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Abelian and non-Abelian bosonization in the path-integral framework

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Abelian and non-Abelian bosonization of two-dimensional models is discussed within the path-integral framework. Concerning the Abelian case, the equivalence between the massive Thirring and the sine-Gordon models is rederived in a very simple way by making a chiral change in the fermionic path-integral variables. The massive Schwinger model is also studied using the same technique. The extension of this bosonization approach to the solution of non-Abelian models is performed in a very natural way, showing the appearance of the Wess-Zumino functional through the Jacobian associated with the non-Abelian chiral change of variables. Relevant features of massless two-dimensional QCD are discussed in this context.

I. INTRODUCTION

Two-dimensional field theories have been widely explored in the last ten years and various phenomena such as dynamical mass generation, asymptotic freedom, and quark confinement, relevant in more realistic models, have been tested. The startling property which was exploited in these studies is related to the possibility of transforming Fermi fields into Bose fields. The existence of such a transformation, called bosonization, provided a powerful tool to obtain nonperturbative information of two-dimensional field theories.

Bosonization has its historical roots in Klaiber's[1] work on the massless Thirring model and Lowenstein and Swieca's[2] investigations on the massless Schwinger[3] model. It found a remarkable application in Coleman's equivalence proof[4] between the massive Thirring and the sine-Gordon theories.

For Abelian models the bosonization prescription is by now very well understood and quite rigorously established.[5–7] On the other hand, when Fermi fields belong to a multiplet transforming under a non-Abelian group, the usual bosonization procedure becomes rather difficult.[8–10] It is only very recently that non-Abelian bosonization in the operator approach has begun to be understood after the works of Polyakov and Wiegman[11] and Witten.[12] (See also Refs. 13–16).

There is an alternative approach to bosonization recently developed using the path-integral framework[17] which has been shown to be very appropriate for non-Abelian theories.[18–20] Basically, this approach parallels, in the path-integral framework, the operator fit of Lowenstein and Swieca,[2] through the use of a chiral change in the fermionic variables. Fujikawa's observation[21] on the noninvariance of the path-integral measure under γ_5 transformations is crucial for this method. In particular, it is the chiral Jacobian which gives rise to the Wess-Zumino term in the study of non-Abelian models.[21–23]

It is the purpose of this work to present a detailed discussion of the path-integral approach to bosonization, both in the Abelian and the non-Abelian cases. To this end we give in Sec. II the path-integral version of Coleman's proof of the equivalence between the massive Thirring and sine-Gordon models.[4] It is important to stress that in our method we first introduce an auxiliary vector field A_μ which is then decoupled from the fermions, exploiting the peculiarities of the algebra of $d=2$ Dirac matrices, as is done in the solution of the massless Schwinger model.[3,2,17] It is precisely this connection which allows us to extend our procedure in a very natural way to the massive Schwinger model. This is done in Sec. III, where the equivalence between this last model and a massive sine-Gordon theory is derived. The results are in complete agreement with those obtained using the operational approach.[24,25,6]

The non-Abelian extension of our bosonization method is discussed in Sec. IV. We show how the Wess-Zumino functional arises after carefully computing the chiral Jacobian associated with the decoupling change in the fermionic variables. Taking as an example massless two-dimensional QCD we compute the fermion determinant and establish the equivalence between this model and a boson theory (related to a certain chiral model). It is worthwhile to note that the role of the (non-Abelian) chiral anomaly becomes apparent in our treatment, showing its relevance in non-Abelian bosonization, exactly as it happens in the Abelian case. We also give a qualitative discussion of the principal features of this non-Abelian theory and indicate the steps to follow in order to complete its solution. A brief summary of our results and conclusions is given at the end of Sec. IV.

II. ABELIAN BOSONIZATION: MASSIVE THIRRING AND SINE-GORDON MODELS

Exactly as in the operator approach, where bosonization was derived by analyzing the equivalence between the massive Thirring and sine-Gordon models,[4] we shall show in this section how an analogous derivation can be performed very simply in the path-integral framework. We work in Euclidean $(1+1)$-dimensional space-time with γ_μ matrices chosen in the form

$$\gamma_0=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix},\ \gamma_1=\begin{bmatrix}0 & i\\ -i & 0\end{bmatrix},\ \gamma_5=i\gamma_0\gamma_1 . \tag{2.1}$$

The following relations hold:

$$\{\gamma_\mu,\gamma_\nu\}=2\delta_{\mu\nu} , \tag{2.2}$$

$$\gamma_\mu\gamma_5=i\epsilon_{\mu\nu}\gamma_\nu , \tag{2.3}$$

with $\epsilon_{01}=-\epsilon_{10}=1$.

The dynamics of the massive Thirring model is determined by the Lagrangian density

$$\mathscr{L}_T=-i\bar\psi\partial\!\!\!/\psi-\tfrac{1}{2}g^2(\bar\psi\gamma_\mu\psi)^2+izm\bar\psi\psi , \tag{2.4}$$

where z is a cutoff-dependent constant. In the path-integral approach the generating functional is given by

$$Z_T=\mathscr{N}\int\mathscr{D}\bar\psi\mathscr{D}\psi\exp\left[-\int d^2x\,\mathscr{L}_T\right] . \tag{2.5}$$

Following Ref. 19 we now use the identity

$$\exp\left[\frac{g^2}{2}\int(\bar\psi\gamma_\mu\psi)^2d^2x\right]=\int\mathscr{D}A_\mu\exp\left[-\int d^2x\,(\tfrac{1}{2}A_\mu A^\mu-g\bar\psi A\!\!\!/\psi)\right] \tag{2.6}$$

in order to eliminate the quartic interaction. Here A_μ is a two-component vector field which (in two dimensions) can be written as

$$A_\mu=-\frac{1}{g}(\epsilon_{\mu\nu}\partial_\nu\phi-\partial_\mu\eta) . \tag{2.7}$$

Using Eqs. (2.5)–(2.7), we get

$$Z_T=\mathscr{N}\int\mathscr{D}\bar\psi\mathscr{D}\psi\mathscr{D}\phi\mathscr{D}\eta\exp\left[-\int d^2x\,\mathscr{L}_{\rm eff}\right] , \tag{2.8}$$

where

$$\mathscr{L}_{\rm eff}=-\bar\psi[i\partial\!\!\!/-\gamma_\mu(\epsilon_{\mu\nu}\partial_\nu\phi-\partial_\mu\eta)]+izm\bar\psi\psi+\frac{1}{2g^2}[(\partial_\mu\phi)^2+(\partial_\mu\eta)^2] . \tag{2.9}$$

