Baker-Campbell-Hausdorff relations and unitarity of SU(2) and SU(1,1) squeeze operators

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For squeeze operators, an alternative to the matrix derivations of Baker-Campbell-Hausdorff relations is presented for the groups SU(2) and SU(1,1). The technique involves the solution of a system of nonlinear, first-order differential equations. By this method, criteria for unitarity of the representations are established, and these apply to both infinite- and to finite-dimensional representations of these groups.

In a recent paper, Fisher, Nieto, and Sandberg[1] offer a proof for the Baker-Campbell-Hausdorff (BCH) relation

$$S(z)=\exp\left[\tfrac{1}{2}(za^\dagger a^\dagger-\bar{z}aa)\right] \tag{1a}$$

$$\begin{aligned}&=\exp\left[\tfrac{1}{2}(e^{i\theta}\tanh r)a^\dagger a^\dagger\right]\\ &\quad\times\exp\left[-2(\ln\cosh r)(\tfrac{1}{2}a^\dagger a+\tfrac{1}{4})\right]\\ &\quad\times\exp\left[-\tfrac{1}{2}(e^{i\theta}\tanh r)aa\right] ,\end{aligned} \tag{1b}$$

where $S(z)$ is the unitary squeeze operator for squeeze states, $z=re^{i\theta}$ is a complex number, and $a^\dagger$ and a are the usual harmonic-oscillator raising and lowering operators. Gilmore[2] gives a similar derivation for real z. These operators arise in diverse areas of physics; in the quantum-optics literature they are referred to as "two-photon operators," whereas in the gravitational-wave-detection literature they are called squeeze operators.[1]

Although the operator (1a) is unitary, it is not clear that (1b) should be unitary, since one of the intermediary steps in the proof involved a nonunitary 2×2 matrix. The reason for the nonunitary character of the finite-dimensional matrix, in this case, is that the operators $L_+=\frac{1}{2}a^\dagger a^\dagger$, $L_-=\frac{1}{2}aa$, and $L_0=\frac{1}{2}a^\dagger a+\frac{1}{4}$ form a realization of the SU(1,1) Lie algebra. It is well known[3] that all unitary representations of SU(1,1) are infinite dimensional. Because of this, the unitary character of (1b) cannot be checked by matrix methods, unlike its compact counterpart,[4] SU(2). The fact that no pathology results from the matrix derivation is perhaps surprising. However, the form of the BCH formulas for SU(1,1) is purely a consequence of the algebraic structure of su(1,1). The question of unitarity is an analytic one. Below we present an interesting alternative development of (1) and other more general BCH formulas for both SU(1,1) and SU(2) to illustrate the technique. Then we derive the necessary and sufficient conditions for the BCH relations on these groups to be unitary.

To begin, let us briefly sketch, in general terms, what we mean by a BCH formula. Let G be a connected Lie group with Lie algebra $\mathscr{G}$ spanned by a set of generators $\{L_j,\ j=1,\dots,n\}$, where n is the dimension of the Lie algebra. The generators satisfy the commutation relations

$$[L_i,L_j]=\sum_{k=1}^{n}C_{ij}^{k}L_k , \tag{2}$$

where the C_{ij}^k are structure constants. We can obtain the group G, or one of its subgroups, by exponentiating elements of the Lie algebra.[4–6] Two common parametrizations which are used are

$$U(\boldsymbol{\alpha})=\exp\left[\sum_{j=1}^{n}\alpha_j L_j\right] , \tag{3}$$

or

$$U_2(\boldsymbol{\beta})=\prod_{j=1}^{n}\exp(\beta_j L_j) , \tag{4}$$

where the parameters α_j or β_j may be real or complex. The parameters $\boldsymbol{\alpha}$ in (3) are called canonical coordinates of the first kind; in (4), the $\boldsymbol{\beta}$ are referred to as canonical coordinates of the second kind. Other noncanonical parametrizations are possible. BCH formulas express analytical and algebraic relationships interrelating these different parametrization schemes of G. For example,[5,6] if X,Y are elements in $\mathscr{G}$, we have a general expression for a BCH formula

$$\begin{aligned}&\exp(X)\exp(Y)\\ &\quad=\exp\{X+Y+\tfrac{1}{2}[X,Y]\\ &\qquad+\tfrac{1}{12}([[X,Y],Y]+[X,[X,Y]])+\cdots\} .\end{aligned} \tag{5}$$

