\\
&+\langle a_+a_+a_+^\dagger a_-\rangle+(1\leftrightarrow 2)+\langle a_+a_-^\dagger a_+^\dagger a_+^\dagger\rangle+(1\leftrightarrow 2)\\
&+\langle a_+a_-^\dagger a_-a_+^\dagger\rangle+(1\leftrightarrow 2)]\\
+\theta(t'-t)[\,&\langle a_-a_-a_-^\dagger a_-^\dagger\rangle+\langle a_+^\dagger a_+^\dagger a_-^\dagger a_-^\dagger\rangle+\langle a_-a_-a_+a_+\rangle\\
&+\langle a_-a_+^\dagger a_-^\dagger a_-^\dagger\rangle+(1\leftrightarrow 2)+\langle a_-a_-a_-^\dagger a_+\rangle+(1\leftrightarrow 2)\\
&+\langle a_+^\dagger a_-a_-^\dagger a_+\rangle+(1\leftrightarrow 2)]\}\ .
\end{aligned}
\tag{4.66}
$$

Now we have to calculate $G_0{}^{-1}G$:

$$
\begin{aligned}
&\left[-\frac{\partial^2}{\partial t^2}-(\omega_1+\omega_2)^2\right]G(t-t',k_1,k_2,k_1',k_2')\\
&\quad=-d_{11'22'}\left[i\frac{\partial}{\partial t}\{i\delta(t-t')[\cdots]-i\delta(t-t')[\cdots]\}+i\delta(t-t')\left[i\frac{\partial}{\partial t}[\cdots]-i\frac{\partial}{\partial t}[\cdots]\right]\right.\\
&\qquad\left.-\theta(t-t')\frac{\partial^2}{\partial t^2}[\cdots]-\theta(t'-t)\frac{\partial^2}{\partial t^2}[\cdots]\right]\ .
\end{aligned}
\tag{4.67}
$$

The first term in the large square brackets in (4.67) [the terms proportional to $(\partial/\partial t)\delta(t-t')$] is zero. This is connected with *PCT* invariance.

Let T be the antiunitary time inversion operator. It is easy to see that Cutkosky's model is invariant under time inversion: $[T,H]=0$ with H according to (4.6), (4.7), and

$$Ta(t,k)T^{-1}=\pm a(-t,-k)\ . \tag{4.68}$$

The theory is also invariant under the parity operation P with

$$Pa(t,k)P^{-1}=a(t,-k)\ , \tag{4.69}$$

and under charge conjugation C with

$$Ca_\pm(t,k)C^{-1}=a_\mp(t,k),\ \ Ca_\pm^\dagger(t,k)C^{-1}=a_\mp^\dagger(t,k)\ . \tag{4.70}$$

Of course, Lorentz invariance also implies invariance under time translation $D_s\colon t\to t+s$.

To show that the first term in the large square brackets in (4.67) vanishes, we do not need all these symmetries separately. Only *PCT* invariance and invariance under time translation is necessary. Especially the correlated vacuum has to be *PCT* invariant (if it is unique, this holds true). Then we get for $\langle a_+a_-^\dagger a_+^\dagger a_+^\dagger\rangle$, for example,

$$
\begin{aligned}
\langle\tilde 0|a_{1+}(t,k_1)a_{2-}^\dagger(t,-k_2)a_{1+}^\dagger(t',k_1')a_{2+}^\dagger(t',k_2')|\tilde 0\rangle&=\langle\tilde 0|a_{1-}(-t,k_1)a_{2+}^\dagger(-t,-k_2)a_{1-}^\dagger(-t',k_1')a_{2-}^\dagger(-t',k_2')|\tilde 0\rangle^+\\
&=\langle\tilde 0|a_{1-}(t,k_1')a_{2-}(t,k_2')a_{1-}^\dagger(t',k_1)a_{2+}(t',-k_2)|\tilde 0\rangle\ ,
\end{aligned}
\tag{4.71}
$$

where we used in a first step invariance under *PCT* and in a second step under $D_{t+t'}$. Thus we get for $t=t'$:

$$\langle a_+a_-^\dagger a_+^\dagger a_+^\dagger\rangle_{t=t'}=\langle a_-a_-a_-^\dagger a_+\rangle_{t=t'}\ . \tag{4.72}$$

In a completely analogous manner one shows the equality of all pairs of corresponding vacuum expectation values in the large square brackets in (4.67), their sign being always opposite, this term vanishes.

Because C invariance holds also alone we have

$$
\begin{aligned}
\langle\tilde 0|a_-(t,k_1')a_-(t,k_2')a_-^\dagger(t',k_1)a_+(t',-k_2)|\tilde 0\rangle&=\langle\tilde 0|a_+(t,k_1')a_+(t,k_2')a_+^\dagger(t',k_1)a_-(t',-k_2)|\tilde 0\rangle\\
&=\langle\tilde 0|a_+(t',k_1)a_-^\dagger(t',-k_2)a_+^\dagger(t,k_1')a_+^\dagger(t,k_2')|\tilde 0\rangle^+\ .
\end{aligned}
\tag{4.73}
$$

Together with (4.72) this shows, that for $t=t'$ $\langle a_+a_-^\dagger a_+^\dagger a_+^\dagger\rangle$ is real. Of course, the same applies for all other equal-time vacuum expectation values of four operators (these may be a, b, j or p operators): they are real.

Now we want to calculate the second term in the large square brackets in (4.67), i.e., the terms proportional to $\delta(t-t')$. Using *PCT* invariance on the one hand, i.e., equations analogous to (4.72), and on the other hand dropping all terms which are not of order g_1g_2 (or g^0) according to the discussion of operators on wrong and correct positions (p or j operators are never on wrong positions because they contain a_+ and $a_-^\dagger$ or a_- and $a_+^\dagger$) the second term in the large square brackets in (4.67) becomes

$$2[(\omega_1+\omega_2)\langle a_+a_+a_+^\dagger a_+^\dagger\rangle+\langle p_+a_+a_+^\dagger a_+^\dagger\rangle+\langle a_+p_+a_+^\dagger a_+^\dagger\rangle$$
$$+(\omega_1+\omega_2)\langle a_+a_+a_-a_-\rangle-(\omega_1+\omega_2)\langle a_-^\dagger a_-^\dagger a_+^\dagger a_+^\dagger\rangle-\langle p_-^\dagger a_-^\dagger a_+^\dagger a_+^\dagger\rangle-\langle a_-^\dagger p_-^\dagger a_+^\dagger a_+^\dagger\rangle$$
$$+(\omega_1+\omega_2)\langle a_+a_+a_+^\dagger a_-\rangle+\langle p_+a_+a_+^\dagger a_-\rangle+(1\leftrightarrow 2)$$
$$+(\omega_1-\omega_2)\langle a_+a_-^\dagger a_-^\dagger a_+^\dagger\rangle+\langle p_+a_-^\dagger a_+^\dagger a_+^\dagger\rangle-\langle a_+p_-^\dagger a_+^\dagger a_+^\dagger\rangle+(1\leftrightarrow 2)$$
$$+(\omega_1-\omega_2)\langle a_+a_-^\dagger a_-a_+^\dagger\rangle-\langle a_+p_-^\dagger a_-a_+^\dagger\rangle+(1\leftrightarrow 2)]\ , \quad (4.74)$$

where "$+(1\leftrightarrow 2)$" refers to all terms of the line. We will use (4.18) and

$$\langle a_+(k_1)a_+(k_2)a_-(-k_1')a_-(-k_2')\rangle=\langle[a_+a_+,a_-a_-]\rangle+\langle a_-(-k_1')a_-(-k_2')a_+(k_1)a_+(k_2)\rangle$$
$$=\langle a_-^\dagger(k_1)a_-^\dagger(k_2)a_+^\dagger(-k_1')a_+^\dagger(-k_2')\rangle$$
$$=\langle a_-^\dagger(-k_1)a_-^\dagger(-k_2)a_+^\dagger(k_1')a_+^\dagger(k_2')\rangle \quad (4.75)$$

because the commutator vanishes and because of *PCT* invariance and *P* invariance alone. Here for the first time, we use more than only *PCT* invariance alone, but this is not a crucial point of our method because we only use it to simplify the formula. If *P* invariance would not hold we would have had to calculate the corresponding terms separately. A similar result applies for $\langle a_+a_-^\dagger a_-a_+^\dagger\rangle$ and $\langle a_-^\dagger a_+a_+^\dagger a_-\rangle$. Thus (4.74) can be simplified to give

$$2[\tfrac{1}{2}(\omega_1+\omega_2)\langle a_+a_+a_+^\dagger a_+^\dagger\rangle+(\omega_1+\omega_2)\langle a_+a_+a_+^\dagger a_-\rangle+(\omega_1-\omega_2)\langle a_+a_-^\dagger a_+^\dagger a_+^\dagger\rangle]+(1\leftrightarrow 2)\ . \quad (4.76)$$

We have $\langle a_+a_+a_+^\dagger a_-\rangle=\langle[a_+a_+,a_+^\dagger a_-]\rangle+\langle a_+^\dagger a_-a_+a_+\rangle$.

In the last term three operators are in a wrong position. Its contribution is $\sim g^4$ and thus we drop it. The commutator yields $\delta^3(k_1-k_1')\langle a_{2-}a_{2+}\rangle$. This vacuum expectation value is of order ${g_2}^2$ or g^4 as can be seen from Fig. 7. Thus $\langle a_+a_+a_+^\dagger a_-\rangle$ gives no contribution of order g_1g_2. The same can be shown for $\langle a_+a_-^\dagger a_+^\dagger a_+^\dagger\rangle$. We are left with

$$(\omega_1+\omega_2)\langle a_+a_+a_+^\dagger a_+^\dagger\rangle=(\omega_1+\omega_2)(\langle[a_+a_+,a_+^\dagger a_+^\dagger]\rangle+\langle a_+^\dagger a_+^\dagger a_+a_+\rangle)\ ,$$

where in the last term four operators are in wrong positions. The commutator yields $\delta^3(k_2-k_2')\langle a_{1+}a_{1+}^\dagger\rangle+\delta^3(k_1-k_1')\langle a_{2+}^\dagger a_{2+}\rangle$. The second vacuum expectation value is again of the order ${g_2}^2$ and the first one gives

$$\langle a_{1+}a_{1+}^\dagger\rangle=\langle[a_{1+},a_{1+}^\dagger]\rangle+\langle a_{1+}^\dagger a_{1+}\rangle=\delta^3(k_1-k_1')+O({g_1}^2)\ .$$

Equation (4.76) becomes

$$2(\omega_1+\omega_2)\delta^3(k_1-k_1')\delta^3(k_2-k_2')\ . \quad (4.77)$$

We obtain the same result as in Sec. IV A when we neglected all vacuum correlations.