At this point we perform a change in the fermion variables which corresponds, in the path-integral framework, to the bosonization realization in the operator approach.[2,17] The change of variables takes the form

$$\psi(x)=\exp[\gamma_5\phi(x)+i\eta(x)]\chi(x) ,$$
$$\bar\psi(x)=\bar\chi(x)\exp[\gamma_5\phi(x)-i\eta(x)] \tag{2.10}$$

and it has been chosen so as to cancel the coupling between scalars and fermions in the kinetic term of $Z_{\rm eff}$. Indeed, using Eq. (2.10) the Lagrangian (2.9) can be written in the form

$$\mathscr{L}_{\rm eff}=-\bar\chi i\partial\!\!\!/\chi+izm\bar\chi e^{2\gamma_5\phi}\chi+\frac{1}{2g^2}[(\partial_\mu\phi)^2+(\partial_\mu\eta)^2] . \tag{2.11}$$

As we see, the scalar field η is completely decoupled from the rest and this fact remains valid at the quantum level. We can write the generating functional in terms of the new variables provided the corresponding Jacobians are taken into account,

$$\mathscr{D}\bar\psi\mathscr{D}\psi=J_F\mathscr{D}\bar\chi\mathscr{D}\chi , \tag{2.12a}$$

$$\mathscr{D}A_\mu=J_A\mathscr{D}\phi\mathscr{D}\eta . \tag{2.12b}$$

The fermion Jacobian is nontrivial due to the noninvariance of the measure under chiral changes (this being related to the axial anomaly). It can be computed following Fujikawa's procedure.[21] We sketch the calculation in an appendix and only state here the final result,

$$J_F=\exp\left[-\frac{1}{2\pi}\int d^2x(\partial_\mu\phi)^2\right] . \tag{2.13}$$

Concerning the change (2.12b), it trivially yields

$$J_A=\frac{1}{g^2}\det\nabla^2 \tag{2.14}$$

which can be absorbed in the normalization constant. We then have

$$Z_T=\mathscr{N}\int\mathscr{D}\bar\chi\mathscr{D}\chi\mathscr{D}\phi\mathscr{D}\eta\times\exp\left\{-\int d^2x\left[-\bar\chi i\partial\!\!\!/\chi+izm\bar\chi e^{2\gamma_5\phi}\chi+\left[\frac{1}{2g^2}+\frac{1}{2\pi}\right](\partial_\mu\phi)^2+\frac{1}{2g^2}(\partial_\mu\eta)^2\right]\right\} . \tag{2.15}$$

The addition of a source term

$$\mathscr{L}_{\rm source}=\bar\theta e^{\gamma_5\phi+i\eta}\chi+\bar\chi e^{\gamma_5\phi-i\eta}\theta \tag{2.16}$$

allows the computation of any Green's function in terms of the new fields.[19] However, this is not necessary in order to prove the equivalence between this model and the sine-Gordon theory. This can be done just by making a perturbative expansion (in the mass) in the generating functional

$$Z_T=\mathscr{N}\int\mathscr{D}\phi\mathscr{D}\bar\chi\mathscr{D}\chi\times\exp\left\{-\int d^2x\left[-\bar\chi i\partial\!\!\!/\chi+\frac{1}{2\lambda^2}(\partial_\mu\phi)^2\right]\right\}\times\sum_{n=0}^{\infty}\frac{(-izm)^n}{n!}\prod_{j=1}^{n}\int d^2x_j\bar\chi(x_j)\times e^{2\gamma_5\phi(x_j)}\chi(x_j) , \tag{2.17}$$

where we have defined

$$\lambda^2=\frac{g^2}{1+g^2/\pi} . \tag{2.18}$$

We then get

$$Z_T=\sum_{n=0}^{\infty}\frac{(-izm)^n}{n!}\left\langle\prod_{j=1}^{n}\int d^2x_j\bar\chi(x_j)e^{2\gamma_5\phi(x_j)}\chi(x_j)\right\rangle_0 , \tag{2.19}$$

where $\langle\ \rangle_0$ means the vacuum expectation value (VEV) in a theory of free fermions and massless free scalars. Note that owing to the presence of λ in the scalar Lagrangian, the scalar propagator is defined through the identity

$$\frac{1}{\lambda^2}\Box\Delta_F(x)=-\delta^2(x)\ , \tag{2.20a}$$

$$\Delta_F(x)=-\frac{\lambda^2}{2\pi}\ln(\mu x)\ . \tag{2.20b}$$

In order to avoid infrared divergences, we shall follow the usual procedure[4] of adding a small mass μ to Eq. (2.20a) then getting for the scalar propagator, instead of (2.20b),

$$\Delta_F(x)=\frac{\lambda^2}{2\pi}K_0(\mu x)\ . \tag{2.21}$$

Of course, we shall take $\mu^2\rightarrow 0$ at the end of our computations. On the other hand, the fermion propagator is just the free fermion Green's function

$$G_F(x)=\frac{i}{2\pi}\frac{\gamma_\mu x^\mu}{x^2}\ . \tag{2.22}$$

In order to compute (2.19) one just separates the boson factor from the free fermionic part by writing

$$\bar{\chi}e^{2\gamma_5\phi}\chi=e^{2\phi}\bar{\chi}\frac{1+\gamma_5}{2}\chi+e^{-2\phi}\bar{\chi}\frac{1-\gamma_5}{2}\chi \tag{2.23}$$

and uses the well-known identity

$$\begin{aligned}\left\langle \exp\left[i\sum_i \beta_i\phi(x_i)\right]\right\rangle_{0\ \text{bosonic}}&=\exp\left\{-\tfrac{1}{2}\sum_{i,j}\beta_i\beta_j[\Delta_F(\mu,x_i-x_j)-\Delta_F(\Lambda,x_i-x_j)]\right\}\\&=\left[\frac{\mu}{\rho}\right]^{(\lambda^2/4\pi)\left[\sum_i\beta_i\right]^2}\left[\frac{\rho}{\Lambda}\right]^{(\lambda^2/4\pi)\sum_i\beta_i^2}\prod_{i>j}(\rho c\,|x_i-x_j|)^{(\lambda^2/2\pi)\beta_i\beta_j}\ ,\end{aligned} \tag{2.24}$$

where Λ is a large mass introduced to cut off the theory and the arbitrary mass ρ is included as a normal-ordering mass.

