Yang-Mills theories in the light-cone gauge

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(Received 27 August 1984)

We develop the Hamiltonian quantization of Yang-Mills theories in the light-cone gauge, obtaining the well-defined prescription for the gluon propagator, previously proposed in the literature. A Hilbert space with indefinite metric emerges in which the role of the residual gauge freedom is clarified. It is possible to define consistently a subspace with positive-semidefinite inner product where Gauss's law and Poincaré covariance are recovered and the perturbative S matrix is unitary.

I. INTRODUCTION

Algebraic gauges, characterized by a constant vector n_μ, have been used quite often in the last ten years in spite of their lack of manifest Lorentz covariance. The main motivation of such a choice is the triviality of the Faddeev-Popov determinant as well as the possibility of interpreting the quanta of the vector field as partons[1] in theories without spontaneous symmetry breaking.

A common feature of these gauges is the presence of "spurious" singularities in the boson Feynman propagator and thereby in all the perturbative Green's functions of the theory.[2] These singularities are related to the residual gauge freedom, still present after the algebraic-gauge condition is imposed.

In the spacelike case ($n^2<0$) we have shown[3] that this residual gauge is directly connected to the spatial asymptotic behavior of the potentials and can be consistently eliminated by imposing a specific boundary condition; as a consequence the Dirac procedure can be performed to accommodate redundant degrees of freedom and, moreover, it turns out that the components of the vector potentials cannot be assumed to vanish at spatial infinity,[4] preventing Euclidean compactification. Any choice of the asymptotic behavior for the potentials gives rise to a specific prescription for the spurious singularity in the free boson propagator; in particular, the principal-value prescription[2] is very convenient to set up renormalized Green's functions.[5]

In the present paper we want to discuss the light-cone choice $n^2=0$. As is known, this gauge turns out to be useful in several applications, the latest one being in connection with the quantization of the supersymmetric Yang-Mills theories.[6,7]

Several approaches to quantization have appeared in the literature, most of them dealing with light-cone variables.[8,9] None of them has been able to *derive* in a consistent way the expression for the boson propagator. In addition the meaning and the treatment of the residual gauge freedom have never been clearly understood. In particular the use of Dirac brackets requires inversion of the differential operator $n\cdot\partial$, but, as known from the spacelike case,[3] this point is quite subtle if only physical degrees of freedom are present.

We shall show that the residual gauge freedom in this case has a quite different meaning; in particular it cannot (and does not have to) be eliminated, at variance with the spacelike case, by means of boundary conditions.

As a matter of fact the residual gauge freedom manifests itself at the quantum level in the presence of a "ghost" field, propagating along a generating line of the light cone. This picture, which directly follows from a Hamiltonian canonical quantization, will lead unavoidably to the well-defined prescription in handling the spurious singularity that was proposed in Refs. 7 and 10.

The space of states emerging from this treatment is an indefinite-metric Hilbert space. It is possible to select a physical subspace, stable under the Poincaré-group generators, and with a positive-semidefinite metric, by imposing in it the vanishing of the Gauss operator.

In Sec. II we develop the classical Hamiltonian formalism; Sec. III is devoted to a detailed analysis of the quantization of the radiation fields, leading to a well-defined expression for the free boson propagator.

In Sec. IV we discuss the interacting case and prove the equivalence with a Lagrangian path-integral formulation, which, however, is unable by itself to provide a prescription for the spurious singularity.

Final comments are given in the conclusions (Sec. V), while some technical developments are reported in the Appendix.

II. HAMILTONIAN FORMULATION

We start from the Lagrangian density

$$\mathscr{L}=-\tfrac{1}{4}G^a_{\mu\nu}G^{a,\mu\nu}-\bar\psi(-i\partial\!\!\!/+m)\psi+g\bar\psi A\!\!\!/^a\tau^a\psi-\lambda^a n^\mu A^a_\mu\ , \tag{2.1}$$

where

$$G^a_{\mu\nu}=\partial_\mu A^a_\nu-\partial_\nu A^a_\mu+gf^{abc}A^b_\mu A^c_\nu\ , \tag{2.2}$$

τ^a is a basis in the fundamental representation of the Lie algebra of the internal-symmetry group, λ^a are Lagrange multipliers, and n_μ is the lightlike vector (1, 0, 0, 1). Our metric tensor $g_{\mu\nu}$ is $(1,\ -1,\ -1,\ -1)$. The canonical momenta are

$$\pi_0^a=0\ ,$$
$$\pi_{,i}^a=G_{0i}^a,\quad i=1,2,3\ ,$$
$$\pi_\psi=i\bar{\psi}\gamma^0\ , \tag{2.3}$$
$$\pi_{\bar{\psi}}=0\ ,$$
$$\pi_\lambda^a=0\ ,$$

and therefore the canonical Hamiltonian density becomes, after an integration by parts,

$$\mathscr{H}=\frac{1}{2}(G_{0i}^aG_{0i}^a+\tfrac{1}{2}G_{jk}^aG_{jk}^a)$$
$$-A_0^a(D_i^{ab}G_{0i}^b+g\bar{\psi}\gamma_0\tau^a\psi)+gA_i^a\bar{\psi}\gamma_i\tau^a\psi$$
$$+\bar{\psi}(i\gamma_k\partial_k+m)\psi+\lambda^a n^\mu A_\mu^a\ , \tag{2.4}$$