To calculate the terms in (4.67) with the θ functions we define the abbreviation

$$g(a_+p_+^\dagger a_+^\dagger a_-)=\theta(t-t')\langle a_+p_-^\dagger a_+^\dagger a_-\rangle+\theta(t'-t)\langle a_-a_+^\dagger a_-^\dagger p_+\rangle \quad (4.78)$$

and analogously $g(a_+a_-^\dagger p_+^\dagger a_-)$, etc. To know the order of such a g function we can again apply the method of the wrong positions: in $g(a_+p_-^\dagger a_+^\dagger a_-)$, e.g., a_{2-} is in a wrong position, this yields a factor g_2; $p_{2-}^\dagger$ explicitly contains a coupling constant g_2 and thus $g(a_+p_-^\dagger a_+^\dagger a_-)$ is of order ${g_2}^2$. We again drop all terms not of order g_1g_2 (or g^0) and obtain

$$\int G_0^{-1}(t-t'',k_1,k_2,k_1'',k_2'')G(t''-t',k_1'',k_2'',k_1',k_2')dt''d^3k_1''d^3k_2''$$
$$=\delta(t-t')\delta^3(k_1-k_1')\delta^3(k_2-k_2')-i\frac{2\omega_1\omega_2}{\omega_1+\omega_2}d_{11'22'}r(t-t',k_1,k_2,k_1',k_2')$$

with

$$r(t-t',k_1,k_2,k_1',k_2')=[g(j_+a_+a_+^\dagger a_+^\dagger)+2\omega_2 g(p_+a_+a_+^\dagger a_+^\dagger)+g(p_+p_+a_+^\dagger a_+^\dagger)+g(j_-^\dagger a_-^\dagger a_+^\dagger a_+^\dagger)$$
$$+2\omega_2 g(p_-^\dagger a_-^\dagger a_+^\dagger a_+^\dagger)+g(p_-^\dagger p_-^\dagger a_+^\dagger a_+^\dagger)+g(j_+a_+a_+^\dagger a_-)+2\omega_2 g(p_+a_+a_+^\dagger a_-)$$
$$-4\omega_1\omega_2 g(a_+a_-^\dagger a_+^\dagger a_+^\dagger)+g(j_+a_-^\dagger a_+^\dagger a_+^\dagger)+g(a_+j_-^\dagger a_+^\dagger a_+^\dagger)-2\omega_1 g(a_+p_-^\dagger a_+^\dagger a_+^\dagger)$$
$$-2\omega_2 g(p_+a_-^\dagger a_+^\dagger a_+^\dagger)-2g(p_+p_-^\dagger a_+^\dagger a_+^\dagger)-4\omega_1\omega_2 g(a_+a_-^\dagger a_-a_+^\dagger)+g(a_+j_-^\dagger a_-a_+^\dagger)$$
$$-2\omega_1 g(a_+p_-^\dagger a_-a_+^\dagger)]+(1\leftrightarrow 2)\ . \quad (4.79)$$

To get M to order g_1g_2, we have to apply ${G_0}^{-1}$ from the right to $R=-2i\omega_1\omega_2 d_{11'22'}(\omega_1+\omega_2)^{-1}r$. All calculations are very similar to those already outlined above. Using *PCT* invariance and (4.18) we obtain the following result:

$$M(t-t',k_1,k_2,k_1',k_2')=\frac{(\omega_1\omega_2\omega_{1'}\omega_{2'})^{1/2}}{(\omega_1+\omega_2)(\omega_{1'}+\omega_{2'})}\{-i\delta(t-t')[-2(\omega_{1'}+\omega_{2'})\langle(j_++j_-^\dagger)a_+a_+^\dagger a_+^\dagger\rangle$$
$$-2(\omega_{1'}+\omega_{2'})\langle(j_++j_-^\dagger)a_-^\dagger a_+^\dagger a_+^\dagger\rangle$$
$$+2(\omega_{2'}-\omega_{1'})\langle(j_++j_-^\dagger)a_+a_+^\dagger a_-\rangle]$$
$$+g[(j_++j_-^\dagger)a_+a_+^\dagger(j_+^\dagger+j_-)]$$
$$-4\omega_{1'}\omega_{2'}g[(j_++j_-^\dagger)a_+a_+^\dagger a_-]-4\omega_1\omega_2 g[a_-^\dagger a_+a_+^\dagger(j_+^\dagger+j_-)]$$
$$+16\omega_1\omega_2\omega_{1'}\omega_{2'}g(a_+a_-^\dagger a_-a_+^\dagger)\}+(1\leftrightarrow 2)\ . \tag{4.80}$$

Notice that no term containing a p is left. Using (4.19) this may also be written as

$$M(t-t',k_1,k_2,k_1',k_2')=\frac{(\omega_1\omega_2\omega_{1'}\omega_{2'})^{1/2}}{(\omega_1+\omega_2)(\omega_{1'}+\omega_{2'})}\{-i\delta(t-t')[-4\omega_1(\omega_{1'}+\omega_{2'})\langle p_+a_+a_+^\dagger a_+^\dagger\rangle$$
$$-4\omega_1(\omega_{1'}+\omega_{2'})\langle p_+a_-^\dagger a_+^\dagger a_+^\dagger\rangle$$
$$-4\omega_1(\omega_{1'}-\omega_{2'})\langle p_+a_+a_+^\dagger a_-\rangle]$$
$$+4\omega_1\omega_{2'}g(p_+a_+a_+^\dagger p_+^\dagger)-8\omega_1\omega_{1'}\omega_{2'}g(p_+a_+a_+^\dagger a_-)$$
$$-8\omega_1\omega_2\omega_{2'}g(a_-^\dagger a_+a_+^\dagger p_+^\dagger)+16\omega_1\omega_2\omega_{1'}\omega_{2'}g(a_+a_-^\dagger a_-a_+^\dagger)\}+(1\leftrightarrow 2)\ . \tag{4.81}$$

We first calculate the equal-time vacuum expectation values, i.e., the quantity in square brackets in (4.81), explicitly. For $t=t'$ we have

$$\langle p_+a_+a_+^\dagger a_-\rangle=c^1_{k_1,k_3}\langle[a_+(k_3)+a_-^\dagger(-k_3)][b(k_1-k_3)+b^\dagger(k_3-k_1)]a_+(k_2)a_+^\dagger(k_1')a_-(-k_2')\rangle$$
$$=c^1\langle a_+(b+b^\dagger)a_+a_+^\dagger a_-\rangle+O(g^4)\ ,$$

because in the other term two a operators are in wrong positions and c^1 explicitly contains a g_1. Furthermore, this is equal to

$$c^1\langle[a_{1+}(b+b^\dagger)a_{2+},a_{1+}^\dagger a_{2-}]\rangle+O(g^4)=c^1_{k_1,k_3}\langle[a_{1+}(k_1),a_{1+}^\dagger(k_1')][b(k_1-k_3)+b^\dagger(k_3-k_1)]a_{2+}(k_2)a_{2-}(-k_2')\rangle$$

and finally we obtain

$$\langle p_+a_+a_+^\dagger a_-\rangle_{t=t'}=\frac{g_1}{[(4\pi)^3\omega_1\omega_{1'}\omega^\mu]^{1/2}}\langle[b(k_1-k_1')+b^\dagger(k_1'-k_1)]a_{2+}(k_2)a_{2-}(-k_2')\rangle_{t=t'}\ . \tag{4.82}$$

For $\langle p_+a_+a_+^\dagger a_+^\dagger\rangle$ we have in a very similar way for $t=t'$:

$$\langle p_+a_+a_+^\dagger a_+^\dagger\rangle=c^1\langle(a_{1+}+a_{1-}^\dagger)(b+b^\dagger)a_{2+}a_{1+}^\dagger a_{2+}^\dagger\rangle$$
$$=c^1\langle[(a_{1+}+a_{1-}^\dagger)(b+b^\dagger)a_{2+},a_{1+}^\dagger a_{2+}^\dagger]\rangle+O(g^4)$$
$$=g_1[(4\pi)^3\omega_1\omega_{1'}\omega^\mu]^{-1/2}\langle(b+b^\dagger)a_{2+}a_{2+}^\dagger\rangle$$
$$=g_1[(4\pi)^3\omega_1\omega_{1'}\omega^\mu]^{-1/2}\{\langle[(b+b^\dagger)a_{2+},a_{2+}^\dagger]\rangle+\langle a_{2+}^\dagger(b+b^\dagger)a_{2+}\rangle\}\ .$$

The commutator yields a product of a δ^3 distribution with the vacuum expectation value $\langle(b+b^\dagger)\rangle$. It is easy to see graphically that $\langle b\rangle$ and $\langle b^\dagger\rangle$ are of order g^3 and together with the explicit factor g_1, this yields an order g^4. In the other vacuum expectation value both a_2 operators are in a wrong position, giving a factor ${g_2}^2$. Thus, to order g_1g_2 we obtain

$$\langle p_+a_+a_+^\dagger a_+^\dagger\rangle_{t=t'}=0\ . \tag{4.83}$$

In a very similar way we get for $\langle p_+a_-^\dagger a_+^\dagger a_+^\dagger\rangle_{t=t'}$

$$g_1[(4\pi)^3\omega_1\omega_{1'}\omega^\mu]^{-1/2}\langle[b(k_1-k_1')+b^\dagger(k_1'-k_1)]a_{2-}^\dagger(-k_2)a_{2+}^\dagger(k_2')\rangle\ .$$

Using *PCT* invariance and *P* invariance alone we find

$$\langle p_+a_-^\dagger a_+^\dagger a_+^\dagger\rangle=\langle p_+a_+a_+^\dagger a_-\rangle\quad\text{for } t=t'\ . \tag{4.84}$$

(Here again it is not crucial that *P* invariance alone does hold.) Thus the quantity in square brackets in (4.81) becomes

$$-\frac{8\omega_1\omega_{1'}g_1}{[(4\pi)^3\omega_1\omega_{1'}\omega^\mu]^{1/2}}\langle[b(k_1-k_1')+b^\dagger(k_1'-k_1)]a_{2+}(k_2)a_{2-}(-k_2')\rangle_{t=t'} . \quad (4.85)$$

To continue we have to use again the equations of motion. Therefore we put

$$f(t-t')=\langle[b(t,k_1-k_1')+b^\dagger(t,k_1'-k_1)]a_{2+}(t,k_2)a_{2-}(t',-k_2')\rangle \quad (4.86)$$

and get

$$f(t-t')\left[i\frac{\overleftarrow{\partial}}{\partial t'}-\omega_{2'}\right]=q(t-t')=\langle[b(t,k_1-k_1')+b^\dagger(t,k_1'-k_1)]a_{2+}(t,k_2)p_{2-}(t',-k_2')\rangle . \quad (4.87)$$

$q(t-t')$ contains explicitly a coupling constant g_2 because p_2 does so. Since we are only interested in calculating (4.84) up to order g_1g_2 we can replace all operators in $q(t-t')$ by the corresponding ones of free fields and the correlated vacuum by the uncorrelated one. Using (4.26) we obtain

$$q(t-t')=\frac{g_2\delta^3(K-K')}{[(4\pi)^3\omega_2\omega_{2'}\omega^\mu]^{1/2}}e^{-i(\omega_2+\omega^\mu)(t-t')} . \quad (4.88)$$

This is of the form

$$h(i\partial/\partial t')f(t-t')=C\,e^{ia(t-t')} , \quad (4.89)$$

where h is an operator function of $i\partial/\partial t'$. Fourier transformation according to (4.19) yields $h(-E)f(E)=(2\pi)^{1/2}C\delta(E+a)$, from what follows $f(E)=(2\pi)^{1/2}C\delta(E+a)/h(a)$ and

$$f(t-t')=C\,e^{ia(t-t')}/h(a) . \quad (4.90)$$

We are interested in the case $t=t'$ and obtain

$$\langle[b(k_1-k_1')+b^\dagger(k_1'-k_1)]a_{2+}(k_2)a_{2-}(-k_2')\rangle_{t=t'}=-\frac{g_2\delta^3(K-K')}{[(4\pi)^3\omega_2\omega_{2'}\omega^\mu]^{1/2}}\frac{1}{(\omega_2+\omega_{2'}+\omega^\mu)} . \quad (4.91)$$

Together with (4.81) and (4.85) we get for the static part of M

$$-i\delta(t-t')\frac{8g_1g_2\delta^3(K-K')\omega_1\omega_{1'}}{(4\pi)^3(\omega_1+\omega_2)(\omega_{1'}+\omega_{2'})\omega^\mu(\omega_2+\omega_{2'}+\omega^\mu)}+(1\leftrightarrow 2) . \quad (4.92)$$

Now we have to calculate the Green's functions g, i.e., the dynamic part of M. Since $g(p_+a_+a_+^\dagger p_+^\dagger)$ is already explicitly of second order we can again replace all operators by the free ones and the correlated vacuum by the uncorrelated one and we get

$$g(p_+a_+a_+^\dagger p_+^\dagger)=\frac{g_1g_2\delta^3(K-K')}{(4\pi)^3(\omega_1\omega_2\omega_{1'}\omega_{2'})^{1/2}\omega^\mu}[\theta(t-t')e^{-i(\omega_2+\omega_{1'}+\omega^\mu)(t-t')}+\theta(t'-t)e^{i(\omega_2+\omega_{1'}+\omega^\mu)(t-t')}] . \quad (4.93)$$