We have also considered the restricted region

$$\Lambda\,|x_i-x_j|\gg 1\ , \tag{2.25}$$

$$\mu\,|x_i-x_j|\ll 1\ . \tag{2.26}$$

This last condition is required in order to circumvent the trivial infrared problems which arise when one performs a mass perturbation expansion about a massless theory.[4]

Note that if $\sum_i\beta_i\neq 0$, then (2.24) vanishes in the limit $\mu\rightarrow 0$. We shall then restrict our analysis to the case

$$\sum_i\beta_i=0\ . \tag{2.27}$$

We then obtain

$$\begin{aligned}Z_T=\sum_{k=0}^{\infty}\frac{(-izm)^{2k}}{(k!)^2}\int&\left[\prod_{i=1}^{k}d^2x_id^2y_i\right]\left\langle\exp\left[2\sum_i[\phi(x_i)-\phi(y_i)]\right]\right\rangle_{0\ \text{bosonic}}\\&\times\left\langle\prod_{i=1}^{k}\bar{\chi}(x_i)\frac{1+\gamma_5}{2}\chi(x_i)\bar{\chi}(y_i)\frac{1-\gamma_5}{2}\chi(y_i)\right\rangle_{0\ \text{fermionic}}\ .\end{aligned} \tag{2.28}$$

The fermionic part is readily computed by writing

$$\bar{\chi}\frac{1+\gamma_5}{2}\chi=\bar{\chi}_1\chi_1\ , \tag{2.29}$$

$$\bar{\chi}\frac{1-\gamma_5}{2}\chi=\bar{\chi}_2\chi_2\ , \tag{2.30}$$

where

$$\chi=\begin{bmatrix}\chi_1\\ \chi_2\end{bmatrix},\quad \bar{\chi}=(\bar{\chi}_1,\bar{\chi}_2) \tag{2.31}$$

and using now Eq. (2.24) for the boson part we get

$$Z_T=\sum_{k=0}^{\infty}\frac{m^{2k}}{k!^2}\int\frac{\prod_{i>j}^{k}(\rho^2c^2\,|x_i-x_j|\,|y_i-y_j|\,)^{2-2\lambda^2/\pi}\prod_{i=1}^{k}d^2x_id^2y_i}{\prod_{i,j}^{k}(\rho c\,|x_i-y_j|\,)^{2-2\lambda^2/\pi}}\ . \tag{2.32}$$

We shall compare Eq. (2.32) with the corresponding one arising in the sine-Gordon case. The generating functional for this last model is given by

$$Z_{\rm SG}=\mathcal{N}\int \mathcal{D}\varphi \exp\left[-\int d^2x\left[\tfrac{1}{2}(\partial_\mu\varphi)^2-\frac{\alpha_0}{\beta^2}\cos\beta\varphi+\gamma_0\right]\right] . \tag{2.33}$$

Performing a perturbative expansion in α_0 we get

$$Z_{\rm SG}=\sum_{k=0}^{\infty}\left[\frac{1}{k!}\right]^2\left\langle\left[\frac{\alpha_0}{\beta^2}\right]^{2k}\int\prod_{i=1}^{k}e^{i\beta\varphi(x_i)}e^{-i\beta\varphi(y_i)}d^2x_i d^2y_i\right\rangle_0 , \tag{2.34}$$

where use has been made of Eq. (2.27). We then proceed exactly as for the boson part of the Thirring model obtaining

$$Z_{\rm SG}=\sum_{k=0}^{\infty}\left[\frac{1}{k!}\right]^2\left[\frac{\alpha}{\beta^2}\right]^{2k}\int\frac{\prod_{i>j}^{k}(c^2M^2|x_i-x_j||y_i-y_j|)^{\beta^2/2\pi}\prod_{i=1}^{k}d^2x_i d^2y_i}{\prod_{i,j}^{k}(cM|x_i-y_j|)^{\beta^2/2\pi}} , \tag{2.35}$$

where we have defined the renormalized constant

$$\alpha=\frac{\alpha_0}{2}\left[\frac{M}{\Lambda}\right]^{\beta^2/4\pi} . \tag{2.36}$$

Here Λ is, as above, a certain cutoff, and M is an arbitrary mass used to normal-order the scalar theory.

We can easily see that both generating functionals (2.32) and (2.35) are identical provided we make the identifications

$$\frac{\beta^2}{4\pi}=\frac{1}{1+g^2/\pi} , \tag{2.37}$$

$$\frac{\alpha}{\beta^2}=m , \tag{2.38}$$

$$M=\rho , \tag{2.39}$$

which are, of course, the ones obtained by Coleman in his original work.[4]

Note that Eq. (2.37) does not depend on the way one has performed the renormalization in both theories [Eq. (2.39)]. In contrast, Eq. (2.38) does depend on the renormalization convention and so it has no independent meaning.

We conclude that it is possible to study the massive Thirring model in terms of a bosonic Lagrangian by considering the usual bosonization relations:

$$-i\bar\psi\partial\!\!\!/\psi=\tfrac{1}{2}(\partial_\mu\varphi)^2 , \tag{2.40}$$

$$\bar\psi\gamma_\mu\psi=\frac{i}{\sqrt{\pi}}\epsilon_{\mu\nu}\partial_\nu\varphi , \tag{2.41}$$

$$imz\bar\psi\psi=-\frac{\alpha_0}{\beta^2}\cos\beta\varphi . \tag{2.42}$$

Thus we see that the corresponding sine-Gordon Lagrangian is

$$\mathcal{L}_{\rm SG}=\tfrac{1}{2}(\partial_\mu\varphi)^2\left[1+\frac{g^2}{\pi}\right]-\frac{\alpha_0}{\beta^2}\cos\beta\varphi+\gamma_0 . \tag{2.43}$$

Note that one can always define a new scalar field φ' so that

$$\varphi'=\left[1+\frac{g^2}{\pi}\right]^{1/2}\varphi \tag{2.44}$$

and then

$$\mathcal{L}_{\rm SG}=\tfrac{1}{2}(\partial_\mu\varphi')^2-\frac{\alpha_0}{\beta^2}\cos\left[\frac{\beta^2\varphi'}{2\sqrt{\pi}}\right]+\gamma_0 . \tag{2.45}$$

Using now the free value $\beta=2\sqrt{\pi}$ ($g=0$), the usual bosonization identification is obtained:

$$\mathcal{L}_{\rm SG}=\tfrac{1}{2}(\partial_\mu\varphi)^2-\frac{\alpha_0}{\beta^2}\cos(2\sqrt{\pi}\varphi)+\gamma_0 . \tag{2.46}$$

III. ABELIAN BOSONIZATION: MASSIVE SCHWINGER AND SINE-GORDON MODELS

Let us now consider the path-integral bosonization for massive quantum electrodynamics in two space-time dimensions,

$$\mathcal{L}_{\rm SM}=-\bar\psi(i\partial\!\!\!/+eA\!\!\!/)\psi+\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}+im_0\bar\psi\psi . \tag{3.1}$$