where

$$D_i^{ab}=\delta^{ab}\partial_i+gf^{acb}A_i^c\ . \tag{2.5}$$

Following the standard Dirac procedure,[9,11,12] we get the equations of motion

$$D_\nu^{ab}G^{i\nu,b}=-n^i\lambda^a+g\bar{\psi}\tau^a\gamma^i\psi\ , \tag{2.6}$$
$$(-i\nabla\!\!\!\!/+m)\psi=0\ , \tag{2.7}$$

where

$$\nabla_\mu=\partial_\mu-ig\tau^aA_\mu^a\ , \tag{2.8}$$

and the set of primary and secondary constraints

$$\pi_0^a=0,\quad \pi_\lambda^a=0\ , \tag{2.9}$$
$$D_i^{ab}G_{0i}^b+g\bar{\psi}\tau^a\gamma_0\psi=\lambda^a\ , \tag{2.10}$$
$$A_0^a=A_3^a\ . \tag{2.11}$$

This set is clearly second class and therefore the Dirac brackets can be defined in the usual way. At this stage we remark that we have a dynamical system with a redundant number of canonical degrees of freedom, namely, A_i^a and G_{0i}^a (besides the fermion ones). In particular we notice that Gauss's law [i.e., the vanishing of the left-hand side of Eq. (2.10)] is not satisfied.

In the quantum treatment Gauss's law will be restored by selecting a suitable subspace of the Hilbert space. In this "physical" subspace the Poincaré covariance is recovered (see the Appendix).

From Eqs. (2.6), (2.7), (2.10), and (2.11) we derive the equation for the Lagrange multipliers

$$n\cdot\partial\lambda^a=0\ , \tag{2.12}$$

which does not contain the interaction even in the non-Abelian case, at variance with the equation for $\partial^\mu A_\mu^a$ in the covariant gauge.

From Eqs. (2.6) and (2.10) we also get

$$D_\nu^{ab}n\cdot\partial A^{b,\nu}=g\bar{\psi}\tau^a n\!\!\!/\psi\ , \tag{2.13}$$

which, in the "free" case ($g=0$), reduces to

$$n\cdot\partial\partial\cdot U^a=0\ , \tag{2.14}$$

U_μ^a being the "radiation" potentials, related to the radiation fields by

$$F_{\mu\nu}^a=\partial_\mu U_\nu^a-\partial_\nu U_\mu^a\ . \tag{2.15}$$

From Eq. (2.11) it is immediate to realize that our system possesses classically a residual gauge freedom involving functions χ^a which do not depend on the variable $x^-=x_+$. We keep this freedom which entails the use of a redundant number of canonical variables; we find indeed canonical equal-time commutation relations between the coordinates A_i^a and their conjugate momenta G_{0i}^a,

$$[A_i^a(t,\mathbf{x}),G_{oj}^b(t,\mathbf{y})]_-=i\delta^{ab}\delta_{ij}\delta^{(3)}(\mathbf{x}-\mathbf{y})\ . \tag{2.16}$$

Fermionic degrees of freedom are treated in the standard way.

III. THE FREE-FIELD ALGEBRA

In this section we are concerned with radiation fields ($g=0$). It is convenient to perform a four-dimensional Fourier transform of the equations of motion and of the constraints,

$$\begin{aligned}&(k^\mu k^\nu-g^{\mu\nu}k^2)U_\nu^a(k)=n^\mu\Lambda^a(k),\\&n^\mu U_\mu^a=0\ ,\end{aligned} \tag{3.1}$$

where

$$U_\mu^a(x)=\frac{1}{(2\pi)^{3/2}}\int d^4k\, e^{ikx}U_\mu^a(k)\ , \tag{3.2}$$

and Λ^a are the Lagrange multipliers in the present case. The Fourier transforms of Eqs. (2.12) and (2.14) are

$$n\cdot k\Lambda^a(k)=0\ , \tag{3.3}$$
$$n\cdot kk\cdot U^a(k)=0\ . \tag{3.4}$$

Equations (3.3) and (3.4) can easily be solved in terms of two sets of independent distributions,

$$\Lambda^a(k)=\Lambda^a(k_3,\mathbf{k})\delta(n\cdot k)=\Lambda^a(\mathbf{k})\delta(n\cdot k)\ , \tag{3.5}$$
$$ikU^a(k)=U^a(k_3,\mathbf{k})\delta(n\cdot k)=U^a(\mathbf{k})\delta(n\cdot k)\ , \tag{3.6}$$

the i being introduced for convenience. Then we can solve Eq. (3.1) and get

$$U_\mu^a(k)=T_\mu^a(k)\delta(k^2)+n_\mu\frac{\delta(n\cdot k)}{k_\perp{}^2}\Lambda^a(\mathbf{k})$$
$$+\frac{ik_\mu}{k_\perp{}^2}U^a(\mathbf{k})\delta(n\cdot k)\ , \tag{3.7}$$

where $k_\perp{}^2=k_1{}^2+k_2{}^2$ and

$$n_\mu T^{a,\mu}=k^\mu T_\mu^a=0\ . \tag{3.8}$$

We introduce the vectors $\epsilon_\mu^{(1)}=(0,1,0,0)$ and $\epsilon_\mu^{(2)}=(0,0,1,0)$. Then the polarization vectors