To obtain explicit expressions for the other Green's functions g we have to apply the equations of motion

$$\begin{aligned} g(p_+a_+a_+^\dagger a_-)(t-t')\left[\left[i\frac{\overleftarrow{\partial}}{\partial t'}\right]^2-(\omega_{1'}-\omega_{2'})^2\right]&=-i\delta(t-t')[2(\omega_{2'}-\omega_{1'})\langle p_+a_+a_+^\dagger a_-\rangle+2\langle p_+a_+a_+^\dagger p_-\rangle] \\ &\quad+\theta(t-t')[\langle p_+a_+a_+^\dagger j_-\rangle-2\omega_{1'}\langle p_+a_+a_+^\dagger p_-\rangle] \\ &\quad+\theta(t'-t)[\langle a_-j_+^\dagger p_-^\dagger a_-\rangle-2\omega_{1'}\langle a_-p_+^\dagger p_-^\dagger a_-\rangle]\equiv q(t-t') , \end{aligned} \quad (4.94)$$

where we already took into account the consequences of PCT invariance. $\langle p_+a_+a_+^\dagger a_-\rangle_{t=t'}$ is given in (4.82) and all other terms are explicitly of order g_1g_2. They are easily calculated as usual:

$$\begin{aligned} q(t-t')=\frac{g_1g_2\delta^3(K-K')}{(4\pi)^3(\omega_1\omega_2\omega_{1'}\omega_{2'})^{1/2}\omega^\mu}\Bigg[&-i\delta(t-t')\left[-\frac{2(\omega_{2'}-\omega_{1'})}{\omega_2+\omega_{2'}+\omega^\mu}+2\right] \\ &+(\omega_{2'}-\omega_2-\omega^\mu-2\omega_1')[\theta(t-t')e^{i(\omega_2+\omega_{1'}+\omega^\mu)(t-t')}+\theta(t'-t)e^{i(\omega_2+\omega_{1'}+\omega^\mu)(t-t')}]\Bigg] . \end{aligned} \quad (4.95)$$

As in (4.89), (4.90) $h(i\partial/\partial t')g(t-t')=q(t-t')$ yields upon Fourier transformation $g(E)=q(E)/h(-E)$ and we obtain

$$g(p_+a_+a_+^\dagger a_-)(E)=\frac{q(E)}{E^2-(\omega_{1'}-\omega_{2'})^2}$$
$$=\frac{g_1g_2\delta^3(K-K')}{(4\pi)^3(\omega_1\omega_2\omega_{1'}\omega_{2'})^{1/2}\omega^\mu}\left[\frac{i}{(2\pi)^{1/2}}\left[\frac{2(\omega_{2'}-\omega_{1'})}{\omega_2+\omega_{2'}+\omega^\mu}-2\right]+\frac{i}{(2\pi)^{1/2}}(\omega_{2'}-\omega_2-\omega^\mu-2\omega_{1'})\right.$$
$$\left.\times\left[\frac{1}{E-\omega_2-\omega_{1'}-\omega^\mu}-\frac{1}{E+\omega_2+\omega_{1'}+\omega^\mu}\right]\frac{1}{E^2-(\omega_{1'}-\omega_{2'})^2}\right]. \tag{4.96}$$

$g(a_-^\dagger a_+a_+^\dagger p_+^\dagger)$ is calculated in a very similar way by applying $[(i\partial/\partial t)^2-(\omega_1-\omega_2)^2]$ to $g(t-t')$. We obtain

$$g(a_-^\dagger a_+a_+^\dagger p_+^\dagger)(E)=\frac{g_1g_2\delta^3(K-K')}{(4\pi)^3(\omega_1\omega_2\omega_{1'}\omega_{2'})^{1/2}\omega^\mu}\frac{1}{E^2-(\omega_1-\omega_2)^2}\left[\frac{i}{(2\pi)^{1/2}}\left[-\frac{2(\omega_2-\omega_1)}{\omega_1+\omega_{1'}+\omega^\mu}+2\right]\right.$$
$$+\frac{i}{(2\pi)^{1/2}}(\omega_1-\omega_{1'}-\omega^\mu-2\omega_2)$$
$$\left.\times\left[\frac{1}{E-\omega_2-\omega_{1'}-\omega^\mu}-\frac{1}{E+\omega_2+\omega_{1'}+\omega^\mu}\right]\right]. \tag{4.97}$$

For $g(a_+a_-^\dagger a_-a_+^\dagger)$ we get

$$\left[\left[i\frac{\partial}{\partial t}\right]^2-(\omega_1-\omega_2)^2\right][g(a_+a_-^\dagger a_-a_+^\dagger)(t-t')+(1\leftrightarrow2)]$$
$$=i\delta(t-t')[2(\omega_1-\omega_2)(\langle a_+a_-^\dagger a_-a_+^\dagger\rangle-\langle a_-^\dagger a_+a_+^\dagger a_-\rangle-2\langle p_-^\dagger a_+a_+^\dagger a_-\rangle-2\langle a_+p_-^\dagger a_-a_+^\dagger\rangle]$$
$$+g(j_-^\dagger a_+a_+^\dagger a_-)-2\omega_2 g(p_-^\dagger a_+a_+^\dagger a_-)+(1\leftrightarrow2)\ . \tag{4.98}$$

As mentioned between Eqs. (4.75) and (4.76), $\langle a_+a_-^\dagger a_-a_+^\dagger\rangle=\langle a_-^\dagger a_+a_+^\dagger a_-\rangle$ for $t=t'$ and using (4.18) $\langle p_-^\dagger a_+a_+^\dagger a_-\rangle=\langle p_+a_+a_+^\dagger a_-\rangle$ which is given by (4.82) and (4.91). From this we obtain $\langle a_+p_-^\dagger a_-a_+^\dagger\rangle$ by exchanging 1 and 2. Due to (4.18) $g(p_-^\dagger a_+a_+^\dagger a_-)=g(p_+a_+a_+^\dagger a_-)$ which is given by (4.96) and in a completely analogous way $g(j_-^\dagger a_+a_+^\dagger a_-)$ is seen to be equal to $(\omega_1-\omega_{1'}-\omega^\mu)g(p_+a_+a_+^\dagger a_-)$. Thus we obtain

$$g(a_+a_-^\dagger a_-a_+^\dagger)(E)+(1\leftrightarrow2)=\frac{g_1g_2\delta^3(K-K')}{(4\pi)^3(\omega_1\omega_2\omega_{1'}\omega_{2'})^{1/2}\omega^\mu}\frac{1}{E^2-(\omega_1-\omega_2)^2}$$
$$\times\left\{\frac{2i}{(2\pi)^{1/2}}\frac{1}{\omega_2+\omega_{2'}+\omega^\mu}+(\omega_1-\omega_{1'}-\omega^\mu-2\omega_2)\frac{i}{(2\pi)^{1/2}}\right.\cdot$$
$$\times\left[\frac{2(\omega_{2'}-\omega_{1'})}{\omega_2+\omega_{2'}+\omega^\mu}-2+(\omega_{2'}-\omega_2-\omega^\mu-2\omega_{1'})\right.$$
$$\left.\left.\times\left[\frac{1}{E-\omega_2-\omega_{1'}-\omega^\mu}-\frac{1}{E+\omega_2+\omega_{1'}+\omega^\mu}\right]\right]\frac{1}{E^2-(\omega_{1'}-\omega_{2'})^2}\right\}+(1\leftrightarrow2)\ . \tag{4.99}$$

We collect the results of Eqs. (4.92), (4.93), (4.96), (4.97), and (4.99) to get $M(E)$. Also performing a decomposition into partial fractions with respect to ω^μ we finally obtain the following expression:

$$M(E,k_1,k_2,k_1',k_2')=-\frac{g_1g_2\delta^3(K-K')4i}{(4\pi)^3(\omega_1+\omega_2)(\omega_{1'}+\omega_{2'})\omega^\mu(2\pi)^{1/2}}$$
$$\times\left[\frac{\omega_1\omega_{2'}}{\omega^\mu+\omega_2+\omega_{1'}-E}\left[1+\frac{2\omega_{1'}}{E-\omega_{1'}+\omega_{2'}}\right]\left[1+\frac{2\omega_2}{E+\omega_1-\omega_2}\right]\right.$$
$$+\frac{\omega_1\omega_{2'}}{\omega^\mu+\omega_2+\omega_{1'}+E}\left[1-\frac{2\omega_{1'}}{E+\omega_{1'}-\omega_{2'}}\right]\left[1-\frac{2\omega_2}{E-\omega_1+\omega_2}\right]$$

$$+\frac{2\omega_1\omega_{1'}}{\omega^\mu+\omega_2+\omega_{2'}}\left[1+\frac{2\omega_{2'}(\omega_{2'}-\omega_{1'})}{E^2-(\omega_{1'}-\omega_{2'})^2}\,\frac{\omega_1+\omega_2+\omega_{1'}-\omega_{2'}}{\omega_1-\omega_2+\omega_{1'}-\omega_{2'}}+\frac{2\omega_2(\omega_2-\omega_1)}{E^2-(\omega_1-\omega_2)^2}\,\frac{\omega_{1'}+\omega_{2'}+\omega_1-\omega_2}{\omega_{1'}-\omega_{2'}+\omega_1-\omega_2}\right]\Bigg]+(1\leftrightarrow2)\ . \tag{4.100}$$

One remarks that M is symmetric under permutation of 1 and 2 as well as under exchange of primed and unprimed quantities. We also find again the symmetry $E\leftrightarrow -E$ related to a particle-antiparticle symmetry.

It is interesting to interpret a typical term of M before any decomposition into partial fractions. Let us consider, for example, the contribution of $g(p_+a_+a_+^\dagger a_-)$ according to (4.96). One term out of it is up to factors independent of E:

$$\left[\frac{1}{E-\omega_2-\omega_{1'}-\omega^\mu}-\frac{1}{E+\omega_2+\omega_{1'}+\omega^\mu}\right]\left[\frac{1}{E-\omega_{1'}+\omega_{2'}}-\frac{1}{E+\omega_{1'}-\omega_{2'}}\right]. \tag{4.101}$$

Fourier transformation yields up to factors independent of t

$$\int[\theta(t-t'')e^{-i(\omega_2+\omega_{1'}+\omega^\mu)(t-t'')}+\theta(t''-t)e^{i(\omega_2+\omega_{1'}+\omega^\mu)(t-t'')}][\theta(t''-t')e^{-i(\omega_{2'}-\omega_{1'})(t''-t')}+\theta(t'-t'')e^{i(\omega_{2'}-\omega_{1'})(t''-t')}]dt''\ , \tag{4.102}$$

where we supposed that the imaginary part of $\omega_{2'}$ is always more negative than the one of $\omega_{1'}$ to define properly the Fourier transform of $(E-\omega_{1'}+\omega_{2'})^{-1}-(E+\omega_{1'}-\omega_{2'})^{-1}$. The expression (4.102) is the product of two Feynman propagators from t' to t'' and from t'' to t. Time t'' is integrated over. Consider, for example, the integrand for $t>t'>t''$:

$$e^{-i(\omega_2+\omega_{1'}+\omega^\mu)(t-t'')}e^{i(\omega_{2'}-\omega_{1'})(t''-t')}=e^{-i(\omega_2+\omega^\mu)(t-t'')}e^{-i\omega_{1'}(t-t')}e^{-i\omega_{2'}(t'-t'')}\ . \tag{4.103}$$

This describes the free propagation of particle 2 with momentum k_2 and of the exchange particle with momentum k_1-k_1' from t'' to t, of particle 1 with momentum k_1' from t' to t and of antiparticle 2 with momentum $-k_2'$ from t'' to t'. This situation is shown in Fig. 8(a). This is exactly one of the graphs which could not appear in Sec. IV A when we neglected all vacuum correlations but which has to be taken into account because of the possibility of spontaneous pair creation in the vacuum. Other terms stemming from $g(a_+a_-^\dagger a_-a_+^\dagger)$ correspond to graphs like the one shown in Fig. 8(b).