We start from the generating functional

$$Z_{\rm SM}=\mathcal{N}\int\mathcal{D}A_\mu\mathcal{D}\bar\psi\mathcal{D}\psi\exp\left[-\int d^2x\,\mathcal{L}_{\rm SM}\right] \tag{3.2}$$

and in analogy with the Thirring model case we perform a decoupling change of variables

$$\psi=e^{\gamma_5\phi}\chi , \tag{3.3a}$$

$$\bar\psi=\bar\chi e^{\gamma_5\phi} , \tag{3.3b}$$

$$A_\mu=-\frac{1}{e}\epsilon_{\mu\nu}\partial_\nu\phi , \tag{3.3c}$$

where we are working in the Lorentz gauge. One then gets

$$Z_{\rm SM}=\mathcal{N}\int\mathcal{D}\phi\,\mathcal{D}\bar\chi\,\mathcal{D}\chi\exp\left[-\int d^2x\,\mathcal{L}_{\rm eff}\right] , \tag{3.4}$$

where

$$\mathcal{L}_{\rm eff}=-i\bar\chi\partial\!\!\!/\chi+\frac{1}{2e^2}\phi\Box\Box\phi-\frac{1}{2\pi}\phi\Box\phi+im_0\bar\chi e^{2\gamma_5\phi}\chi \tag{3.5}$$

and the nontrivial Jacobian J_F has been computed as in the Thirring case, adding a term

$$\mathscr{L}_m = -\frac{1}{2\pi}\phi\Box\phi$$

in the effective Lagrangian.

We can now follow the procedure of the preceding section defining the scalar propagator

$$\left[\frac{1}{e^2}\Box\Box - \frac{1}{\pi}\Box\right]\tilde{\Delta}_F(x) = \delta^2(x) , \tag{3.6}$$

$$\tilde{\Delta}_F(x) = -\tfrac{1}{2}\left[K_0\left[\frac{e}{\sqrt{\pi}}x\right] + \ln\left[\frac{ecx}{\sqrt{\pi}}\right]\right]$$

$$\equiv -\pi\left[\Delta_F\left[\frac{e}{\sqrt{\pi}},x\right] - \Delta_F(0,x)\right] .$$

As we see $\tilde{\Delta}_F$ corresponds to the propagator of a free scalar field of mass $e/\sqrt{\pi}$ and a massless free field. This last is a manifestation in the path-integral approach of the zero-mass gauge excitation which appears in the Lowenstein-Swieca solution for the massless Schwinger model.[2]

Note that due to the particular form of the propagator (3.6) there are no ultraviolet divergences. On the other hand, one has to introduce, as in the preceding section, a mass μ^2 in order to avoid infrared divergences and take $\mu^2 \to 0$ at the end of the computation. We shall not repeat this part of the computations since they are in complete analogy with those detailed in Sec. II.

The boson part in Eq. (3.4) can be separated using again an identity of the form

$$\left\langle \exp\left[-i\sum_i \beta_i\phi(x_i)\right]\right\rangle_0 = \exp\left[-\tfrac{1}{2}\sum_{i,j}\beta_i\beta_j\tilde{\Delta}_F(x_i-x_j)\right] . \tag{3.7}$$

Concerning the fermion part it is computed exactly as in the Thirring model case. We then get for the generating functional of the massive Schwinger model the following expression:

$$Z_{\rm SM} = \sum_{k=0}^{\infty}\left[\frac{1}{k!}\right]^2\left[\frac{m_0}{2\pi}\right]^{2k}\int B_k \frac{\prod_{i>j}^{k}|x_i-x_j|^2|y_i-y_j|^2\left[\prod_{i=1}^{k}d^2x_id^2y_i\right]}{\prod_{i,j}^{k}|x_i-y_j|^2} , \tag{3.8}$$

where B_k is the scalar contribution to $Z_{\rm SM}$,

$$B_k = \left[\frac{ec}{\sqrt{\pi}}\right]^{2k}\frac{\prod_{i>j}^{k}|x_i-x_j|^{-2}|y_i-y_j|^{-2}}{\prod_{i,j}^{k}|x_i-y_j|^{-2}}$$

$$\times\exp\left\{-2\sum_{i>j}\left[K_0\left[\frac{e}{\sqrt{\pi}}|x_i-x_j|\right] + K_0\left[\frac{e}{\sqrt{\pi}}|y_i-y_j|\right] - K_0\left[\frac{e}{\sqrt{\pi}}|x_i-y_j|\right]\right]\right\} . \tag{3.9}$$

We then see that the massless excitation [see Eq. (3.6)] cancels out the free-fermion contribution to $Z_{\rm SM}$. We then get

$$Z_{\rm SM} = \sum_{k=0}^{\infty}\left[\frac{mec}{\sqrt{\pi}}\right]^{2k}\left[\frac{1}{k!}\right]^2\int\left[\prod_{i=1}^{k}d^2x_id^2y_i\right]\exp\left\{-2\sum_{i>j}\left[K_0\left[\frac{e}{\sqrt{\pi}}|x_i-x_j|\right] + K_0\left[\frac{e}{\sqrt{\pi}}|y_i-y_j|\right] - K_0\left[\frac{e}{\sqrt{\pi}}|x_i-y_j|\right]\right]\right\} , \tag{3.10}$$

where $m = m_0/2\pi$.

One can now proceed to the bosonization identification. It is evident that the generating functional (3.10) coincides with the one for a massive sine-Gordon model with Lagrangian density

$$\mathscr{L}_{\rm SG} = \tfrac{1}{2}(\partial_\mu\varphi)^2 + \frac{e^2}{2\pi}\varphi^2 - \frac{\alpha}{\beta^2}\cos\beta\varphi + \gamma \tag{3.11}$$

provided the following identifications hold:

$$\frac{\alpha}{\beta^2} = m\frac{ec}{\sqrt{\pi}} , \tag{3.12}$$

$$\beta^2 = 4\pi . \tag{3.13}$$

Of course, the analysis of the massless Schwinger model can be performed by taking $m=0$ in Eq. (3.12) and hence the isomorphism between this model and a free massive (with mass $e/\sqrt{\pi}$) scalar theory becomes apparent. [There is also the massless gauge excitation; see the discussion in Refs. (2), (18) and (19)].