$$e_\mu^{(\alpha)}(\mathbf{k})=\left[g_\mu^\nu-\frac{n_\mu k^\nu+k_\mu n^\nu}{n\cdot k}\right]\epsilon_\nu^{(\alpha)}\equiv d_\mu^\nu\epsilon_\nu^{(\alpha)}\,,$$
$$k^2=k_0{}^2-k_3{}^2-k_\perp{}^2=k_0{}^2-\mathbf{k}^2=0,\quad \alpha=1,2, \tag{3.9}$$

are orthogonal to n_μ and k_μ and satisfy the normalization condition

$$e_\mu^{(\alpha)}e^{\mu(\beta)}=-\delta^{\alpha\beta}\,. \tag{3.10}$$

We notice that, on shell ($k^2=0$),

$$\frac{1}{n\cdot k}=\frac{1}{k_0-k_3}\overset{(k^2=0)}{=}\frac{k_0+k_3}{k_\perp{}^2}\,. \tag{3.11}$$

The potentials T_μ^a can be expanded on the basis $e_\mu^{(\alpha)}$,

$$T_\mu^{a\pm}(k)=\sum_{\alpha=1}^{2}e_\mu^{(\alpha)}(\mathbf{k})t_\alpha^{a\pm}(\mathbf{k})\,, \tag{3.12}$$

and the frequencies can be quantized in the usual way. For dimensional reasons it is useful to define the quantities

$$g^a(\mathbf{k})=k_\perp{}^{-3/2}\Lambda(\mathbf{k})\,,$$
$$f^a(\mathbf{k})=k_\perp{}^{-1/2}U^a(\mathbf{k})\,, \tag{3.13}$$

which satisfy the conjugation properties

$$[g^a(\mathbf{k})]^*=g^a(-\mathbf{k})\,,$$
$$[f^a(\mathbf{k})]^*=f^a(-\mathbf{k})\,. \tag{3.14}$$

They can be decomposed as

$$g^a(\mathbf{k})=\theta(k_3)g_+^a(\mathbf{k})+\theta(-k_3)g_-^a(-\mathbf{k})\,, \tag{3.15}$$

$$f^a(\mathbf{k})=i\theta(k_3)f_+^a(\mathbf{k})-i\theta(-k_3)f_-^a(-\mathbf{k})\,, \tag{3.16}$$

into components $g_\pm^a$ and $f_\pm^a$ having support in the half-space $k_3>0$ with the conjugation properties

$$[g_+^a(\mathbf{k})]^*=g_-^a(\mathbf{k})\,,$$
$$[f_+^a(\mathbf{k})]^*=f_-^a(\mathbf{k})\,. \tag{3.17}$$

Actually the decompositions (3.15) and (3.16) are rather to be thought of as decompositions *in frequencies* owing to the support condition $k_0=k_3$, and this will have far-reaching consequences in constructing the Hilbert space of our system.

The equal-time commutation relations (2.16) entail the commutators

$$[\partial^\mu U_\mu^a(t,\mathbf{x}),\partial_i F_{0i}^b(t,\mathbf{y})]_-=i\delta^{ab}\partial_\perp{}^2\delta^{(3)}(\mathbf{x}-\mathbf{y})\,, \tag{3.18}$$

which is in turn realized by imposing on $g_\pm^a$ and $f_\pm^a$ the algebra

$$[g_\mp^a(\mathbf{k}),f_\pm^b(\mathbf{k}')]=\pm\delta^{ab}\delta^{(3)}(\mathbf{k}-\mathbf{k}')\,, \tag{3.19}$$

all the other commutators vanishing. We define a vacuum state $|0\rangle$ as a state annihilated by all the negative-frequency operators.

The "free" propagator is

$$D_{\mu\nu}^{ab}(x-y)=\langle 0\,|\,T[U_\mu^a(x)U_\nu^b(y)]\,|\,0\rangle\,, \tag{3.20}$$

T being, as usual, the Dyson operator. According to the decomposition (3.7) we get

$$D_{\mu\nu}^{ab}(x-y)=\langle 0\,|\,T[T_\mu^a(x)T_\nu^b(y)]\,|\,0\rangle+\langle 0\,|\,T[\Gamma_\mu^a(x)\Gamma_\nu^b(y)]\,|\,0\rangle\,, \tag{3.21}$$

with $\Gamma_\mu^a(x)$ corresponding to the second and third terms in Eq. (3.7). Cross terms vanish owing to the algebra (3.19). The expansion (3.12) leads to

$$\langle 0\,|\,T[T_\mu^a(x)T_\nu^b(0)]\,|\,0\rangle=\frac{\sigma^{ab}}{(2\pi)^3}\int\frac{d^3k}{2\,|\,\mathbf{k}\,|}\sum_{\alpha=1}^{2}e_\mu^{(\alpha)}(\mathbf{k})e_\nu^{(\alpha)}(\mathbf{k})\left[\theta(x_0)e^{-i\,|\mathbf{k}|\,x_0-i\mathbf{k}\cdot\mathbf{x}}+\theta(-x_0)e^{i\,|\mathbf{k}|\,x_0-i\mathbf{k}\cdot\mathbf{x}}\right] \tag{3.22}$$

which represents the propagation of the transverse degrees of freedom. One can easily check that

$$\sum_{\alpha=1}^{2}e_\mu^{(\alpha)}(\mathbf{k})e_\nu^{(\alpha)}(\mathbf{k})=-g_{\mu\nu}+\frac{n_\mu k_\nu+n_\nu k_\mu}{k_\perp{}^2}(k_0+k_3) \tag{3.23}$$

with $k_0=|\,\mathbf{k}\,|$. We notice that, owing to the singularity $k_\perp{}^2$ in Eq. (3.23), the distribution in Eq. (3.22) has to be regularized, e.g., by means of dimensional regularization.