For many groups, the series on the right in Eq. (5) is nonterminating and convergence to a recognizable form is not always obvious. To get around this problem, a technique exploiting finite-dimensional matrix representations of the algebra $\mathscr{G}$ has been used (see, for example, Ref. 3). As has been mentioned in the preceding paragraph, this second method is not always free of potential difficulties. We present a third way of tackling the calculation by illustrating it for SU(2) and SU(1,1).

Let us define the structure of these two Lie algebras according to Barut,[3(a)]

$$[L_-,L_+]=2\epsilon L_0,\ \ [L_0,L_\pm]=\pm L_\pm\ , \tag{6}$$

where $\epsilon=+1$ for su(1,1) and $\epsilon=-1$ for su(2). We have the additional constraint that L_0 is self-adjoint but L_+ and L_- are adjoints, i.e.,

$$(L_-)^\dagger=L_+,\ \ (L_0)^\dagger=L_0\ . \tag{7}$$

Now, we can define a representation of SU(1,1) or SU(2) in the following way:[4]

$$U_1(\lambda)=\exp[\lambda(\tau L_+ -\bar{\tau}L_- +i\alpha L_0)],\ \ U_1(0)=I\ , \tag{8}$$

where λ is a real parameter and I is the identity operator. The parameters (τ,α) are canonical coordinates of the first kind, where τ is complex and α is real. Such a representation is unitary, for it is easy to check that $U_1^\dagger(\lambda)=U_1{}^{-1}(\lambda)$. We can choose a second representation

$$U_2(\lambda)=\exp[p_2(\lambda)L_+]\exp[p_0(\lambda)L_0]\exp[p_1(\lambda)L_-] \tag{9}$$

subject to the constraint $U_2(0)=I$, that is, $p_j(0)=0$, $j=0,1,2$. The parameters (p_1,p_0,p_2) are canonical coordinates of the second kind. Note that the exponentials in (9) may be chosen in other orders as desired.

For what choice of the $p_j(\lambda)$ will $U_1(\lambda)\ =\ U_2(\lambda)$? To determine this we make use of a simple extension of some of the ideas of Wei and Norman[7] as elaborated by Wilcox.[8] Differentiating (8) and (9), and requiring that $U_1(\lambda)=U_2(\lambda)$, we have

$$\begin{aligned}(\tau L_+ -\bar{\tau}L_- +i\alpha_0 L_0)U_1&=(\tau L_+ -\bar{\tau}L_- +i\alpha_0 L_0)U_2\\ &=p_2'L_+e^{p_2L_+}e^{p_0L_0}e^{p_1L_-}\\ &\quad+p_0'e^{p_2L_+}L_0e^{p_0L_0}e^{p_1L_-}\\ &\quad+p_1'e^{p_2L_+}e^{p_0L_0}L_-e^{p_1L_-}\ ,\end{aligned} \tag{10}$$

where primes indicate differentiation with respect to λ. Multiplying from the right by

$$U_2{}^{-1}=\exp(-p_1L_-)\exp(-p_0L_0)\exp(-p_2L_+)$$

we get

$$\begin{aligned}\tau L_+ -\bar{\tau}L_- +i\alpha_0 L_0&=p_2'L_+ +p_0'e^{p_2L_+}L_0e^{-p_2L_+}\\ &\quad+p_1'e^{p_2L_+}e^{p_0L_0}L_-e^{-p_0L_0}e^{-p_2L_+}\ .\end{aligned} \tag{11}$$