We shall end this section with a brief discussion on the way the θ vacuum can be very easily incorporated in our approach to bosonization. As it is well known, the θ vacuum can be studied by adding to the generating functional of the Schwinger model, Eq. (3.2), a θ term

$$Z[\theta]=\mathcal{N}\int \mathcal{D}A_\mu \mathcal{D}\bar{\psi}\mathcal{D}\psi \times\exp\left[-\int d^2x\left[\mathcal{L}_0+\frac{e\theta}{4\pi}\epsilon_{\mu\nu}F_{\mu\nu}\right]\right] . \tag{3.14}$$

In principle, for the massless theory any reference to θ can be eliminated from Z by making a finite chiral rotation of fermions,

$$\psi=e^{\gamma_5\alpha}\chi , \quad \bar{\psi}=\bar{\chi}e^{\gamma_5\alpha} , \tag{3.15}$$

since it gives rise to a Jacobian (see the Appendix)

$$J=\exp\left[\frac{\alpha e}{2\pi}\int \epsilon_{\mu\nu}F_{\mu\nu}d^2x\right] \tag{3.16}$$

and hence the choice $\alpha=\theta/2$ cancels the θ term in (3.14), thus giving

$$Z_{m=0}[\theta]=Z_{m=0}[0] . \tag{3.17}$$

However, as is discussed in Ref. 28, quantities involving chiral nonsinglet operators keep trace of the θ term. For example, in order to compute the quantity

$$\langle \bar{\psi}(x)\psi(x)\bar{\psi}(0)\psi(0)\rangle \tag{3.18}$$

one has to add a source term to the Lagrangian

$$Z_{m=0}[\theta,j]=\int \mathcal{D}A_\mu\mathcal{D}\bar{\psi}\mathcal{D}\psi \times\exp\left[-\int d^2x\left[\mathcal{L}_0+\frac{e\theta}{4\pi}F_{\mu\nu}\epsilon_{\mu\nu}+j\bar{\psi}\psi\right]\right] \tag{3.19}$$

and this source term keeps trace of the θ angle when the rotation (3.15) is performed. This aspect is discussed at length in Ref. 29.

It is now evident what happens in the massive theory. The chiral rotation (3.15) eliminates the θ term from $Z_m[\theta]$ through the Jacobian, but it changes the mass term

$$im_0\bar{\psi}(x)\psi(x)\rightarrow im_0\bar{\psi}(x)e^{\gamma_5\theta}\psi(x) . \tag{3.20}$$

At this point one has to proceed as in the $\theta=0$ case, performing transformations (3.3) and then making an expansion in the mass. The only change in the effective Lagrangian (3.5) is the presence of a mass term of the form

$$\mathcal{L}_{\text{mass}}=im_0\bar{\chi}e^{2\gamma_5[\phi(x)+\theta/2]}\chi . \tag{3.21}$$

All the analysis then follows exactly as before, except that now the cosine term in the equivalent sine-Gordon theory is changed [cf. Eq. (3.11)]:

$$\mathcal{L}_{\text{SG}}=\tfrac{1}{2}(\partial_\mu\varphi)^2+\frac{e^2}{2\pi}\varphi^2-\frac{\alpha}{\beta^2}\cos\beta(\varphi-\theta/2)+\gamma . \tag{3.22}$$

Equation (3.22) makes contact with the usual treatment of the θ-vacuum Schwinger model.[25]

IV. NON-ABELIAN BOSONIZATION

We shall show in this section how the path-integral bosonization approach developed above can be naturally extended to the non-Abelian case.

As we stated in the introduction the usual bosonization procedure turns out to be very awkward in the case of fermion theories with non-Abelian symmetries.[8–10] An alternative bosonization procedure has been proposed by Witten,[12] and from his work and those of other authors[11–16] interesting connections with the Wess-Zumino functional—originally constructed as an effective action for chiral anomalies—were discovered. In the path-integral framework it has been pointed out by Gamboa Saraví, Schaposnik, and Solomin[20] how this relation emerges in a very natural way by extending the chiral change of variables (2.10) or (3.3) to the non-Abelian case.

Consider for simplicity the case of massless QCD in two dimensions. The Euclidean Lagrangian for this model is

$$\mathcal{L}=-\bar{\psi}\not{D}\psi+\tfrac{1}{4}F_{\mu\nu}F_{\mu\nu}+\text{gauge-fixing terms} , \tag{4.1}$$

where $\not{D}=i\not{\partial}+e\not{A}$, and A_μ takes values in the Lie algebra of SU(2) [the extension to SU(N) is trivial]. The massless fermions are taken in the fundamental representation of SU(2).

As it was stressed in Ref. 18 there exists a non-Abelian analog of the decoupling change of variables (2.10) or (3.3). Indeed, if one makes the following transformation,

$$\psi=U_5\chi , \quad \bar{\psi}=\bar{\chi}U_5 , \tag{4.2}$$

where

$$U_5=e^{\gamma_5\phi} \tag{4.3}$$

and $\phi=\phi^a t^a$ [takes values in the Lie algebra of SU(2), generated by the t^a's], it is straightforward to check that the fermion Lagrangian decouples completely from gauge fields, i.e.,

$$\mathcal{L}_F=-\bar{\psi}\not{D}\psi=-\bar{\chi}i\not{\partial}\chi . \tag{4.4}$$

Although Eq. (4.4) holds in an arbitrary gauge it is simpler and more instructive to work in the decoupling gauge, defined by the relation

$$\not{A}=-\frac{i}{e}(\not{\partial}U_5)U_5^{-1} \tag{4.5}$$

[Eq. (4.5) becomes $A_\mu=(1/e)\epsilon_{\nu\mu}\partial_\nu\phi$ in the Abelian case, and hence the decoupling gauge coincides with the Lorentz gauge for the Schwinger model].

That the choice of the decoupling gauge is possible can be proved following Roskies's work[26] by considering the $j=i\gamma_5$ complexification of SU(2), SL(2,c). Indeed, U_5 can be taken as an element of the form

$$U_5=e^{-ij\phi} , \tag{4.6}$$

that is, a positive-definite Hermitian matrix of determinant one. We shall call G_5 the set of all such elements, $G_5 \subset SL(2,c)$.

Defining

$$x_{\pm} = x_1 \pm x_0 \tag{4.7}$$

Eq. (4.5) can be written as

$$A_+ = -2i(\partial_- U_5)U_5^{-1} . \tag{4.8}$$

Note that in our way to bosonization the role of fermion currents, which in the operator approach are written in terms of scalar fields [for example, $J_\mu = (i/\pi)\epsilon_{\mu\nu}\partial_\nu\phi$ in the Abelian case, see Eq. (2.41)], is played now by the gauge field. We then see that in our approach to non-Abelian bosonization we have naturally arrived to the analog of currents $J_\pm$ introduced in Refs. 11 and 12, except for the fact that $x_\pm$ are defined by Eq. (4.7) and are not then true light-cone coordinates. (Precisely this difference simplifies considerably our treatment.)