For the second term in Eq. (3.21) the algebra (3.19) gives

$$\langle 0\,|\,T[\Gamma_\mu^a(x)\Gamma_\nu^b(0)]\,|\,0\rangle=-i\frac{\delta^{ab}}{(2\pi)^4}\int\frac{d^4k\,e^{ikx}}{k_0-k_3+i\epsilon\,\mathrm{sign}k_3}\left[\frac{n_\mu k_\nu+n_\nu k_\mu}{k_\perp{}^2}\right]_{k_0=k_3}, \tag{3.24}$$

ϵ being, as usual, a positive quantity which eventually will tend to zero.

Again this distribution in four dimensions is ill defined owing to the $k_\perp{}^2$ singularity. Equation (3.22) can now be set in the form

$$\langle 0\,|\,T[T_\mu^a(x)T_\nu^b(0)]\,|\,0\rangle=\frac{i\delta^{ab}}{(2\pi)^4}\int\frac{d^4k\,e^{ik\cdot x}}{k^2+i\epsilon}\left[-g_{\mu\nu}+\frac{n_\mu k_\nu+n_\nu k_\mu}{k_\perp{}^2}(k_0+k_3)-\frac{n_\mu g_{\nu0}+n_\nu g_{\mu0}}{k_\perp{}^2}k^2\right], \tag{3.25}$$

whereas the "longitudinal" part becomes

$$\langle 0 \mid T[\Gamma_\mu^a(x)\Gamma_\nu^b(0)] \mid 0\rangle = -\frac{i\delta^{ab}}{(2\pi)^4}\int \frac{d^4k\, e^{ik\cdot x}}{k^2+k_\perp{}^2+i\epsilon}\left[\frac{n_\mu k_\nu + k_\mu n_\nu}{k_\perp{}^2}(k_0+k_3) - \frac{n_\mu g_{\nu 0}+n_\nu g_{\mu 0}}{k_\perp{}^2}(k^2+k_\perp{}^2)\right], \qquad (3.26)$$

corresponding to the propagation of a "ghost" field along a generating line of the light cone, plus a contact term. Adding the two Green's functions together as in Eq. (3.21), we get

$$D_{\mu\nu}^{ab}(x) = \frac{i\delta^{ab}}{(2\pi)^4}\int \frac{d^4k\, e^{ik\cdot x}}{k^2+i\epsilon}\left[-g_{\mu\nu} + \frac{n_\mu k_\nu + n_\nu k_\mu}{n\cdot k}\right], \qquad (3.27)$$

where $1/n\cdot k$ is to be understood as the distribution

$$\frac{1}{n\cdot k} = \frac{k_0+k_3}{k^2+k_\perp{}^2+i\epsilon}\ . \qquad (3.28)$$

We note that Eq. (3.28) holds for our special choice $n_\mu = (1,0,0,1)$; in general, if $n_\mu = (n_0, \mathbf{n})$, $n_0{}^2 = \mathbf{n}^2$, Eq. (3.28) should be replaced by

$$\frac{1}{n\cdot k} = \frac{n_0 k_0 + \mathbf{n}\cdot\mathbf{k}}{(n_0k_0)^2 - (\mathbf{n}\cdot\mathbf{k})^2 + i\epsilon}\ .$$

This prescription was proposed in Refs. 7 and 10. As a matter of fact, the product of distributions in Eq. (3.27) is well defined and can be Wick rotated. In particular no singularity arises at $k_\perp = 0$ in internal lines, where there is no need of infrared dimensional regularization at variance with previous treatments.[13]

It is worthwhile to notice that, from the decomposition in frequencies of the ghost field $\Gamma_\mu^a(x)$,

$$\Gamma_\mu^a(x) = \frac{1}{(2\pi)^{3/2}}\int \frac{d^3k}{k_\perp{}^2}\theta(k_3)\{e^{ik_3x_0 - i\mathbf{k}\cdot\mathbf{x}}[n_\mu k_\perp{}^{3/2} g_+^a(\mathbf{k}) - k_\mu k_\perp{}^{1/2} f_+^a(\mathbf{k})]$$
$$+ e^{-ik_3x_0 + i\mathbf{k}\cdot\mathbf{x}}[n_\mu k_\perp{}^{3/2} g_-^a(\mathbf{k}) - k_\mu k_\perp{}^{1/2} f_-^a(\mathbf{k})]\}\ , \qquad (3.29)$$

where $k_0 = k_3$, the one-ghost state components

$$[n_\mu k_\perp{}^{3/2} g_+^a(\mathbf{k}) - k_\mu k_\perp{}^{1/2} f_+^a(\mathbf{k})] \mid 0\rangle \qquad (3.30)$$

have negative or vanishing norm. As we have anticipated, the Hilbert space of the Fock states has an indefinite metric. We can consistently define a physical subspace $\mathscr{H}_p$ in the radiation case by imposing the condition

$$g_-^a(\mathbf{k}) \mid \Phi\rangle = 0, \quad \forall \mid \Phi\rangle \in \mathscr{H}_p\ . \qquad (3.31)$$

In the Appendix we show that this subspace is stable under the Poincaré algebra and has a positive-semidefinite metric. The Gauss operator is zero in this subspace; in particular its average value vanishes for any localized physical state.