We also want to give another form of M, obtained by decomposition into partial fractions with respect to E:

$$M(E,k_1,k_2,k_1',k_2')=-\frac{g_1g_2\delta^3(K-K')4i}{(4\pi)^3(\omega_1+\omega_2)(\omega_{1'}+\omega_{2'})\omega^\mu(2\pi)^{1/2}}\ .$$

$$\times\left[\left[\frac{1}{E+\omega_2+\omega_{1'}+\omega^\mu}-\frac{1}{E-\omega_2-\omega_{1'}-\omega^\mu}\right]\omega_1\omega_{2'}\left[1+\frac{2\omega_{1'}}{\omega_2+\omega_{2'}+\omega^\mu}\right]\left[1+\frac{2\omega_2}{\omega_1+\omega_{1'}+\omega^\mu}\right.\right.$$

$$\left.\left.+\frac{2\omega_1\omega_{1'}}{\omega_2+\omega_{2'}+\omega^\mu}\right]\right]+(1\leftrightarrow2)\ . \tag{4.104}$$

The integral kernel is again obtained according to (4.57) as $2\pi G_0(E)M(E)$ and the propagator $G(E)$ is solution of

$$G(E)=G_0(E)+(G_0M)(E)G(E)\ . \tag{4.105}$$

In our calculation leading to M to order g_1g_2 we did not introduce any approximation: To this order M is exact. For the following discussion we will call this M of order g_1g_2 (times a factor 2π) $M^{(2)}$: $M^{(2)}=2\pi M(E)$ with $M(E)$ from (4.100) or (4.104). We reserve the symbol $M(E)$ for the memory function including all orders (times 2π), i.e.,

$$M=M^{(2)}+M^{(4)}+M^{(6)}+\cdots\ , \tag{4.106}$$

where all graphs contributing only to mass renormalization have to be deleted. The irreducible kernel K of the Bethe-Salpeter equation (also without graphs connected with mass renormalization) can also be written as a sum of terms of different orders:

$$K=K^{(2)}+K^{(4)}+K^{(6)}+\cdots\ , \tag{4.107}$$

where $K^{(2)}$ is just the kernel of the ladder approximation. Now it is possible, at least in principle, to solve the Bethe-Salpeter equation with the complete kernel K and to put $t_1=t_2=t$ and $t_1'=t_2'=t'$ in the four-time Green's function obtained. Thus we get a two-time Green's function which we will call $G_F(t-t')$. On the other hand we

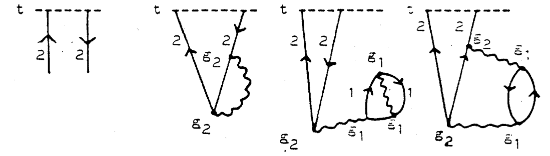

FIG. 7. Interpretation of the vacuum expectation values $\langle a_{2-}a_{2+}\rangle_{t=t'}$ similar to Fig. 5.

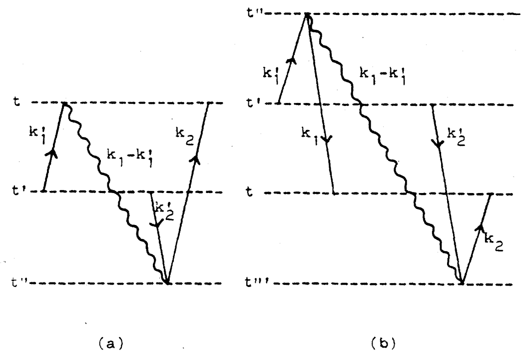

FIG. 8. Graphs described by $M(t-t')$.

can also solve, in principle, our equation $G=G_0+G_0MG$ with the complete M. The two-time Green's function obtained will be simply called $G(t-t')$. If now the perturbation series are convergent we must have $G=G_F$. If we consider G and G_F as a power series of the coupling constants g we have an identity to all orders. In zeroth order we have of course $G_F^{(0)}(t-t')=G_0(t-t')=G^{(0)}(t-t')$. In second order we have

$$G_F^{(2)}(t-t')=\int G_0(t,t,t_1'',t_2'')K^{(2)}(t_1'',t_2'',t_1''',t_2''') \times G_0(t_1''',t_2''',t',t')dt_1''dt_2''dt_1'''dt_2''' \quad (4.108)$$

and

$$G^{(2)}(t-t')=\int G_0(t-t'')M^{(2)}(t''-t''') \times G_0(t'''-t')dt''dt''' , \quad (4.109)$$

where $M(t-t')$ is the Fourier transform of (4.100) or (4.104). Below we will show that $G_F^{(2)}(t-t')$ from (4.108) or, equivalently, calculated by the ordinary Feynman rules is indeed identical with $G^{(2)}(t-t')$ according to (4.109).

But in reality we are not able to solve the Bethe-Salpeter equation with the complete kernel K and most of the time we restrict ourselves to the ladder approximation. We did not calculate the complete M either but only $M^{(2)}$. Now the question arises whether the propagators $G_F^{(2*)}$ and $G^{(2*)}$ obtained by iteration of the corresponding equations with the kernels of second order $K^{(2)}$ and $M^{(2)}$ are also identical or not. We certainly cannot expect to obtain the same propagators the different iteration schemes leading to a shift between the different orders. This is connected with the definition of the irreducible kernel. We will explain this is some more detail.

We consider some typical graphs emerging from the iteration of the Bethe-Salpeter equation with an irreducible kernel of second order (ladder approximation) as shown in Fig. 9. The graphs 9(a) and 9(b) are described by

$$G_0(t_4-t_3)M^{(2)}(t_3-t_2)G_0(t_2-t_1)$$

and

$$G_0(t_6-t_5)M^{(2)}(t_5-t_4)G_0(t_4-t_3)M^{(2)}(t_3-t_2)G_0(t_2-t_1) .$$

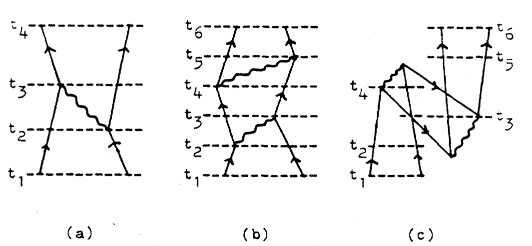

FIG. 9. Graphs of the ladder approximation.

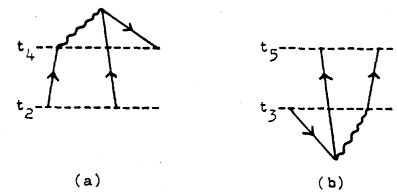

FIG. 10. Subgraphs from Fig. 9(c) described by $M^{(2)}(t-t')$.

Already we could describe these graphs 9(a) and 9(b) with the memory function from Sec. IV A when we neglected vacuum correlations. Graph 9(c), however, can only be described if we take into account vacuum correlations. This is done by

$$G_0(t_6-t_5)M^{(2)}(t_5-t_3)G_0(t_3-t_4)M^{(2)}(t_4-t_2)G_0(t_2-t_1) .$$

Here $M^{(2)}$ describes the subgraphs shown in Fig. 10, similar to those we showed in Fig. 8.

However, in the ladder approximation graphs as shown in Fig. 11(a) are also present. This graph 11(a) cannot be described by an expression of the form $G_0M^{(2)}G_0M^{(2)}G_0$, but only by an expression $G_0(t_5-t_4)M^{(2)}(t_4-t_3)M^{(2)}(t_3-t_2)G_0(t_2-t_1)$.

This may be obtained as a limiting case from $G_0M^{(2)}G_0M^{(2)}G_0$ with all times equal in the second G_0. But with respect to the integration over these times this is satisfied only on a set of measure zero, thus we cannot obtain the graph 11(a) by iteration with a kernel $M^{(2)}$. On the other hand it is not difficult to see that this graph can be obtained as $G_0M^{(4)}G_0$. This is what we called the shift between different orders: All graphs emerging from an iteration with kernel $M^{(2)}$, i.e., $G_0+G_0M^{(2)}G_0+G_0M^{(2)}G_0M^{(2)}G_0+\cdots$ are also contained in those emerging from iteration of the Bethe-Salpeter equation with kernel $K^{(2)}$: $G_0+G_0K^{(2)}G_0+G_0K^{(2)}G_0K^{(2)}G_0$

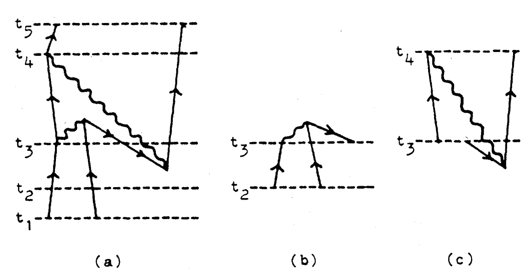

FIG. 11. A graph of the ladder approximation (a) and two subgraphs that can be described by $M^{(2)}(t-t')$: (b) and (c).

$+\cdots$. But the latter contains in addition graphs which are only found in $G_0M^{(4)}G_0$ or $G_0M^{(6)}G_0$, etc. We can also say that the definition of the irreducibility has changed. Whereas the graph 11(a) is considered as reducible for the Bethe-Salpeter equation, it has to be regarded as irreducible in the memory function approach. One must not forget this discussion when comparing the binding energies found numerically for the ladder approximation (kernel $K^{(2)}$) and with a kernel $M^{(2)}$. This will be done in Sec. IV C. Before, we will show that $G_F^{(2)}=G^{(2)}$ as we claimed above, instead of showing directly that $G_F^{(2)}=G_0M^{(2)}G_0$. We will demonstrate the equivalent equation (cf. Logunov and Tavkhelidze[3] and Fishbane and Namyslowski[30]):

$$\int G_0{}^{-1}(t-t'')G_F^{(2)}(t''-t''')G_0{}^{-1}(t'''-t')dt''dt'''=M^{(2)}(t-t') . \tag{4.110}$$

We obtain $G_F^{(2)}$ according to the usual Feynman rules, putting $t_1=t_2=t$, $t_1'=t_2'=t'$, and Fourier transformation into momentum space:

$$\begin{aligned}G_F^{(2)}(t-t',k_1,k_2,k_1',k_2')=&\frac{g_1g_2\delta^3(K-K')}{4(4\pi)^3\omega_1\omega_2\omega_{1'}\omega_{2'}\omega^\mu}\\&\times\int dt_1dt_2[\theta(t-t_1)e^{-i\omega_1(t-t_1)}+\theta(t_1-t)e^{i\omega_1(t-t')}]\\&\times[\theta(t-t_2)e^{-i\omega_2(t-t_2)}+\theta(t_2-t)e^{i\omega_2(t-t_2)}]\\&\times[\theta(t_1-t_2)e^{-i\omega^\mu(t_1-t_2)}+\theta(t_2-t_1)e^{i\omega^\mu(t_1-t_2)}]\\&\times[\theta(t_1-t')e^{-i\omega_{1'}(t_1-t')}+\theta(t'-t_1)e^{i\omega_{1'}(t_1-t')}]\\&\times[\theta(t_2-t')e^{-i\omega_{2'}(t_2-t')}+\theta(t'-t_2)e^{i\omega_{2'}(t_2-t')}] .\end{aligned} \tag{4.111}$$

When calculating $G_0{}^{-1}G_F^{(2)}G_0{}^{-1}$ we apply $\partial^2/\partial t^2$ and $\partial^2/\partial t'^2$ to $G_F(t-t')$ yielding terms with δ distributions making one (or both) integrations over the intermediate times trivial. Other terms contain products of θ functions, e.g., $\theta(t-t_1)\theta(t_2-t)\theta(t_1-t')\theta(t'-t_2)$ demanding $t_2\geq t_1$ and $t_1\geq t_2$. They are satisfied only on the set of measure zero with $t_1=t_2$. Thus, these terms give no contributions. Others contain $\theta(t-t_1)\theta(t_2-t)\theta(t'-t_1)\theta(t_2-t')$, demanding $t_2\geq t\geq t_1$ and $t_2\geq t'\geq t_1$, and therefore we can also write instead of the product of the four θ functions: $\theta(t_2-t)\theta(t-t')\theta(t'-t_1)+\theta(t_2-t')\theta(t'-t)\theta(t-t_1)$. In a similar way, one changes other products of θ functions. One obtains a sum of several terms. One of them is, for example, up to factors independent of t:

$$\theta(t-t')e^{-i(\omega_2+\omega_{1'}+\omega^\mu)(t-t')}\int dt_2\theta(t'-t_2)e^{i(\omega_2+\omega_{2'}+\omega^\mu)(t_2-t')}+\theta(t'-t)e^{i(\omega_2+\omega_{1'}+\omega^\mu)(t-t')}\int dt_2\theta(t_2-t')e^{-i(\omega_2+\omega_{2'}+\omega^\mu)(t_2-t')} . \tag{4.112}$$