In order to write the generating functional in terms of the new variables we have to take into account as before the change in the fermionic measure under transformation (4.2):

$$\mathcal{D}\bar{\psi}\,\mathcal{D}\psi = J_F \mathcal{D}\bar{\chi}\,\mathcal{D}\chi . \tag{4.9}$$

It is interesting to note that the Jacobian J_F coincides with the fermionic determinant

$$\det \not{D} = \int \mathcal{D}\bar{\psi}\,\mathcal{D}\psi\, e^{-\int \bar{\psi}\not{D}\psi d^2x} = J_F \int \mathcal{D}\bar{\chi}\,\mathcal{D}\chi\, e^{-\int \bar{\chi} i\not{\partial}\chi d^2x} = J_F \det i\not{\partial} . \tag{4.10}$$

As we shall see, this relation allows one to make contact with the Polyakov-Wiegman solution of the nonlinear σ model.[11]

In computing J_F we consider an extended U_5 transformation depending on a parameter t ($t \in [0,1]$):

$$U_5(x,t) = e^{(1-t)\gamma_5\phi(x)} . \tag{4.11}$$

The whole transformation (4.3) is then built up by iteration from the infinitesimal one, varying t from 0 to 1.

The method described in the appendix for the Abelian Jacobian can be extended to the non-Abelian case following Refs. 18–20. One then gets

$$\ln J_F = -\frac{e^2}{2\pi} \int d^2x \,\mathrm{tr}\left[\tfrac{1}{2}\boldsymbol{A}\boldsymbol{A} + \int_0^1 dt\, \gamma_5 \boldsymbol{A}_t \phi \boldsymbol{A}_t\right] , \tag{4.12}$$

where tr means trace both in Lorentz and SU(2) indices and

$$\boldsymbol{A}_t = -\frac{i}{e}[\partial U_5(x,t)]U_5^{-1}(x,t) . \tag{4.13}$$

One can then infer from the first term in (4.12) that the Schwinger mechanism[3] also takes place in QCD_2, giving a mass m ($m^2 = e^2/2\pi$) to the scalar fields (see the discussion below). Concerning the second term, it is related to the Wess-Zumino functional in two dimensions. Indeed, using Eq. (4.13), the second term in Eq. (4.12) can be written as

$$-\frac{e^2}{2\pi}\mathrm{tr}\int \gamma_5 \boldsymbol{A}_t\phi\boldsymbol{A}_t dt\, d^2x - \frac{e^2}{2\pi}\int d^2x\, \mathrm{tr}\boldsymbol{A}\boldsymbol{A} = \frac{i}{\pi}\int_0^1 dt \int d^2x\, \mathrm{tr}^c\left[\frac{-i}{4}\partial_\mu U \partial_\mu U^{-1} + [(\partial_t U)U^{-1}(\partial_\mu U)U^{-1}(\partial_\nu U)U^{-1}\epsilon_{\mu\nu}]\right] \tag{4.14}$$

with $U(x,t) = e^{t\phi(x)}$, and tr^c indicates an SU(2) trace. The relation between these results and the Wess-Zumino functional can be better discussed by considering the analytic continuation of U to an SU(2) element.

Let us now rewrite the Jacobian (4.12) in the form

$$\ln J_F = \frac{1}{\pi}\mathrm{tr}\left[\tfrac{1}{4}\int d^2x[\partial_\mu U(x,1)][\partial_\mu U^{-1}(x,1)] + \frac{i}{2}\epsilon_{\mu\nu}\int_0^t dt \int d^2x[\partial_t U(x,t)]U^{-1}(x,t)[\partial_\mu U(x,t)]U^{-1}(x,t)[\partial_\nu U(x,t)]U^{-1}(x,t)\right] . \tag{4.15}$$

By considering the analytic continuation of U to an element $U_c = e^{it\phi} \in SU(2)$ one can discover the relation between the second term in the Jacobian (4.15) and the Wess-Zumino functional.[22] This term is endowed with deep topological meaning since it corresponds to the Chern-Simons secondary invariant in differential geometry. Again, the chiral transformation makes manifest the role of topology in bosonization [for details see Ref. (20)].

In order to establish the rigorous equivalence between QCD_2 and a certain bosonic model one now has to follow the steps described in the preceding section for Abelian theories, i.e., to write an effective Lagrangian constructed from the original one in terms of fields U and $\chi, \bar{\chi}$, including the Jacobian (4.14),

$$\int \mathcal{L}_{\mathrm{eff}} d^2x = \int d^2x[-\bar{\chi}i\not{\partial}\chi + \tfrac{1}{4}F_{\mu\nu}(U)F_{\mu\nu}(U)] + \ln J_F . \tag{4.16}$$

We leave for a forthcoming paper[27] the detailed discussion of the effective action (4.16). However, one can give a qualitative picture by making a perturbative expansion of the form

$$U = 1 + 2\phi^a t^a + O(\phi^2) . \tag{4.17}$$

In this approximation the Jacobian reads

$$\ln J_F = -\frac{4}{\pi}\mathrm{tr}\int d^2x[(\partial_\mu\phi)^2 + \tfrac{1}{6}\epsilon_{\mu\nu}\phi(\partial_\mu\phi)(\partial_\nu\phi) + \text{higher-order terms}] . \tag{4.18}$$

This Jacobian resembles the effective Lagrangian discussed by Witten[23] in order to describe low-energy hadron phenomenology. However, in the present case the effective Lagrangian contains the $F_{\mu\nu}{}^2$ term and reads

$$\mathscr{L}_{\text{eff}}=\frac{1}{e^2}\,\text{tr}\left[\phi\left[\Box\Box+\frac{e^2}{2\pi}\Box\right]\phi+\frac{e^2}{6\pi}\phi(\partial_\mu\phi)(\partial_\nu\phi)\epsilon_{\mu\nu}+2\phi\partial_\mu\Box\phi\partial_\nu\phi\epsilon_{\mu\nu}\right]. \tag{4.19}$$

As in the Abelian case we have gotten an effective Lagrangian with high-order derivative terms. The free Lagrangian $\mathscr{L}_0$,

$$\mathscr{L}_0=\frac{1}{e^2}\,\text{tr}\left[\phi\left[\Box\Box+\frac{e^2}{2\pi}\Box\right]\phi\right], \tag{4.20}$$

corresponds to N^2-1 [$2^2-1=3$ for SU(2)] massive scalars (with mass $m=e/2\sqrt{\pi}$) and N^2-1 massless gauge excitations [the propagator associated to the Lagrangian (4.20) takes the form (3.6)].