The unitarity sum, when restricted only to the physical states, does exhibit a singularity $k_\perp{}^{-2}$ coming from the polarization sum in Eq. (3.25). The "physical" wave functions have therefore to be chosen in the subspace of $\mathscr{S}$ of functions vanishing at $k_\perp = 0$. This subspace is dense in L_2 in the topology of L_2.

IV. THE INTERACTING CASE

We notice that Eq. (2.12) holds also in the presence of interaction. As a consequence a decomposition in frequencies for λ^a analogous to the one in Eq. (3.15),

$$\lambda^a(\mathbf{k}) = k_\perp{}^{3/2}[\theta(k_3)\gamma_+^a(\mathbf{k}) + \theta(-k_3)\gamma_-^a(-\mathbf{k})] \qquad (4.1)$$

is still possible. However, at variance with the free-field case, the algebra satisfied by the operators $\gamma_\pm^a(\mathbf{k})$ is unknown. Nevertheless, the very existence of a splitting such as in Eq. (4.1) is by itself a sufficient condition which allows in a general way to select a subspace of the Hilbert space where Poincaré covariance of the theory is recovered, as explained in the sequel.

From the behavior of the Lagrangian density (2.1) under infinitesimal Poincaré transformations, taking the equations of motion into account, it is possible to identify as generators a set $\Sigma(t)$ of operators (see the Appendix).

Owing to the noncovariant gauge choice, the equal-time commutation relations of those operators do not realize the usual Poincaré algebra. In particular some of them are not conserved in time. We can remedy this situation by defining physical states as those belonging to a subspace $\mathbf{H}_p$ of the extended Hilbert space $\mathbf{H}$ characterized by the condition

$$\gamma_-^a \mid \Phi\rangle = 0, \quad \forall \mid \Phi\rangle \in \mathbf{H}_p\ . \qquad (4.2)$$

We show in the Appendix that $\mathbf{H}_p$ is stable under the elements of $\Sigma(t)$. As a consequence their restrictions to $\mathbf{H}_p$ are well defined and moreover they obey the usual Poincaré algebra.

Now we can face the problem of the unitarity and covariance of the perturbative S matrix in this gauge; we note, however, that the reasoning given below is of a formal character since, strictly speaking, the perturbative scattering matrix does not exist owing to the well-known infrared problems.

It is worth emphasizing that the very same Eq. (2.12) allows us to select in a simple way perturbative incoming and outgoing physical states. To this aim we introduce the generators

$$\lambda[\alpha]=\int d^3x\,\lambda^a(0,\mathbf{x})\alpha^a(\mathbf{x})\,,$$
$$\Lambda[\alpha]=\int d^3x\,\Lambda^a(0,\mathbf{x})\alpha^a(\mathbf{x})\,, \tag{4.3}$$

α^a being suitable test functions. Owing to Eq. (2.12) we have

$$\exp(iHt)\lambda[\alpha]\exp(-iHt)\equiv\lambda[\alpha_t]=\int d^3x\,\lambda^a(0,\mathbf{x})\alpha^a(x_\perp,x_3-t)\,, \tag{4.4}$$

$$\exp(iH_0t)\Lambda[\alpha]\exp(-iH_0t)\equiv\Lambda[\alpha_t]=\int d^3x\,\Lambda^a(0,\mathbf{x})\alpha^a(x_\perp,x_3-t)\,. \tag{4.5}$$

Now we give a formal argument[14] concerning the commutation of the perturbative S matrix with $\Lambda[\alpha]$. If we set

$$S=\lim_{\substack{t'\to-\infty\\ t''\to+\infty}} e^{iH_0t''}e^{-iH(t''-t')}e^{-iH_0t'} \tag{4.6}$$

a straightforward calculation gives

$$[S,\Lambda[\alpha_t]]=\lim_{\substack{t''\to+\infty\\ t'\to-\infty}}\{e^{iH_0t''}\rho[\alpha_{t-t''}]e^{-iH(t''-t')}e^{-iH_0t'}-e^{iH_0t''}e^{-iH(t''-t')}\rho[\alpha_{t-t'}]e^{-iH_0t'}\}\,, \tag{4.7}$$

where

$$\rho[\alpha]=gf^{abc}\int d^3x\,A_i^b(0,\mathbf{k})G_{0i}^c(0,\mathbf{x})\alpha^a(\mathbf{x})\,. \tag{4.8}$$

If we take matrix elements of Eq. (4.7) between normalizable states, the limit on the right-hand side of Eq. (4.7) should vanish, as those states are supposed to spread out in their time evolution.