From the well-known theorem[5,6]

$$e^ABe^{-A}=B+[A,B]+\frac{1}{2!}[A,[A,B]]+\cdots\ , \tag{12}$$

and the commutation relations (6), we obtain

$$\begin{aligned}\tau L_+ -\bar{\tau}L_- +i\alpha L_0&=\{p_1'e^{-p_0}\}L_- +\{p_0'-2\epsilon p_1'p_2e^{-p_0}\}L_0\\ &\quad+\{p_2'-p_0'p_2+\epsilon p_1'p_2{}^2e^{-p_0}\}L_+\ .\end{aligned} \tag{13}$$

We identify the coefficients of the respective basis elements of the Lie algebra and obtain a system of coupled nonlinear equations,

$$p_1'e^{-p_0}=-\bar{\tau}\ , \tag{14a}$$

$$p_0'-2\epsilon p_2p_1'e^{-p_0}=i\alpha\ , \tag{14b}$$

$$p_2'-p_0'p_2+\epsilon p_2{}^2p_1'e^{-p_0}=\tau\ , \tag{14c}$$

with initial conditions $p_j(0)=0$, $j=0,1,2$. Substituting (14a) into (14b), we obtain

$$p_0'+2\epsilon\bar{\tau}p_2=i\alpha\ . \tag{15}$$

Together, Eqs. (14a), (14c), and (15) imply

$$p_2'-i\alpha p_2+\epsilon\bar{\tau}p_2{}^2=\tau\ , \tag{16}$$

a Riccati equation for $p_2(\lambda)$. If we can solve (16) then $p_0(\lambda)$ will follow from (15) and p_1 from (14a).

To solve (16) we use a method described by Ince.[9] Making the substitutions, first $p_2=y/\epsilon\bar{\tau}$, $y(0)=0$, then $y=u'/u$, $u'(0)=0$, we transform (16) into the second-order, ordinary differential equation,

$$u''-i\alpha u'-\epsilon|\tau|^2u=0\ , \tag{17}$$

with constant coefficients. Subject to the initial conditions this equation has the solution

$$u(\lambda)=Ae^{i\alpha\lambda/2}\left[\cosh(\hat{\sigma}_\epsilon\Delta_\epsilon\lambda)-\frac{i\alpha}{2\hat{\sigma}_\epsilon\Delta_\epsilon}\sinh(\hat{\sigma}_\epsilon\Delta_\epsilon\lambda)\right]\ , \tag{18}$$

where A is a constant of integration and

$$\Delta_\epsilon{}^2=\left|\epsilon|\tau|^2-\frac{\alpha^2}{4}\right|\ , \tag{18a}$$

$$\sigma_\epsilon=\mathrm{sgn}\left[\epsilon|\tau|^2-\frac{\alpha^2}{4}\right]\ ,\ \ \hat{\sigma}_\epsilon=\sqrt{\sigma_\epsilon}\ . \tag{18b}$$

Therefore, we get for $p_2(\lambda)$ the expression

$$p_2(\lambda)=\frac{\tau\sinh(\hat{\sigma}_\epsilon\Delta_\epsilon\lambda)}{\hat{\sigma}_\epsilon\Delta_\epsilon\cosh(\hat{\sigma}_\epsilon\Delta_\epsilon\lambda)-(i\alpha/2)\sinh(\hat{\sigma}_\epsilon\Delta_\epsilon\lambda)}\ . \tag{19}$$

It is important to note, at this juncture, that the solution (19) to the Riccati equation (16) is an interesting example of the nonlinear superposition rule discussed by Anderson, Harnad, and Winternitz.[10] Although our solution was obtained directly, their nonlinear superposition principles for systems of first-order ordinary differential equations will be instrumental in finding BCH relations for higher-dimensional groups such as SL(n,$\mathbb{R}$) and O(p,q).