Performing the integrations over t_2 this yields

$$\frac{-i}{\omega_2+\omega_{2'}+\omega^\mu}[\theta(t-t')e^{-i(\omega_2+\omega_{1'}+\omega^\mu)(t-t')}+\theta(t'-t)e^{i(\omega_2+\omega_{1'}+\omega^\mu)(t-t')}] . \tag{4.113}$$

The other integrals are evaluated in an analogous manner and we obtain

$$\begin{aligned}\int G_0{}^{-1}&(t-t'')G_F^{(2)}(t''-t''')G_0{}^{-1}(t'''-t')dt''dt'''\\&=-\frac{4g_1g_2\delta^3(K-K')}{(4\pi)^3(\omega_1+\omega_2)(\omega_{1'}+\omega_{2'})\omega^\mu}\left[2i\omega_1\omega_{1'}\frac{\delta(t-t')}{\omega_2+\omega_{2'}+\omega^\mu}-4\omega_1\omega_{2'}\left[1+\frac{2\omega_{1'}}{\omega_2+\omega_{2'}+\omega^\mu}\right]\left[1+\frac{2\omega_2}{\omega_1+\omega_{1'}+\omega^\mu}\right]\right.\\&\qquad\left.\times[\theta(t-t')e^{-i(\omega_2+\omega_{1'}+\omega^\mu)(t-t')}+\theta(t'-t)e^{i(\omega_2+\omega_{1'}+\omega^\mu)(t-t')}]\right]+(1\leftrightarrow 2)\\&\qquad[=M(t-t',k_1,k_2,k_1',k_2')] .\end{aligned} \tag{4.114}$$

But this is nothing else than the Fourier transform of (4.104), i.e., $M(t-t',k_1,k_2,k_1',k_2')$, Q.E.D.

We will again numerically calculate the binding energies in the two limiting cases $m_1=m_2$ and $m_2\to\infty$. Therefore, we will give the expressions of $(G_0M)(E)$ in these limits. We put again $g_1g_2=16\pi m_1m_2\lambda$.

(a) $m_1 = m_2 = 2m$. As in Sec. IV A, we introduce total and relative variables and write ω for ω_k and ω' for $\omega_{k'}$. After factorizing the $\delta^3(K-K')$ distribution, we get in the center-of-mass system ($\mathbf{K}=0$)

$$(G_0M)(E,K=0,k,k') = -\frac{2m^2\lambda}{\pi^2\omega(E^2-4\omega^2)\omega^\mu}\left[\frac{(1-2\omega/E)(1-2\omega'/E)}{\omega^\mu+\omega+\omega'+E}+\frac{(1+2\omega/E)(1+2\omega'/E)}{\omega^\mu+\omega+\omega'-E}+\frac{2(1-4\omega\omega'/E^2)}{\omega^\mu+\omega+\omega'}\right]. \quad (4.115)$$

For weak coupling, i.e., for small λ, the dominating term is the one with $\omega^\mu+\omega+\omega'-E$ in the denominator. Comparing this term with the corresponding one in (4.61) we see that now a factor $(1+2\omega/E)(1+2\omega'/E)$ replaces $[1+E/(2\omega)][1+E/(2\omega')]$. For bound states $E<4\leq\omega,\omega'$ and thus the term in (4.115) is greater than the one in (4.61). A greater $(G_0M)(E)$ means more binding energy. Thus we can predict, at least for small λ, that the binding energy increases when taking into account vacuum correlations.

(b) $m_2\to\infty$. We get

$$(G_0M)(E,K=0,k,k') = -\frac{m\lambda}{2\pi^2\omega(m-B-\omega)\omega^\mu}\left[\frac{\omega}{m-B+\omega}\frac{1}{\omega^\mu+\omega'-m+B}+\frac{\omega'}{m-B+\omega'}\frac{1}{\omega^\mu+\omega-m+B}+\left(1-\frac{\omega}{m-B+\omega}-\frac{\omega'}{m-B+\omega'}\right)\frac{1}{\omega^\mu+\omega+\omega'}\right]. \quad (4.116)$$

Unlike the kernel for equal masses, this expression yields in the limit $\lambda\to 0$ the corresponding one (4.62) without taking into account vacuum correlations. Because for bound states $\omega/(m-B+\omega)>\frac{1}{2}$, $(m-B<1,\omega>1)$, vacuum correlations increase the binding energy also in this case.

C. Numerical results

In this section we will present the binding energies found numerically, compare them with those of the Bethe-Salpeter equation and discuss the results. One always obtains the binding energy B by determining the total energy K_0 or E that allows for a nontrivial solution of (2.4) or (4.63). We take the kernels K and M to order g_1g_2 (ladder approximation) and call the values from the Bethe-Salpeter equation with this kernel [Eq. (2.10)] BS values, those from the corresponding instantaneous approximation (2.13) IA values, those obtained by the memory function approach MF values if vacuum correlations are taken into account and MFN if we neglect vacuum correlations. The BS and IA values have been calculated by Silvestre-Brac *et al.*[19] Because BS values are easily available only for a mass of the exchange particle $\mu=0$ [due to a $O(4)$ symmetry appearing in this case] we restrict ourselves to this case. We consider the two limiting cases $m_1=m_2$ and $m_2\to\infty$. The integral kernels (4.61), (4.62) and (4.115), (4.116) have a very sharp maximum at $k'=k$ if λ is small and if $\mu \lesssim \lambda^2 m$, especially for $\mu=0$, and thus the numerical calculations are more complicated if we want to obtain the same precision. For an angular quantum number $l\neq 0$ the computing time is again increased, thus we restricted ourselves to $l=0$, what nevertheless reveals the essential features. For λ or μ not too small it is no problem to compute the energies also for $l\neq 0$. We determined the energies for the first three levels $n=1,2,3$ and for three different coupling constants λ, one being characteristic of electromagnetic coupling: $\lambda=1/137.036$, one corresponding to a coupling of quarks to gluons, we took $\lambda=1$, and an intermediate one of $\lambda=0.1$. Details concerning the numerical procedure can be found in the Appendix.

In Table I we present the binding energies for the three different λ, for $n=1$, 2, and 3 and for the two cases $m_1=m_2$ and $m_2\to\infty$, always for MFN, MF, BS, IA, and NR (NR being the nonrelativistic Schrödinger energy levels for a Coulomb potential: $B=m\lambda^2/2n^2$). We always have $\mu=0$ and $l=0$. The upper row refers to the values for $m_2\to\infty$ and the lower one to those for $m_1=m_2$.

We see immediately that the binding energies of the memory function approach are much closer to those of the Bethe-Salpeter equation than the energies of the instantaneous approximation. Looking more closely at the values we find the MFN values are always a little smaller then the BS values and the MF values lie between the MFN and the BS values, being still a little bit smaller than BS, while the IA values are always much greater than the BS values. For BS, B/m is always a little greater for $m_1=m_2$ than for $m_2\to\infty$, but this difference is small. For IA this difference is considerably more pronounced. For MFN it is a little greater than for BS but much smaller than for IA; it has a reversed sign, i.e., B/m is greater for $m_2\to\infty$ than for $m_1=m_2$. In the case of MF this difference also has a reversed sign, but its magnitude is very small. For $\lambda=1$ and $n=1$ the differences are -5% for MFN, -0.5% for MF, 2% for BS, and 20% for IA. The differences increase with the binding energy when going from MFN to IA. For $\lambda=0.1$ the corresponding differences are -0.3%, -0.3%, 0.06%, and 1%. The differences for MF and BS are closer than the differences for BS and IA.

The MF values lie between the MFN and the BS values. For small λ there is no observable difference between MF and MFN values. For $\lambda=0.1$ a difference is obvious but the MF values are still closer to MFN than to BS. For $\lambda=1.0$ finally they are closer to BS than to MFN. This

TABLE I. The binding energies B/m for the first three bound states $n=1$, 2, and 3 ($l=0,\mu=0$) and for three different coupling strengths λ. The upper row refers to $m_2\rightarrow\infty$, the lower one to $m_1=m_2$. The various approximation MFN, MF, BS, IA, and NR are explained in the text.

		MFN	MF	BS	IA	NR
$\lambda=\frac{1}{137.036}$	$n=1$	0.2533×10^{-4} 0.2533×10^{-4}	0.2533×10^{-4} 0.2533×10^{-4}	0.2544×10^{-4} 0.2544×10^{-4}	0.2659×10^{-4} 0.2659×10^{-4}	0.2663×10^{-4}
	$n=2$	0.6335×10^{-5} 0.6335×10^{-5}	0.6335×10^{-5} 0.6335×10^{-5}	0.6366×10^{-5} 0.6366×10^{-5}	0.6616×10^{-5} 0.6616×10^{-5}	0.6656×10^{-5}
	$n=3$	0.2816×10^{-5} 0.2816×10^{-5}	0.2816×10^{-5} 0.2816×10^{-5}	0.2830×10^{-5} 0.2830×10^{-5}	0.2941×10^{-5} 0.2941×10^{-5}	0.2958×10^{-5}
$\lambda=0.1$	$n=1$	0.3498×10^{-2} 0.3490×10^{-2}	0.3533×10^{-2} 0.3526×10^{-2}	0.3696×10^{-2} 0.3698×10^{-2}	0.4926×10^{-2} 0.4966×10^{-2}	0.5×10^{-2}
	$n=2$	0.8812×10^{-3} 0.8800×10^{-3}	0.8855×10^{-3} 0.8846×10^{-3}	0.9292×10^{-3} 0.9293×10^{-3}	1.235×10^{-3} 1.239×10^{-3}	1.25×10^{-3}
	$n=3$	0.3924×10^{-3} 0.3920×10^{-3}	0.3937×10^{-3} 0.3934×10^{-3}	0.4136×10^{-3} 0.4136×10^{-3}	0.5498×10^{-3} 0.5512×10^{-3}	0.5556×10^{-3}
$\lambda=1$	$n=1$	0.1289 0.1214	0.1492 0.1484	0.1649 0.1684	0.3421 0.4131	0.5
	$n=2$	0.3415×10^{-1} 0.3286×10^{-1}	0.3682×10^{-1} 0.3636×10^{-1}	0.4244×10^{-1} 0.4267×10^{-1}	1.002×10^{-1} 1.082×10^{-1}	1.25×10^{-1}
	$n=3$	0.1542×10^{-1} 0.1500×10^{-1}	0.1622×10^{-1} 0.1605×10^{-1}	0.1898×10^{-1} 0.1902×10^{-1}	0.4733×10^{-1} 0.4970×10^{-1}	0.5556×10^{-1}

merely reflects the fact that spontaneous pair creation, i.e., vacuum correlations, are more important for large λ than for small λ, and neglecting them is responsible for most of the difference MFN-BS for strong coupling. For weak coupling this is negligible and MFN and MF values are (nearly) identical. The difference between MF and BS is due to the shift between different orders as discussed in Sec. IV B.

However, the difference between MF and BS is small and much less important than the one between BS and IA. For the levels $n=1$ we have the differences given in Table II. On the average, the differences IA−BS are ten times greater than the differences MF−BS. Thus the fact that the memory function approach, unlike the instantaneous approximation, takes into account retardation effects in a natural manner is of enormous importance. One can say that (in the given order of the kernel) the introduction of the memory function is a "ten times" better approximation for the Bethe-Salpeter equation than is the instantaneous approximation.

V. GENERALIZATION TO INTERACTING DIRAC FIELDS

Now, having seen for the simple model of Cutkosky, where the original Bethe-Salpeter equation was solvable at least in the ladder approximation, that the memory function approach, even for strong couplings, yields results very close to the Bethe-Salpeter equation, we want to show how this formalism can be generalized to the physically more interesting case of interacting Dirac fields. Eventual applications being bound states of a particle-antiparticle pair (e^-e^+: positronium $q\bar{q}$: mesons, etc.), this time we will treat the particle-antiparticle (ph) channel explicitly.