Therefore a perturbative physical Fock space $\mathscr{H}_p$ can be consistently defined by the condition

$$\Lambda^{(-)}[\alpha_t]\,|\Phi_p\rangle=0,\quad |\Phi_p\rangle\in\mathscr{H}_p\,, \tag{4.9}$$

which is equivalent to

$$g_-^a(\mathbf{k})\,|\Phi_p\rangle=0\,. \tag{4.10}$$

It is clear *a fortiori* that, following the general method previously outlined and discussed in the Appendix, Poincaré covariance of the physical space $\mathscr{H}_p$ can be proven. [In this case the free-field Lagrangian density has to be used to define generators and symmetrized (or antisymmetrized) products are replaced by normal products.] Moreover it can be explicitly checked that $\mathscr{H}_p$ has a positive-semidefinite inner product. As a consequence unitarity and covariance of the restriction of the perturbative scattering matrix (pseudounitarity of the scattering matrix in the full Hilbert space follows from Hermiticity of the total Hamiltonian) to the physical subspace $\mathscr{H}_p$ is proven, i.e., "ghost" states do not enter in unitarity sums of physical transition amplitudes, and zero-norm physical states do not contribute to physical probabilities.[15] (The subspace $\mathbf{H}_p$ was just introduced in order to show that Poincaré covariance can be recovered in a general way on a purely algebraic basis. It is clear that to establish a precise relation between $\mathbf{H}_p$ and $\mathscr{H}_p$ would entail the nonperturbative solution of the theory and is therefore beyond the scope of this work.)

In the rest of this section we work in the interaction picture. We split the Hamiltonian density of Eq. (2.4) into a free part,

$$\mathscr{H}_0=:\tfrac{1}{2}(F_{0i}^aF_{0i}^a+\tfrac{1}{2}F_{ij}^aF_{ij}^a)-U_3^a\partial_iF_{0i}^a+\bar\psi(i\gamma_k\partial_k+m)\psi:\,, \tag{4.11}$$

and an interaction part,

$$\mathscr{H}_I=:-gU_\mu^a\bar\psi\tau^a\gamma^\mu\psi-gf^{abc}U_0^aU_i^bF_{0i}^c+V[U_j]:\,,\quad U_0^a=U_3^a\,, \tag{4.12}$$

where

$$V[U_j]=\tfrac{1}{2}g(\partial_jU_k^a-\partial_kU_j^a)f^{abc}U_j^bU_k^c+\tfrac{1}{4}g^2f^{abc}f^{apq}U_j^bU_k^cU_j^pU_k^q\,. \tag{4.13}$$

We recall that we have only three independent potentials. Therefore we introduce three external bosonic sources J_i^a together with the fermionic sources η and $\bar\eta$ and set

$$\mathscr{H}_{\rm ext}=-J_i^aU^{a,i}-\bar\eta\psi-\bar\psi\eta=-\mathscr{L}_{\rm ext}\,. \tag{4.14}$$

The generating functional is given by

$$Z[J_i^a,\eta,\bar\eta]=\left\langle 0\left|T\exp\left[-i\int d^4x(\mathscr{H}_I+\mathscr{H}_{\rm ext})\right]\right|0\right\rangle\,. \tag{4.15}$$

A standard calculation leads to the expression

$$Z[J_i^a,\eta,\bar\eta]=\exp\left[-i\int d^4x\left(V[\underline{U}_j]-\tfrac{1}{2}g^2f^{acb}\underline{U}_k^c\underline{U}_0^b f^{apq}\underline{U}_k^p\underline{U}_0^q+gf^{abc}\underline{F}_{0k}^a\underline{U}_k^c\underline{U}_0^b+g\underline{\bar\psi}\gamma^\mu\tau^a\underline{\psi}\underline{U}_\mu^a\right)\right]$$
$$\times\left\langle 0\left|T\exp\left[i\int d^4x(J^{a,i}U_i^a+\bar\psi\eta+\bar\eta\psi)\right]\right|0\right\rangle, \tag{4.16}$$

where

$$\underline{U}_k^a=\frac{\delta}{i\delta J^{a,k}},\quad \underline{U}_0^a=\frac{\delta}{i\delta J^{a,3}}, \tag{4.17a}$$

$$\underline{F}_{0k}^a=\partial_0\frac{\delta}{i\delta J^{a,k}}-\partial_k\frac{\delta}{i\delta J^{a,3}}, \tag{4.17b}$$

$$\underline{\psi}=\frac{\delta}{i\delta\bar\eta},\quad \underline{\bar\psi}=\frac{\delta}{i\delta\eta}. \tag{4.17c}$$

It is known that the boson free functional

$$Z_0=\left\langle 0\left|T\exp\left[i\int d^4x\, J^{a,i}U_i^a\right]\right|0\right\rangle \tag{4.18}$$

is equal to

$$Z_0=\exp\left[\frac{i}{2}\int d^4x\int d^4y\, J_i^a(x)D^{ij}(x-y)J_j^a(y)\right], \tag{4.19}$$

where D^{ij} are the space components of the propagator in Eq. (3.27). An easy calculation shows that

$$Z_0=\mathscr{N}^{-1}\int d[A_i^a]\exp\left[i\int d^4x[A_i^a(x)K_{ij}A_j^a(x)+A_i^aJ^{a,i}]\right], \tag{4.20}$$

where

$$K_{ij}=\tfrac{1}{2}[-\Box\delta_{ij}-\partial_i\partial_j+\partial_0(n_i\partial_j+n_j\partial_i)-n_in_j\Delta]. \tag{4.21}$$