In contrast with the Abelian cases described previously (where the massive scalars were free), here a self-interaction (given by the Wess-Zumino functional and the nonquadratic part of the $F_{\mu\nu}{}^2$ term) is present.

Because of the Wess-Zumino term, the Lagrangian violates both naive parity operation ($P_0\equiv X_0\to X_0$, $X_1\to -X_1$, $U\to U$) and (modulo 2) boson number N_B conservation [$(-1)^{N_B}\equiv U\to U^{-1}$ or $\phi^a\to-\phi^a$] but it is invariant under the product $P_0(-1)^{N_B}$.

As was pointed out above, the fermion determinant (4.14) coincides with the one computed by Polyakov and Wiegman[11] in their solution of the nonlinear σ model. However, their effective Lagrangian has no $F_{\mu\nu}{}^2$ term and hence bosons remain massless. Similarly to the σ-model case, the solution of the chiral Gross-Neveu model leads, in our approach, to a theory of fermions interacting with an effective vector field (see Ref. 19) and after decoupling the bosons remain massless.

In summary, we have been able to develop a path-integral approach to bosonization which parallels, in the Abelian case, the operator prescription established by Coleman[4] and Madelstam.[5] For the massive Thirring model we reobtained Coleman's results in a very simple way. It is interesting to note that it is the contribution of the chiral Jacobian that makes our results meaningful: were the Jacobian absent the theory would become free, since in that case $\beta^2/4\pi=1$ [see Eq. (2.37)]. Our approach reveals then the importance of the chiral anomaly in two-dimensional bosonization, making then apparent the role of topology, usually hidden in the operator procedure.

It is important to stress that the decoupling change of variables which allowed us to establish the equivalence between fermion and boson models in the Abelian case showed how the non-Abelian extension has to be performed. Indeed, in analogy with the Abelian case we were able to decouple fermions from bosons by making a non-Abelian chiral transformation

$$\psi=e^{\gamma_5\phi^a t^a}\chi\,,\qquad \bar\psi=\bar\chi e^{\gamma_5\phi^a t^a}\,. \tag{4.21}$$

The decoupling was possible for the case we analyzed (massless QCD_2) due to the fact that the gauge field can be written as in Eq. (4.5). This in terms of complexified variables $X_\pm$ takes the form

$$A_+=-2i(\partial_-U_5)U_5{}^{-1}\,. \tag{4.22}$$

As we stressed before this equation is the path-integral version of the usual identification allowing to write the fermionic current in terms of boson fields. It makes natural the introduction of currents J_+ and J_- used by Polyakov and Wiegman[11] and Witten.[12] Again the role of the anomaly becomes apparent in the non-Abelian case: it originates the Wess-Zumino term [Eq. (4.14)], establishing a link with other two-dimensional models. In particular, by using the non-Abelian analog of the Gaussian identity (2.6) one can easily extend our treatment to the SU(N) Thirring model. One has to introduce a non-Abelian vector field

$$\exp\left[\frac{g^2}{2}\int d^2x(\bar\psi\gamma_\mu\lambda^a\psi)^2\right]=\int\mathscr{D}A^a_\mu\exp\left[-\int d^2x(\tfrac{1}{2}A^a_\mu A^a_\mu-g\bar\psi A^a\lambda^a\psi)\right],$$

where λ^a are the SU(N) generators and the fermions are taken in the fundamental representation of SU(N). One can then decouple the fermions from the vector field A_μ by making a non-Abelian chiral transformation. The corresponding Jacobian originates also in this case a Wess-Zumino term.

Due to the fact that the SU(N) chiral Gross-Neveu interaction Lagrangian

$$\mathscr{L}=-\frac{g^2}{4N}[(\bar\psi\psi)^2-(\bar\psi\gamma_5\psi)^2]$$

can be written using a Fierz-type transformation in the form

$$\mathscr{L}_{\text{int}}=-\frac{g^2}{2N}(\bar\psi\gamma_\mu t^a\psi)^2$$

[with t^a being the generators of U(N)] this model can also be solved using our method.

All relevant features of the Abelian models discussed in Secs. II and III can be inferred very simply from our treatment. Concerning the non-Abelian model (QCD_2) we were able to show that scalars become massive and self-interacting due in particular to the Wess-Zumino term. From their decoupling one can easily show that at short distances fermions become free. The long-distance behavior of fermion Green's functions as well as the analysis of bosonic correlation functions necessitates a more thorough investigation of the complete effective Lagrangian (4.16). Work on this aspect will be reported elsewhere.

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Note added in proof. After this paper was submitted for publication we discovered the paper of D. Gonzales and A. N. Redlich, Phys. Lett. **147B**, 150 (1984) where similar results are presented.

APPENDIX

We shall now compute, for the Abelian case, the Jacobian J_F associated with the fermion change of variables

$$\begin{aligned} \psi(x) &= U_5(x)\chi(x) , \\ \bar{\psi}(x) &= \bar{\chi}(x)U_5(x) , \\ U_5(x) &= e^{\gamma_5\phi(x)} . \end{aligned} \tag{A1}$$

Let us now consider the normalized eigenvectors of the Hermitian operator $\not{D} = i\not\partial + e\not{A}$:

$$\not{D}\varphi_n(x) = \lambda_n\varphi_n(x) . \tag{A2}$$

The classical fields $\psi(x)$ and $\bar{\psi}(x)$ can be expanded as

$$\begin{aligned} \psi(x) &= \sum_n a_n\varphi_n(x) , \\ \bar{\psi}(x) &= \sum_n \varphi_n^\dagger(x)\bar{b}_n , \end{aligned} \tag{A3}$$

where a_n and $\bar{b}_n$ are elements of a Grassmann algebra. The fermionic part of the functional-integral measure is then defined by

$$\mathscr{D}\bar{\psi}\,\mathscr{D}\psi = \prod_{n,m} d\bar{b}_n da_m . \tag{A4}$$