We define the two-time propagator in this channel as

$$[G(t-t',x_1,x_2,x'_1,x'_2)]_{\alpha\beta\gamma\delta} = -\langle T\psi_\alpha(t,x_1)\bar{\psi}_\gamma(t,x_2)\psi_\delta(t',x'_2)\bar{\psi}_\beta(t',x'_1)\rangle , \quad (5.1)$$

TABLE II. Differences between the B/m values. The upper row refers to $m_2\rightarrow\infty$, the lower one to $m_1=m_2$.

	MF-BS	IA-BS
$\lambda=1/137.036$	0.4%	5%
	0.4%	5%
$\lambda=0.1$	5%	33%
	5%	34%
$\lambda=1$	11%	107%
	13%	145%

where the vacuum expectation value is to be taken with respect to the correlated vacuum. But here we only want to demonstrate the principle, and therefore we again will neglect all vacuum correlations in this section. In (5.1) $\alpha,\beta,\gamma,\delta$ are spinor indices. G is a 16×16 spinor matrix, α and γ indicate the rows, and β and δ the columns; α and β refer to the particle (antiparticle) and γ and δ to the antiparticle (particle). For the field ψ we adopt the following momentum space expansion:[31]

$$\psi(t,x)=\sum_{\pm s}\int\frac{d^3k}{(2\pi)^{3/2}}\left[\frac{m}{\omega(k)}\right]^{1/2}[b(t,k,s)u(k,s)e^{ikx}+d^{\dagger}(t,k,s)v(k,s)e^{-ikx}]\ . \tag{5.2}$$

We will not continue to write the spin variables, unless necessary, k_i standing for k_i and s_i. We recall the following relations for the spinors u and v:

$$\bar{u}(k,s)u(k,s')=\delta_{ss'}=-\bar{v}(k,s)v(k,s')\ ,\quad \bar{v}(k,s)u(k,s')=0=\bar{u}(k,s)v(k,s')\ , \tag{5.3a}$$

$$\sum_{\pm s}u_{\alpha}(k,s)\bar{u}_{\beta}(k,s)=[\Lambda_{+}(k)]_{\alpha\beta}\ ,\quad -\sum_{\pm s}v_{\alpha}(k,s)\bar{v}_{\beta}(k,s)=[\Lambda_{-}(k)]_{\alpha\beta}\ . \tag{5.3b}$$

Λ_+ and Λ_- are the orthogonal projectors on the spaces of positive and negative energy solutions of the Dirac equation. The creation and destruction operators b, d, $b^{\dagger}$ and $d^{\dagger}$ satisfy the usual equal time anticommutation relations.

We remark that for given momenta k_1 and k_2

$$I_{\alpha\beta\gamma\delta}(k_1,k_2)=[\Lambda_{+}(k_1)]_{\alpha\beta}[\Lambda_{-}(k_2)]_{\delta\gamma}+[\Lambda_{-}(k_1)]_{\alpha\beta}[\Lambda_{+}(k_2)]_{\delta\gamma} \tag{5.4}$$

acts like the unity operator in the subspace corresponding to the particle-antiparticle (ph) channel considered here (cf. the discussion concerning N in Sec. III).

Thus, neglecting all vacuum correlations (5.1) and (5.2) yields

$$\begin{aligned}[G(t-t',k_1,k_2,k_1',k_2')]_{\alpha\beta\gamma\delta}=-d_{11'22'}\sum_{s_1s_2s_1's_2'}[&\theta(t-t')\langle b(t,k_1)d(t,k_2)d^{\dagger}(t',k_2')b^{+}(t',k_1')\rangle u_{\alpha}(k_1)\bar{u}_{\beta}(k_1')\bar{v}_{\gamma}(k_2)v_{\delta}(k_2')\\&+\theta(t'-t)\langle b(t',k_2')d(t',k_1')d^{\dagger}(t,k_1)b^{\dagger}(t,k_2)\rangle v_{\alpha}(k_1)\bar{v}_{\beta}(k_1')\bar{u}_{\gamma}(k_2)u_{\delta}(k_2')]\end{aligned}$$

with

$$d_{11'22'}=m^2[\omega(k_1)\omega(k_1')\omega(k_2)\omega(k_2')]^{1/2}\ . \tag{5.5}$$

The function G also contains two other terms $\langle b(t)b^{\dagger}(t)b(t')b^{\dagger}(t')\rangle$ and $\langle b(t')b^{\dagger}(t')b(t)b^{\dagger}(t)\rangle$ corresponding to creation and destruction at the same time and with the same momentum of a particle. They do not contribute to the propagation of the particle-antiparticle pair and we will neglect them.

From Eq. (5.5) the free propagator is easily obtained to be [using (5.3b)]

$$\begin{aligned}[G_0(t-t',k_1,k_2,k_1',k_2')]_{\alpha\beta\gamma\delta}=\frac{m^2}{\omega_1\omega_2}\{&\theta(t-t')e^{-i(\omega_1+\omega_2)(t-t')}[\Lambda_{+}(k_1)]_{\alpha\beta}[\Lambda_{-}(k_2)]_{\delta\gamma}\\&+\theta(t'-t)e^{i(\omega_1+\omega_2)(t-t')}[\Lambda_{-}(k_1)]_{\alpha\beta}[\Lambda_{+}(k_2)]_{\delta\gamma}\}\delta^3(k_1-k_1')\delta^3(k_2-k_2')\end{aligned} \tag{5.6}$$

and Fourier transformation yields

$$\begin{aligned}[G_0(E,k_1,k_2,k_1',k_2')]_{\alpha\beta\gamma\delta}=&\frac{im^2}{(2\pi)^{1/2}\omega_1\omega_2}\delta^3(k_1-k_1')\delta^3(k_2-k_2')\\&\times\left[\frac{[\Lambda_{+}(k_1)]_{\alpha\beta}[\Lambda_{-}(k_2)]_{\delta\gamma}}{E-(\omega_1+\omega_2)+i\epsilon}-\frac{[\Lambda_{-}(k_1)]_{\alpha\beta}[\Lambda_{+}(k_2)]_{\delta\gamma}}{E+(\omega_1+\omega_2)-i\epsilon}\right]\ .\end{aligned} \tag{5.7}$$

It is easy to see that I according to (5.4) indeed acts like the unity operator in the subspace considered (for given momenta):

$$[G(t-t',k_1,k_2,k_1',k_2']_{\alpha\beta\gamma\delta}I_{\beta\mu\delta\nu}(k_1',k_2')=[G(t-t',k_1,k_2,k_1',k_2')]_{\alpha\mu\gamma\nu} \tag{5.8}$$

(greek indices figuring twice are to be summed over) because of (5.3) and (5.5). The same holds if G were replaced by $G_0(t-t')$ or $G_0(E)$. In this spirit we define the pseudoinverse $G_0{}^{-1}$ of G_0 by

$$\int[G_0{}^{-1}(E,k_1,k_2,k_1'',k_2'')]_{\alpha\beta\gamma\delta}[G_0(E,k_1'',k_2'',k_1',k_2')]_{\beta\mu\delta\nu}d^3k_1''d^3k_2''=I_{\alpha\mu\gamma\nu}(k_1,k_2)\delta^3(k_1-k_1')\delta^3(k_2-k_2') \tag{5.9}$$

and in the same way for $G_0{}^{-1}(t-t')$. $G_0{}^{-1}(E)$ is then found immediately and $G_0{}^{-1}(t-t')$ is its Fourier transform [times the usual factor $(2\pi)^{-1}$]:

$$[G_0{}^{-1}(t-t',k_1,k_2,k_1',k_2')]_{\alpha\beta\gamma\delta}=-im^{-2}\omega_1\omega_2\delta(t-t')\delta^3(k_1-k_1')$$
$$\times\delta^3(k_2-k_2')\left\{\left[i\frac{\partial}{\partial t'}-\omega_1-\omega_2\right][\Lambda_+(k_1)]_{\alpha\beta}[\Lambda_-(k_2)]_{\delta\gamma}-\left[i\frac{\partial}{\partial t'}+\omega_1+\omega_2\right][\Lambda_-(k_1)]_{\alpha\beta}[\Lambda_+(k_2)]_{\delta\gamma}\right\} \quad (5.10)$$

Of course, G_0 and $G_0{}^{-1}$ would not be changed if we took vacuum correlations into account.

In the next step we have to apply $G_0{}^{-1}$ to G. Therefore we have to know the equations of motion for b, d, $b^\dagger$, and $d^\dagger$. Let us suppose a given Hamilton operator $H=H_0+H_{\rm int}$ (such a separation into a free part and an interaction part is not necessary but simplifies the formulas). We define

$$p_b(t,k)=[b(t,k),H_{\rm int}],\quad p_d(t,k)=[d(t,k),H_{\rm int}]\ , \quad (5.11)$$

and the equations of motion are

$$i\frac{\partial}{\partial t}b(t,k)=\omega_k b(t,k)+p_b(t,k)\ , \quad (5.12)$$

etc. Because $H_{\rm int}$ explicitly contains at least one coupling constant the same is true for p. We obtain

$$\int[G_0{}^{-1}(t-t'',k_1,k_2,k_1'',k_2'')]_{\alpha\beta\gamma\delta}[G(t''-t',k_1'',k_2'',k_1',k_2')]_{\beta\mu\delta\nu}dt''d^3k_1''d^3k_2''$$
$$=\delta(t-t')\delta^3(k_1-k_1')\delta^3(k_2-k_2')I_{\alpha\mu\gamma\nu}(k_1,k_2)+[R(t-t',k_1,k_2,k_1',k_2')]_{\alpha\mu\gamma\nu}$$

with

$$[R(t-t',k_1,k_2,k_1',k_2')]_{\alpha\beta\gamma\delta}=-i\omega_1\omega_2 m^{-2}\{[G(t-t',p(k_1),k_2,k_1',k_2')]_{\alpha\beta\gamma\delta}+[G(t-t',k_1,p(k_2),k_1',k_2')]_{\alpha\beta\gamma\delta}\}\ , \quad (5.13)$$

where we defined $G(t-t',p(k_1),k_2,k_1',k_2')$ in complete analogy with the last section: the operator having an argument k_1 is replaced by the corresponding p operator.

Finally we have to apply $G_0{}^{-1}$ from the right to R to obtain M to lowest in the coupling constants:

$$[M(t-t',k_1,k_2,k_1',k_2')]_{\alpha\beta\gamma\delta}=i(d_{11'22'})^{-1}\delta(t-t')$$
$$\times\sum_{s_1s_2s_1's_2'}\{[\langle p_b(k_1)d(k_2)d^\dagger(k_2')b^\dagger(k_1')\rangle_{t=t'}$$
$$+\langle b(k_1)p_d(k_2)d^+(k_2')b^\dagger(k_1')\rangle_{t=t'}]u_\alpha(k_1)\bar u_\beta(k_1')\bar v_\gamma(k_2)v_\delta(k_2')$$
$$+[\langle b(k_2')d(k_1')p_d^\dagger(k_1)b^\dagger(k_2)\rangle_{t=t'}+\langle b(k_2')d(k_1')d^\dagger(k_1)p_b(k_2)\rangle_{t=t'}]$$
$$\times v_\alpha(k_1)\bar v_\beta(k_1')\bar u_\gamma(k_2)u_\delta(k_2')\}$$
$$-(d_{11'22'})^{-2}\{[G(t-t',p(k_1),k_2,p(k_1'),k_2')]_{\alpha\beta\gamma\delta}$$
$$+[G(t-t',p(k_1),k_2,k_1',p(k_2'))]_{\alpha\beta\gamma\delta}$$
$$+[G(t-t',k_1,p(k_2),p(k_1'),k_2')]_{\alpha\beta\gamma\delta}$$
$$+[G(t-t',k_1,p(k_2),k_1',p(k_2'))]_{\alpha\beta\gamma\delta}\}\ . \quad (5.14)$$

The Green's functions containing two p operators are of second (and higher) order in the coupling constants. To give the order of the terms with $\delta(t-t')$ as a factor is quite a subtle question in general. For quantum electrodynamics, however, it is not difficult to show that all terms contained in M are at least of order $e_0{}^2$. If we suppose again all masses already renormalized we have to delete (at least to second order) the Green's functions

$$G(t-t',p(k_1),k_2,p(k_1'),k_2')\ \text{and}\ G(t-t',k_1,p(k_2),k_1',p(k_2'))\ .$$

To discuss the static part we must have more detailed information about H_{int}. As an example we will consider a *proton-antiproton pair with strong interaction via π^0 exchange.* The interaction Hamilton operator is

$$H_{\text{int}} = ig_0 \int d^3x :\Psi(x)\gamma_5\Psi(x)\Phi_0(x): , \tag{5.15}$$

where the proton-antiproton field Ψ is given by (5.2) and the π^0 field Φ_0 by (4.2) μ being the mass of π^0. (We call the π^0 destruction and creation operators now a and $a^\dagger$ to avoid confusion with the b and $b^\dagger$ of the proton field.)