Equation (4.20) can be set in a covariant notation

$$Z_0=\mathscr{N}^{-1}\int d[A_\mu^a]\delta(n\cdot A)\exp\left[i\int d^4x(-\tfrac{1}{4}F_{\mu\nu}^aF^{a,\mu\nu}+A_\mu^aJ^{a,\mu})\right], \tag{4.22}$$

if we make the substitution

$$J_\alpha^a\to J_\alpha^a,\quad J_3^a\to J_0^a-J_3^a. \tag{4.23}$$

Then Eq. (4.16) becomes

$$Z[J_\mu^a,\eta,\bar\eta]=\mathscr{N}^{-1}\int d[A_\mu^a,\lambda^a,\psi,\bar\psi]\exp\left[i\int d^4x(\mathscr{L}+\mathscr{L}_{\rm ext})\right]. \tag{4.24}$$

It is useful to remark that the free generating functional can be explicitly exhibited as a Gaussian functional of the external currents either using external sources coupled only to the three independent degrees of freedom, or by coupling external sources also to the fourth component of the potential and to a Lagrange multiplier associated to the gauge condition (see, for instance, Ref. 2).

V. CONCLUSIONS

In this paper we have given a complete and consistent Hamiltonian treatment of non-Abelian theories in the light-cone gauge. By this method we derive the light-cone prescription (3.28) previously proposed in Refs. 7—10; in particular we show that the prescription (3.28) follows directly from the correct quantization of the theory and an understanding of the structure of the perturbative Hilbert space of the states, which turns out to be of indefinite metric. Gauss's law, Poincaré covariance, and unitarity of the perturbative S matrix are recovered in a subspace with positive-semidefinite metric, i.e., in a weak sense.

The next problem to solve is to perform the renormalization of the theory and derive the generalized Ward identities. Some preliminary work in this direction has already appeared.[16] A direct computation of the divergent bosonic part of the self-energy tensor at the second order (we agree with some technical calculations given in Ref. 10) exhibits terms with a structure different from the ones in the original Lagrangian, thereby preventing a simple multiplicative renormalization. In addition a nonlocal term arises, which, however, is harmless being multiplied by $n_\mu n_\nu$ (Ref. 10).

Then a procedure quite analogous to the one of Ref. 5 in the spacelike planar case seems possible. The problem is actually under investigation and the results will be reported elsewhere.

ACKNOWLEDGMENTS

It is a pleasure to thank A. A. Slavnov for very illuminating discussions. This work has been partially supported by the Ministero della Pubblica Istruzione (Italy).

APPENDIX

We give in this appendix the explicit expressions for the generators of the Poincaré transformations, disregarding the fermionic fields, in order to avoid writing complexities unessential in the context of the sequel. We show as well that those operators leave invariant the physical Hilbert space $\mathbf{H}_p$ defined in (4.4) and that they there realize the Poincaré algebra.

We start from the classical expression for the energy-momentum tensor

$$T^{\nu\mu} \equiv \frac{\delta \mathscr{L}}{\delta \partial_\mu A^a_\sigma} \partial^\nu A^a_\sigma - g^{\mu\nu}\mathscr{L}$$
$$= G^{a,\mu i}G^{a,\nu i} - \tfrac{1}{2}g^{\mu\nu}G^a_{0i}G^a_{0i} + \tfrac{1}{4}g^{\mu\nu}G^a_{ij}G^a_{ij}$$
$$- A^{a,\nu}\lambda^a n^\mu - \partial_\sigma(G^{a,\mu\sigma}A^{a,\nu}) , \qquad \text{(A1)}$$

and the angular momentum tensor

$$\mathscr{M}^{\mu\nu\rho} = x^\rho T^{\nu\mu} - x^\nu T^{\rho\mu} + G^{a,\mu\nu}A^{a,\rho} - G^{a,\mu\rho}A^{a,\nu} , \qquad \text{(A2)}$$

which are obtained in the standard way using the equations of motion and the transformation properties of the potentials under the Poincaré group. As expected we have

$$\partial_\mu T^{\nu\mu} = 0 \qquad \text{(A3)}$$

and

$$\partial_\mu \mathscr{M}^{\mu\nu\rho} = \lambda^a(n^\nu A^{a,\rho} - n^\rho A^{a,\nu}) ; \qquad \text{(A4)}$$

the extra term in the last equation is due to the Lorentz-noninvariant gauge-fixing term in (2.1).