For notational reasons, it is simpler to treat the two cases SU(2) and SU(1,1) separately. If $\epsilon=-1$ [SU(2)], then $\Delta_-{}^2=\theta^2=|\tau|^2+\alpha^2/4\geq 0$, since $|\tau|$ and α are real, and $\hat{\sigma}_\epsilon=i$. On the other hand, if $\epsilon=+1$ [SU(1,1)], then $\Delta_+{}^2=\mu^2=||\tau|^2-\alpha^2/4|$ and $\hat{\sigma}=\hat{\sigma}_\epsilon=+1$ if $|\tau|^2>\alpha^2/4$ but $\hat{\sigma}=\hat{\sigma}_\epsilon=i$ if $|\tau|^2<\alpha^2/4$. Therefore, Eq. (19) for $p_2(\lambda)$ reduces to the following:

(i) SU(2)

$$p_2(\lambda)=\frac{\tau \sin(\theta\lambda)}{\theta \cos(\theta\lambda)-(i\alpha/2)\sin(\theta\lambda)} , \quad (20a)$$

(ii) SU(1,1)

$$p_2(\lambda)=\frac{\tau \sinh(\hat{\sigma}\mu\lambda)}{\hat{\sigma}\mu \cosh(\hat{\sigma}\mu\lambda)-(i\alpha/2)\sinh(\hat{\sigma}\mu\lambda)} . \quad (20b)$$

Equations (20) still hold when $\mu=\theta=0$ as can be checked by taking the limit in each case.

Substituting (20) into Eq. (15) for p_0' and integrating,[11] we obtain the following expressions:

(i) SU(2)

$$p_0(\lambda)=-2 \ln \left[\cos(\theta\lambda)-\frac{i\alpha}{\theta}\sin(\theta\lambda)\right] , \quad (21a)$$

(ii) SU(1,1)

$$p_0(\lambda)=-2 \ln \left[\cosh(\hat{\sigma}\mu\lambda)-\frac{i\alpha}{2\mu\hat{\sigma}}\sinh(\hat{\sigma}\mu\lambda)\right] . \quad (21b)$$

With the appropriate $p_0(\lambda)$ from (21), we can integrate the differential equation (14a) to get the following:

(i) SU(2)

$$p_1(\lambda)=\frac{-\bar{\tau} \sin(\theta\lambda)}{\theta \cos(\theta\lambda)-(i\alpha/2)\sin(\theta\lambda)} , \quad (22a)$$

(ii) SU(1,1)

$$p_1(\lambda)=\frac{-\bar{\tau} \sinh(\hat{\sigma}\mu\lambda)}{\hat{\sigma}\mu \cosh(\hat{\sigma}\mu\lambda)-(i\alpha/2)\sinh(\hat{\sigma}\mu\lambda)} . \quad (22b)$$

To obtain the BCH formulas, choose $\lambda=1$. We include here some special cases:

(i) SU(2)

$$\exp(\tau L_+ - \bar{\tau} L_-)=\exp\left[\left[\frac{\tau}{|\tau|}\tan|\tau|\right]L_+\right]\exp[-2(\ln\cos|\tau|)L^0]\exp\left[-\left[\frac{\bar{\tau}}{|\tau|}\tan|\tau|\right]L_-\right] , \quad (23a)$$

$$\exp(\tau L_+ - \bar{\tau} L_-)\exp(i\alpha L_0)=\exp\left[\left[\frac{\tau}{|\tau|}\tan|\tau|\right]L_+\right]\exp[(i\alpha-2\ln\cos|\tau|)L_0]\exp\left[-\left[\frac{\bar{\tau}e^{i\alpha}}{|\tau|}\tan|\tau|\right]L_-\right] , \quad (24a)$$