We obtain for H_{int}

$$\begin{aligned} H_{\text{int}} = ig_0 \sum_{s_1 s_2} \frac{d^3x\, d^3k_1 d^3k_2 d^3k_3}{(2\pi)^{9/2}(2\omega_1\omega_2\omega_3^\mu)^{1/2}} [& b^\dagger(t,k_1,s_1) b(t,k_2,s_2) \bar{u}(k_1,s_1)\gamma_5 u(k_2,s_2) e^{i(k_2-k_1)x} \\ & + b^\dagger(t,k_1,s_1) d^\dagger(t,k_2,s_2) \bar{u}(k_1,s_1)\gamma_5 v(k_2,s_2) e^{-i(k_1+k_2)x} \\ & + d(t,k_1,s_1) b(t,k_2,s_2) \bar{v}(k_1,s_1)\gamma_5 u(k_2,s_2) e^{i(k_1+k_2)x} \\ & - d^\dagger(t,k_2,s_2) d(t,k_1,k_2) \bar{v}(k_1,s_1)\gamma_5 v(k_2,s_2) e^{-i(k_1-k_2)x}][a(t,k_3) + a^\dagger(t,-k_3)] e^{ik_3x} . \end{aligned} \tag{5.16}$$

p_b and p_d are the commutators of b and d with H_{int}. Using

$$[A,BC] = \{A,B\}C - B\{A,C\} \tag{5.17}$$

we get

$$\begin{aligned} p_b(t,k,s) = ig_0 \sum_{s_3} \int & \frac{d^3k_3}{[(2\pi)^3 2\omega_k \omega_3 \omega^\mu_{k-k_3}]^{1/2}} \bar{u}(k,s)\gamma_5 [u(k_3,s_3) b(t,k_3,s_3) + v(-k_3,s_3) d^\dagger(t,-k_3,s_3)] \\ & \times [a(t,k-k_3) + a^\dagger(t,k_3-k)] \end{aligned} \tag{5.18}$$

with

$$\begin{aligned} p_d(t,k,s) = -ig_0 \sum_{s_3} \int & \frac{d^3k_3}{(2\pi)^3 2\omega_k \omega_3 \omega^\mu_{k-k_3})^{1/2}} \\ & \times [d(t,k_3,s_3)\bar{v}(k_3,s_3) + b^\dagger(t,-k_3,s_3)\bar{u}(-k_3,s_3)]\gamma_5 v(k,s) [a(t,k-k_3) + a^\dagger(t,k_3-k)] . \end{aligned}$$

If we neglect all vacuum correlations the static part of M vanishes because all equal-time vacuum expectation values contain only one a operator and thus give no contribution because of (4.4). If all masses are already renormalized M reduces to

$$M(t-t',k_1,k_2,k_1',k_2') = -(d_{11'22'})^{-2}[G(t-t',p(k_1),k_2,k_1',p(k_2')) + G(t-t',k_1,p(k_2),p(k_1'),k_2')] . \tag{5.19}$$

We have

$$\begin{aligned} & [G(t-t',p(k_1),k_2,k_1',p(k_2'))]_{\alpha\beta\gamma\delta} \\ & = -d_{11'22'} \sum_{s_1 s_2 s_1' s_2'} [\theta(t-t')\langle 0 | p_b(t,k_1) d(t,k_2) p_d^\dagger(t',k_2') b^\dagger(t',k_1') | 0\rangle u_\alpha(k_1)\bar{u}_\beta(k_1')\bar{v}_\gamma(k_2) v_\delta(k_2') \\ & \qquad + \theta(t'-t)\langle 0 | p_b(t',k_2') d(t',k_1') p_d^\dagger(t,k_1) b^\dagger(t,k_2) | 0\rangle v_\alpha(k_1)\bar{v}_\beta(k_1')\bar{u}_\gamma(k_2) u_\delta(k_2')] . \end{aligned} \tag{5.20}$$

For $\langle 0 | p_b(t,k_1) d(t,k_2) p_d^\dagger(t',k_2') b^\dagger(t',k_1') | 0\rangle$ we obtain to second order in g_0

$$ig_0(-ig_0)^\dagger \sum_{s_3 s_4} \int \frac{d^3k_3 d^3k_4}{2(2\pi)^3(\omega_1\omega_3\omega^\mu_{1-3}\omega_{2'}\omega_4\omega^\mu_{2'-4})^{1/2}} \langle 0 | b(t,k_3,s_3) a(t,k_1-k_3) d(t,k_2,s_2)$$

$$\times d^\dagger(t',k_4,s_4) a^\dagger(t',k_2'-k_4) b^\dagger(t',k_1') | 0\rangle$$

$$\times \bar{u}(k_1,s_1)\gamma_5 u(k_3,s_3)[\bar{v}(k_4,s_4)\gamma_5 v(k_2',s_2')]^\dagger$$

$$= g_0{}^2 \frac{\delta^3(k_1+k_2-k_1'-k_2')}{(2\pi)^3 2(\omega_1\omega_{1'}\omega_2\omega_{2'})^{1/2}\omega^\mu_{1-1'}} e^{-i(\omega^\mu_{1-1'}+\omega_{1'}+\omega_2(t-t')} \bar{u}(k_1,s_1)\gamma_5 u(k_1',s_1')\bar{v}(k_2',s_2')\gamma_5 v(k_2,s_2) \quad (5.21)$$

and completely analogously

$$\langle 0 | p_b(t',k_2') d(t',k_1') p_d^\dagger(t,k_1) b^\dagger(t,k_2) | 0\rangle = g_0{}^2 \frac{\delta^3(K-K')}{(2\pi)^3 2(\omega_1\omega_{1'}\omega_2\omega_{2'})^{1/2}\omega^\mu} e^{i(\omega^\mu+\omega_{1'}+\omega_2)(t-t')}$$

$$\times \bar{u}(k_2',s_2')\gamma_5 u(k_2,s_2)\bar{v}(k_1,s_1)\gamma_5 v(k_1',s_1') , \quad (5.22)$$

and thus we get

$$[G(t-t',p(k_1),k_2,k_1',p(k_2'))]_{\alpha\beta\gamma\delta}$$

$$= -\frac{m^2 g_0{}^2 \delta^3(K-K')}{(2\pi)^3 2\omega_1\omega_{1'}\omega_2\omega_{2'}\omega^\mu} \sum_{s_1 s_2 s_1' s_2'} [\theta(t-t') e^{-i(\omega^\mu+\omega_{1'}+\omega_2)(t-t')} \bar{u}_\mu(k_1,s_1)\gamma^5_{\mu\nu} u_\nu(k_1',s_1')$$

$$\times \bar{v}_\rho(k_2',s_2')\gamma^5_{\rho\lambda} v_\lambda(k_2,s_2) u_\alpha(k_1,s_1)\bar{u}_\beta(k_1',s_1')\bar{v}_\gamma(k_2,s_2) v_\delta(k_2',s_2')$$

$$+\theta(t'-t) e^{i(\omega^\mu+\omega_{1'}+\omega_2)(t-t')} \bar{v}_\mu(k_1,s_1)\gamma^5_{\mu\nu} v_\nu(k_1',s_1') u_\rho(k_2',s_2')$$

$$\times \gamma^5_{\rho\lambda} u_\lambda(k_2,s_2) v_\alpha(k_1,s_1)\bar{v}_\beta(k_1',s_1')\bar{u}_\gamma(k_2,s_2) u_\delta(k_2',s_2')] . \quad (5.23)$$

But for the first spinor sum we have

$$\sum_{s_1 s_2 s_1' s_2'} \bar{u}_\mu(k_1,s_1)\gamma^5_{\mu\nu} u_\nu(k_1',s_1')\bar{v}_\rho(k_2',s_2')\gamma^5_{\rho\lambda} v_\lambda(k_2,s_2) u_\alpha(k_1,s_1)\bar{u}_\beta(k_1',s_1')\bar{v}_\gamma(k_2,s_2) v_\delta(k_2',s_2')$$

$$= [\Lambda_+(k_1)\gamma^5\Lambda_+(k_1')]_{\alpha\beta}[\Lambda_-(k_2')\gamma^5\Lambda_-(k_2)]_{\delta\gamma} \quad (5.24)$$

and for the second one

$$[\Lambda_-(k_1)\gamma^5\Lambda_-(k_1')]_{\alpha\beta}[\Lambda_+(k_2')\gamma^5\Lambda_+(k_2)]_{\delta\gamma} . \quad (5.25)$$

Therefore (5.23) can be written as

$$[G(t-t',p(k_1),k_2,k_1',p(k_2'))]_{\alpha\beta\gamma\delta} = -\frac{m^2 g_0{}^2 \delta^3(K-K')}{(2\pi)^3 2\omega_1\omega_{1'}\omega_2\omega_{2'}\omega^\mu}$$

$$\times \{\theta(t-t') e^{-i(\omega^\mu+\omega_{1'}+\omega_2)(t-t')} [\Lambda_+(k_1)\gamma^5\Lambda_+(k_1')]_{\alpha\beta}[\Lambda_-(k_2')\gamma^5\Lambda_-(k_2)]_{\delta\gamma}$$

$$+\theta(t'-t) e^{i(\omega^\mu+\omega_{1'}+\omega_2)(t-t')} [\Lambda_-(k_1)\gamma^5\Lambda_-(k_1')]_{\alpha\beta}[\Lambda_+(k_2')\gamma^5\Lambda_+(k_2)]_{\delta\gamma}\} .$$

(5.26)

From this expression we obtain $G(t-t',k_1,p(k_2),p(k_1'),k_2')$ by exchanging ω_1 with ω_2 and $\omega_{1'}$ with $\omega_{2'}$, the projection operators remaining the same. Inserting these two Green's functions into (5.19) yields $M(t-t')$, and Fourier transformation according to (4.21) gives

$$[M(E,k_1,k_2,k_1',k_2')]_{\alpha\beta\gamma\delta}=\frac{ig_0{}^2\delta^3(K-K')}{2(2\pi)^3\omega^2\omega^\mu(2\pi)^{1/2}}\Bigg\{\left[\frac{1}{E-\omega^\mu-\omega_{1'}-\omega_2}+\frac{1}{E-\omega^\mu-\omega_{2'}-\omega_1}\right]$$
$$\times[\Lambda_+(k_1)\gamma^5\Lambda_+(k_1')]_{\alpha\beta}[\Lambda_-(k_2')\gamma^5\Lambda_-(k_2)]_{\delta\gamma}$$
$$-\left[\frac{1}{E+\omega^\mu+\omega_{1'}+\omega_2}+\frac{1}{E+\omega^\mu+\omega_{2'}+\omega_1}\right]$$
$$\times[\Lambda_-(k_1)\gamma^5\Lambda_-(k_1')]_{\alpha\beta}[\Lambda_+(k_2')\gamma^5\Lambda_+(k_2)]_{\delta\gamma}\Bigg\}\,. \tag{5.27}$$

The projectors assure that the initial (momenta k_1' and k_2') and final states (momenta k_1 and k_2) are in the subspace corresponding to the proton-antiproton channel.