We deduce then, again at a classical level, the generators of the Poincaré transformations, namely,

$$P_0 \equiv \int d^3x \, T_{00}$$
$$= \int d^3x (\tfrac{1}{2}G^a_{0i}G^a_{0i} + \tfrac{1}{4}G^a_{ij}G^a_{ij} - n_j A^a_j D^{ab}_i G^b_{0i}) , \qquad \text{(A5)}$$

$$P_i \equiv \int d^3x \, T_{i0} = \int d^3x \, G^a_{0j}\partial_i A^a_j , \qquad \text{(A6)}$$

$$J_i \equiv -\tfrac{1}{2}\epsilon_{ijk} \int d^3x \, \mathscr{M}^{0jk}$$
$$= \epsilon_{ijk} \int d^3x (x^j G^a_{0h}\partial^k A^a_h + A^{a,k}G^{a,0j}) , \qquad \text{(A7)}$$

ϵ_{ijk} being the completely antisymmetric tensor with indices running from 1 to 3,

$$K_i \equiv \int d^3x \, \mathscr{M}_{0i0}$$
$$= x_0 P_i - \int d^3x \, x_i(\tfrac{1}{2}G^a_{0k}G^a_{0k} + \tfrac{1}{4}G^a_{ij}G^a_{ij} - n_k A^a_k \lambda^a) . \qquad \text{(A8)}$$

Using the appropriate Dirac brackets, for the couples A^a_i, G^a_{0i}, and the identities (2.10) and (2.11), we find

$$\{P^\mu, P^\nu\} = 0 , \qquad \text{(A9a)}$$
$$\{P_0, J_i\} = \epsilon_{ijk} n^j \int d^3x \, \lambda^a A^{a,k} , \qquad \text{(A9b)}$$
$$\{P_i, J_j\} = \epsilon_{ijk} P_k , \qquad \text{(A9c)}$$
$$\{P_0, K_i\} = P_i - \int d^3x \, \lambda^a(\delta_{ij} - n_i n_j)A^a_j , \qquad \text{(A9d)}$$
$$\{P_i, K_j\} = \delta_{ij} P_0 , \qquad \text{(A9e)}$$
$$\{J_i, J_j\} = \epsilon_{ijk} J_k , \qquad \text{(A9f)}$$
$$\{J_i, K_j\} = \epsilon_{ijh} K_h + \epsilon_{irs} n_s \int d^3x \, x_j A^a_r \lambda^a , \qquad \text{(A9g)}$$
$$\{K_i, K_j\} = -\epsilon_{ijh} J_h$$
$$+ \int d^3x [x_i(\delta_{jh} - n_j n_h)$$
$$- x_j(\delta_{ih} - n_i n_h)] A^a_h \lambda^a . \qquad \text{(A9h)}$$

Clearly in the space of the physical classical solutions of the equations of motion characterized by Gauss's law $\lambda^a = 0$, we see that (A9a)–(A9h) realize the Poincaré algebra.

When we pass to the quantum theory, we understand that all the (fermionic) bosonic products are (anti) symmetrized, so that the Dirac brackets in (A9a)–(A9h) can be replaced by operator (anti) commutators.

Let us introduce now [see (4.1)]

$$\lambda^a_\pm(x) = \int \frac{d^3k}{(2\pi)^{3/2}} k_\perp{}^{3/2} \theta(\pm k_3) \gamma^a_\pm(\pm \mathbf{k}) e^{i(k_3 x_0 - \mathbf{k}\cdot\mathbf{x})} , \qquad \text{(A10)}$$

where we have set $k_0 = k_3$ thanks to Eq. (2.12). We have, taking Eq. (2.12) into account,

$$[\lambda^a_\pm(x), P_\mu] = i\partial_\mu \lambda^a_\pm(x) , \qquad \text{(A11a)}$$
$$[\lambda^a_\pm(x), J_i(0)] = i\epsilon_{ijk} x_j \partial_k \lambda^a_\pm(x) , \qquad \text{(A11b)}$$
$$[\lambda^a_\pm(x), K_j(0)] = -i(x_0\partial_j - x_j\partial_0)\lambda^a_\pm(x) + i n_j \lambda^a_\pm(x) . \qquad \text{(A11c)}$$

It follows that all the generators of the Poincaré group are physical, i.e., they map $\mathbf{H}_p$ onto itself, so that their restrictions to $\mathbf{H}_p$ can be defined. Moreover between any couple of physical states $|\Phi\rangle$, $|\Phi'\rangle$ we have

$$\langle \Phi' | \tfrac{1}{2}[\lambda^a(x)A^a_j(x) + A^a_j(x)\lambda^a(x)] | \Phi \rangle$$
$$= \tfrac{1}{2}\langle \Phi' | [\lambda^a_+(x), A^a_j(x)]$$
$$+ [A^a_j(x), \lambda^a_-(x)] | \Phi \rangle$$
$$= n_j \int \frac{d^3q}{(2\pi)^3} \frac{\mathbf{n}\cdot\mathbf{q}}{\mathbf{n}^2} \langle \Phi' | \Phi \rangle , \qquad \text{(A12)}$$

since $n_j = -\delta_{j3}$ and

$$[A^a_j(x), \lambda^b(x_0, \mathbf{y})] = -iD^{ab}_j(x) \int \frac{d^3q}{(2\pi)^3} e^{i\mathbf{q}\cdot(\mathbf{x}-\mathbf{y})} \theta(\pm q_3) .$$

When we consider the matrix elements in the physical subspace of the quantum version of Eqs. (A9), we easily verify that the Poincaré algebra is recovered in $\mathbf{H}_p$ [up to a regularization of the infinite c number appearing in Eq. (A12)].

We conclude remarking that the color charge operators

$$Q^a = \int d^3x\, g f^{abc} A_i^b G_{oi}^c \tag{A13}$$

leave the physical subspace $\mathbf{H}_p$ invariant as we have

$$[Q^a, \lambda_\pm^b(x)] = i f^{abc} \lambda_\pm^c(x) \ . \tag{A14}$$

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