The new fields $\chi(x)$ and $\bar{\chi}(x)$ can also be expanded in terms of the φ_n's,

$$\chi(x) = \sum_n a_n'\varphi_n(x) , \tag{A5}$$

where the coefficients of the expansions (A3) and (A5) are related by

$$a_m' = \sum_n c_{mn}a_n \tag{A6}$$

with

$$c_{mn} = \langle \varphi_m \mid U_5{}^{-1} \mid \varphi_n \rangle . \tag{A7}$$

One then easily obtains

$$J_F = (\det c_{mn})^2 . \tag{A8}$$

In order to compute the determinant of the c matrix defined in Eq. (A8) we shall consider the one-parameter-dependent γ_5 transformation

$$\begin{aligned} \psi &= U_5(\alpha)\chi , \\ \bar{\psi} &= \bar{\chi}U_5(\alpha) , \end{aligned} \tag{A9}$$

where $U_5(\alpha) = e^{\alpha\gamma_5\phi}$ and α is a real parameter to be varied from 0 to 1, allowing us to build up the whole transformation (A1) by iteration of the infinitesimal one,

$$1 + \gamma_5\phi(x)\delta\alpha . \tag{A10}$$

In order to follow Fujikawa's procedure[21] at each state of the transformation one must take the eigenvectors which correspond to that stage:

$$\tilde{\not{D}}(\alpha)\tilde{\varphi}_n(x,\alpha) = \tilde{\lambda}_n(\alpha)\tilde{\varphi}_n(x,\alpha) , \tag{A11}$$

where

$$\not{D}(\alpha) = U_5(\alpha)\not{D}U_5(\alpha) = i\not\partial + e\tilde{\not{A}}(\alpha) \tag{A12}$$

with

$$\tilde{\not{A}}(\alpha) = U_5(\alpha)\not{A}\,U_5(\alpha) + \frac{i}{e}U_5(\alpha)\not\partial U_5(\alpha) . \tag{A13}$$

This α-dependent field satisfies

$$\begin{aligned} \tilde{A}(\alpha)\Big|_{\alpha=0} &= \not{A} , \\ \tilde{\not{A}}(\alpha)\Big|_{\alpha=1} &= \not{A}' , \\ \frac{\partial\tilde{\not{A}}(\alpha)}{\partial\alpha} &= -\gamma_5\tilde{\not{D}}(\alpha)U_5(\alpha) . \end{aligned} \tag{A14}$$

Let us now consider the matrix

$$B_{pn}(\beta;\alpha) = \langle \tilde{\varphi}_p(x,\beta+\alpha) \mid U_5{}^{-1}(\alpha) \mid \tilde{\varphi}_n(x,\beta) \rangle \tag{A15}$$

which satisfies

$$B(\beta;\alpha+\delta) = B(\beta+\alpha;\delta)B(\beta;\alpha) . \tag{A16}$$

Taking $\beta=0$ and $\delta=\delta\alpha$, we obtain

$$\frac{d}{d\alpha}[\ln\det B(0,\alpha)] = \frac{\ln\det[B(\alpha,\delta\alpha)]}{\delta\alpha} . \tag{A17}$$

Using the relation

$$|\tilde{\varphi}_n(\alpha+\delta\alpha)\rangle \simeq |\tilde{\varphi}_n(\alpha)\rangle + \sum_p b_{np}\,|\tilde{\varphi}_p(\alpha)\rangle\delta\alpha , \tag{A18}$$

where $b_{np}=0$ if $n=p$, we can rewrite (A17) in the form

$$\frac{d}{d\alpha}\ln\det B(0,\alpha) = -\sum_n \langle \tilde{\varphi}_n(\alpha \mid \gamma_5\phi \mid \tilde{\varphi}_n(\alpha)\rangle = -w(\alpha) . \tag{A19}$$

Taking into account that

$$\begin{aligned} B_{nm}(0,0) &= \langle \varphi_n \mid \varphi_m \rangle = \delta_{nm} , \\ B_{nm}(0,1) &= \langle \varphi_n{}^1 \mid U_5{}^{-1} \mid \varphi_m \rangle , \end{aligned} \tag{A20}$$

we can easily integrate Eq. (A19) obtaining

$$\det\langle \varphi_p^1 \mid U_5{}^{-1} \mid \varphi_m \rangle = e^{-\int_0^1 d\alpha\, w(\alpha)} . \tag{A21}$$

We can write from Eq. (A7)

$$\det c_{mn} = \det\langle \varphi_m \mid \varphi_p{}^1 \rangle \det\langle \varphi_p{}^1 \mid U_5{}^{-1} \mid \varphi_n \rangle . \tag{A22}$$

One now uses Eq. (A18) to show that

$$\det\langle \varphi_m \mid \varphi_p^1 \rangle = 1 . \tag{A23}$$

Inserting Eq. (A21) in Eq. (A8) we obtain

$$J_F=e^{-2\int_0^1 w(\alpha)d\alpha}\ , \tag{A24}$$

where

$$w(\alpha)=\int d^2x\,\phi(x)\sum_n \tilde{\varphi}_n^\dagger(x,\alpha)\gamma_5\tilde{\varphi}_n(x,\alpha)\ . \tag{A25}$$

The summation appearing in this integrand is an ill-defined quantity and we evaluate it by using a well-known regularization procedure

$$\sum_n \tilde{\varphi}_n^\dagger(x,\alpha)\gamma_5\tilde{\varphi}_n(x,\alpha) = \lim_{M\to\infty}\sum_n \tilde{\varphi}_n^\dagger(x,\alpha)\gamma_5 e^{-\tilde{\not{D}}^2(\alpha)/M^2}\tilde{\varphi}_n(x,\alpha) \tag{A26}$$

getting

$$w(\alpha)=-\frac{ie}{8\pi}\int d^2x\ \mathrm{tr}[\gamma_5\gamma_\mu\gamma_\nu\phi(x)\tilde{F}_{\mu\nu}(x,\alpha)]\ , \tag{A27}$$

where

$$\tilde{F}_{\mu\nu}(x,\alpha)=\partial_\mu\tilde{A}_\nu(x,\alpha)-\partial_\nu\tilde{A}_\mu(x,\alpha)\ . \tag{A28}$$

Now one can use γ properties and the antisymmetry of $F_{\mu\nu}$ in order to write

$$w(\alpha)=\frac{e}{2\pi}\int d^2x\,(1-\alpha)A_\nu(x)\epsilon_{\mu\nu}\partial_\mu\phi(x)\ , \tag{A29}$$

where we have used Eq. (A13), for the Abelian case, and neglected a surface term. The fermion Jacobian then reads

$$J_F=\exp\left[-\frac{e}{\pi}\int d^2x\int_0^1 d\alpha(1-\alpha)A_\nu(x)\epsilon_{\mu\nu}\partial_\mu\phi(x)\right]\ . \tag{A30}$$

It is straightforward to integrate in α, and for Abelian vector fields decomposed in transverse and longitudinal parts,

$$A_\nu=-\frac{1}{e}\epsilon_{\nu\mu}\partial_\mu\phi+\frac{a}{e}\partial_\nu\eta\ , \tag{A31}$$

we finally obtain

$$J_F=e^{-(1/2\pi)\int d^2x(\partial_\mu\phi)^2}\ . \tag{A32}$$

The non-Abelian case can be treated in a completely analogous way (see Refs. 18 and 19).

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