The integral kernel $(G_0M)(E)$ is

$$[(G_0M)(E,k_1,k_2,k_1',k_2')]_{\alpha\beta\gamma\delta}=-\frac{g_0{}^2\delta^3(K-K')}{2(2\pi)^3\omega_1\omega_2\omega^\mu}\Bigg\{\frac{1}{E-\omega_1-\omega_2}\left[\frac{1}{E-\omega^\mu-\omega_{1'}-\omega_2}+\frac{1}{E-\omega^\mu-\omega_{2'}-\omega_1}\right]$$
$$\times[\Lambda_+(k_1)\gamma^5\Lambda_+(k_1')]_{\alpha\beta}[\Lambda_-(k_2')\gamma^5\Lambda_-(k_2)]_{\delta\gamma}$$
$$+\frac{1}{E+\omega_1+\omega_2}\left[\frac{1}{E+\omega^\mu+\omega_{1'}+\omega_2}+\frac{1}{E+\omega^\mu+\omega_{2'}+\omega_1}\right]$$
$$\times[\Lambda_-(k_1)\gamma^5\Lambda_-(k_1')]_{\alpha\beta}[\Lambda_+(k_2')\gamma^5\Lambda_+(k_2)]_{\delta\gamma}\Bigg\} \tag{5.28}$$

and a bound state $[\chi(E,k_1,k_2)]_{\alpha\gamma}$ of energy E can be obtained as a solution of the following equation:

$$[\chi(E,k_1,k_2)]_{\alpha\gamma}\int[(G_0M)(E,k_1,k_2,k_1',k_2')]_{\alpha\mu\gamma\nu}[\chi(E,k_1',k_2')]_{\mu\nu}d^3k_1'd^3k_2'\,. \tag{5.29}$$

VI. CONCLUSIONS

Our aim has been to establish an integral equation similar to the Bethe-Salpeter equation, but with no relative time or relative energy appearing, thus facilitating the solution considerably. The equations established in Secs. IV and V with the integral kernel $(G_0M)(E,\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_1',\mathbf{k}_2')$ satisfy this requirement. We have seen how to proceed for scalar Klein-Gordon fields and how the formalism looks like for spinorial Dirac fields. For both cases we explicitly calculated the memory function M and the integral kernel (G_0M) for a given model. For the scalar-scalar model of Cutkosky the Bethe-Salpeter equation was solvable in the ladder approximation for a mass $\mu=0$ of the exchange particle. Thus in this case we could compare the binding energies which we found numerically to those from the Bethe-Salpeter equation. The memory function approach yielded binding energies very close to those of the Bethe-Salpeter equation, while the instantaneous approximation deviates more and more with increasing coupling. Unlike the instantaneous approximation, the formalism of the memory function takes into account retardation effects in a natural way. The numerical results show that this is already important for weak relativistic systems and we saw that in the strong relativistic case ($\lambda=1$) the formalism of the memory function yielded, in the given order, "ten times better results" than the instantaneous approximation. These results obtained within a specific model (and for a long-range force: $\mu=0$) almost certainly continue to be true in the general case, as we may assume according to the discussion about the equivalence of the memory function approach and the Bethe-Salpeter equation at the end of Sec. IV B.

Putting equal pairs of two times is not a relativistically invariant procedure, but we always can privilege the rest frame of the center of mass, since it is a Lorentz frame due to the conservation of total momentum, and put times equal in this frame. Thus we do not have any ambiguity (it is clear that it is impossible to construct any covariant formalism without appearance of relative times). Although our procedure is not covariant, it is completely relativistic [as we saw, e.g., at the end of Sec. IV B when demonstrating the equivalence of $(GM^{(2)}G)(t-t')$ and the Feynman propagator $G_F^{(2)}(t-t')$]. The memory-function approach is as "exact" as is the Bethe-Salpeter equation, only different iteration schemes lead to the appearance of different graphs when iterated with a kernel of a given finite order, explaining thus the small differences in the binding energies.

As we proceeded, we assumed a perturbation theoretic expansion in the coupling constants to be valid. When calculating the kernel of the Bethe-Salpeter equation as the sum of all irreducible Feynman graphs one makes the same assumption. But we used essentially the Heisenberg equations of motion, and perturbation theory only comes into play when determining M by solving the integral equation (3.6) approximatively by (3.7). Therefore it is conceivable, that M may be obtained directly from (3.6) and that a formalism without any perturbative expansions is possible. However, this can only be decided by further calculations.

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APPENDIX: THE NUMERICAL SOLUTION

In this appendix we will outline how we proceeded for the numerical solution of Eq. (4.63). The method is the same whether we have the kernels with or without vacuum correlations, whether $m_1=m_2$ or $m_2\to\infty$. Therefore we will only consider here the kernel (4.61) when we neglected vacuum correlations and for $m_1=m_2$.

We introduce reduced quantities by dividing all quantities having a dimension of a mass like μ, E, B, k, and ω by the reduced mass m. For simplicity we continue to call these reduced quantities μ, E, B, k, and ω. It is not difficult to see that the only variable parameter in Eq. (4.63) with the corresponding kernels is the coupling parameter λ, and in particular for a given level B/m it is only a function of λ (and of m_1/m_2 of course).

Because of the rotational invariance of our problem we search solutions of the form

$$\phi(\mathbf{k})=\frac{\phi_1(k)}{k}Y_{lm}(\theta,\phi)\ . \tag{A1}$$

The integral kernel $(G_0M)(E,\mathbf{k},\mathbf{k}')$ can be developed over $Y_{lm}(\theta\phi)Y_{l'm'}(\theta',\phi')$ as any function of $\mathbf{k}$ and $\mathbf{k}'$. We call the coefficients $g_{lml'm'}(E,k,k')$, they are given by

$$g_{lml'm'}(E,k,k')=\int (G_0M)(E,k,k')Y^{\dagger}_{lm}(\theta,\phi)Y_{l'm'}(\theta',\phi')\times d\Omega\, d\Omega' \quad \text{with } d\Omega=\sin\theta\, d\theta\, d\phi\ . \tag{A2}$$

If we define

$$f_l(E,k,k')=kk'g_{l0l0}(E,k,k')\ , \tag{A3}$$

it is easy to show that the original equation (4.63) becomes

$$\phi_l(k)=\int_0^\infty f_l(E,k,k')\phi_l(k')dk'\ . \tag{A4}$$

This integral equation in only one radial variable admits a normalizable solution only for certain discrete values of the total energy E.

To find g_l we have to calculate the integral (A2). Besides for $l=0$, this is not easy analytically (if not impossible). Of course, it can be calculated numerically for all values of E, k, and k' needed, but this increases the computation times considerably. This is why we restricted ourselves to $l=0$, where all essential features can be seen. For $l=0$ ($l'=0$) the Y_{lm} and $Y_{l'm'}$ are constant. Thus we can put the axis $\theta=0$ parallel to the direction of $\mathbf{k}$ when performing the integration in (A2). $(G_0M)(E,\mathbf{k},\mathbf{k})$ then depends only on the angle between k and k' (besides E,k,k'). This angle is just θ'. ω^μ is the only quantity depending on θ' and if we put $x=\cos\theta'$ we have

$$\omega^\mu=\omega^\mu(x)=(\mu^2+k^2+k'^2-kk'x)^{1/2}$$

and

$$d\omega^\mu(x)/dx=-kk'/\omega^\mu(x)$$

and thus

$$f_0(E,k,k')=-2\pi\int_{\omega^\mu(-1)}^{\omega^\mu(1)}(G_0M)(E,k,k',\omega^\mu)\omega^\mu d\omega^\mu\ . \tag{A5}$$

If (G_0M) is given by (4.61) one has

$$f_0(E,k,k')=\frac{4\lambda}{\pi\omega_k(E^2-4\omega_k^2)}\left\{[1-E/(2\omega_k)][1-E/(2\omega_{k'})]\ln\frac{(k^2+k'^2-2kk'+\mu^2)^{1/2}+\omega_k+\omega_{k'}+E}{(k^2+k'^2+2kk'+\mu^2)^{1/2}+\omega_k+\omega_{k'}+E}\right.$$
$$\left.+[1+E/(2\omega_k)][1+E/(2\omega_{k'})]\ln\frac{(k^2+k'^2+2kk'+\mu^2)^{1/2}+\omega_k+\omega_{k'}-E}{(k^2+k'^2+2kk'+\mu^2)^{1/2}+\omega_k+\omega_{k'}-E}\right\}. \tag{A6}$$

We have $\omega_k=(4+k^2)^{1/2}$ and $E=4-B$. For a weak relativistic system, i.e., for small λ, $\omega_k+\omega_{k'}-E$ is very small (of order λ^2). If furthermore μ is small, i.e., $\mu\lesssim\lambda^2$, and especially for $\mu=0$, the numerator of the second logarithm has a very sharp minimum for $k'=k$ leading to very peaked f_0. For μ or λ not too small Eq. (A4) can be solved by going to a standard discrete limit and diagonalization, but for small λ and small μ, the strong peak moving with k may lead to large errors if, e.g., a simple trapezoidal rule is used. To get rid of this difficulty we split f_0 into a term having the same peaked behavior at $k'=k$ as f_0 but being simple enough, so that an integration over a product with spline functions can be performed analytically, and another term which is small and varies sufficiently little, so that an integration by a trapezoidal rule induces only small errors. For $\mu=0$ we define

$$P(E,k,k')=\frac{4\lambda[1+E/(2\omega_k)][1+E/(2\omega_{k'})]}{\pi\omega_k(E^2-4\omega_k{}^2)} \,, \quad \text{(A7)}$$

$$Q(E,k,k')=\ln\frac{|k-k'|+\omega_k+\omega_{k'}-E}{|k+k'|+\omega_k+\omega_{k'}-E} \,, \quad \text{(A8)}$$

$$R(E,k,k')=\frac{4\lambda[1-E/(2\omega_k)][1-E/(2\omega_{k'})]}{\pi\omega_k(E^2-4\omega_k{}^2)} \times\ln\frac{|k-k'|+\omega_k+\omega_{k'}+E}{|k+k'|+\omega_k+\omega_{k'}+E} \,, \quad \text{(A9)}$$

$$Q^0(E,k,k')=\ln\frac{|k-k'|+2\omega_k-E}{|k+k'|+2\omega_{k'}E} \,. \quad \text{(A10)}$$

We have $f_0(E,k,k')=P(E,k,k')Q(E,k,k')+R(E,k,k')$. The peaked part is PQ. The function $P(E,k,k)Q^0(E,k,k')$ has the same peaked behavior as $P(E,k,k')Q(E,k,k')$ as $k'\to k$, but has the advantage that a product with any polynom in k' can be integrated analytically.

We develop $\phi_0(k)$ over spline functions where linear ones are sufficient for our problem (S_N approximates the asymptotic behavior of ϕ_0):

$$\phi_0(k)=\sum_{j=1}^{N} S_j(k)\phi^j, \quad \phi^j=\phi_0(k_j) \quad \text{(A11)}$$

and thus we obtain

$$\phi^i=\sum_{j=1}^{N} F_{ij}\phi^j \quad \text{(A12)}$$

with

$$F_{ij}=P(k_i,k_i)\int_0^\infty Q^0(k_i,k')S_j(k')dk' +\Delta_j[P(k_i,k_j)Q(k_i,k_j)-P(k_i,k_i)Q^0(k_i,k_j) +R(k_i,k_j)] \,.$$

Here is $\Delta_j=(k_{j+1}-k_{j-1})/2$ if $j\neq 1$ and $\Delta_1=(k_2-k_1)/2$ (we choose $k_1=0$). It has been advantageous to have the first k_i only little spaced and to increase their distance with increasing i.

The integrations over Q^0S_j are performed analytically for all i and j. The system of linear algebraic equations (A12) admits a nontrivial solution only if the determinant of $F_{ij}-\delta_{ij}$ vanishes. Thus the energies E are solutions of

$$\det[F(E)-1]=0 \,. \quad \text{(A13)}$$

We did the calculations for $N=25$, 50, 100, and 200. When passing from $N=100$ to $N=200$ the energies remain stable with a precision of about 10^{-4}, and thus we can suppose that the values for $N=200$ have at least this precision. For great λ (f_0 is not peaked) we obtain the same precision already for smaller λ.

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