(ii) SU(1,1)

$$\exp(\tau L_+ - \bar{\tau} L_-)=\exp\left[\left[\frac{\tau}{|\tau|}\tanh|\tau|\right]L_+\right]\exp[-2(\ln\cosh|\tau|)L_0]\exp\left[-\left[\frac{\bar{\tau}}{|\tau|}\tanh|\tau|\right]L_-\right] , \quad (23b)$$

$$\exp(\tau L_+ - \bar{\tau} L_-)\exp(i\alpha L_0)=\exp\left[\left[\frac{\tau}{|\tau|}\tanh|\tau|\right]L_+\right]\exp[(i\alpha-2\ln\cosh|\tau|)L_0]\exp\left[-\left[\frac{\bar{\tau}e^{i\alpha}}{|\tau|}\tanh|\tau|\right]L_-\right] . \quad (24b)$$

Note that (23b) is exactly the identity (1) where $\tau=re^{i\theta}$. The BCH formulas for SU(2) can be compared to those of Gilmore,[4] keeping in mind the presence of the imaginary factor i in his exponents.

Finally, we wish to establish the necessary and sufficient conditions that the BCH relation between (8) and (9) in the form of (20), (21), and (22) preserves unitarity. Clearly a necessary condition is that U_1 is unitary since we have $U_2^\dagger=U_1^\dagger=U_1^{-1}=U_2^{-1}$. But is it sufficient? One can ask under what conditions the following will hold:

$$\begin{aligned} U_2^{-1}(\lambda)&=\exp[-p_1(\lambda)L_-]\exp[-p_0(\lambda)L_0]\exp[-p_2(\lambda)L_+] \\ &=\exp[\bar{p}_2(\lambda)L_+]\exp[\bar{p}_0(\lambda)L_0]\exp[\bar{p}_1(\lambda)L_-]=U_2^\dagger(\lambda) . \end{aligned} \quad (25)$$

To answer this let us proceed more generally. What relationship must exist between the parameters $q_j(\lambda)$ and $r_j(\lambda)$, $j=0,1,2$, such that

$$\begin{aligned} \exp[-\lambda(\tau L_+ - \bar{\tau} L_- + i\alpha L_0)]&=\exp[q_2(\lambda)L_+]\exp[q_0(\lambda)L_0]\exp[q_1(\lambda)L_-] \\ &=\exp[r_1(\lambda)L_-]\exp[r_0(\lambda)L_0]\exp[r_2(\lambda)L_+] \end{aligned} \quad (26)$$

subject to the constraint that $q_j(0)=r_j(0)=0$, $j=0,1,2,?$ Repeating the analysis above, we get the systems of nonlinear equations

$$q_1' e^{-q_0} = \bar{\tau} , \tag{27a}$$

$$q_0' - 2\epsilon q_2 q_1' e^{-q_0} = -i\alpha , \tag{27b}$$

$$q_2' - q_0' q_2 + \epsilon q_2{}^2 q_1' e^{-q_0} = -\tau , \tag{27c}$$

and

$$r_2' e^{r_0} = -\tau , \tag{28a}$$

$$r_0' + 2\epsilon r_1 r_2' e^{r_0} = -i\alpha , \tag{28b}$$

$$r_1' + r_0' r_1 + \epsilon r_1{}^2 r_2' e^{r_0} = \bar{\tau} \tag{28c}$$

with associated Riccati equations for q_2,

$$q_2' + i\alpha q_2 - \epsilon\bar{\tau} q_2{}^2 = -\tau , \tag{29}$$

and for r_1,

$$r_1' - i\alpha r_1 + \epsilon\tau r_1{}^2 = \bar{\tau} . \tag{30}$$

Consistency of (29) and (30) requires that

$$r_1' + \bar{q}_2' - i\alpha(r_1 + \bar{q}_2) + \epsilon\tau(r_1{}^2 - \bar{q}_2{}^{-2}) = 0 ,$$

which implies that $r_1 + \bar{q}_2 = 0$, or

$$r_1 = -\bar{q}_2 . \tag{31a}$$

Equations (27a), (27b), (28a), and (28b) imply that

$$\bar{q}_0' - 2\epsilon\tau\bar{q}_2 = i\alpha = -(r_0' - 2\epsilon\tau r_1) .$$

Because of (31a), $r_0' = -\bar{q}_0'$, and so

$$r_0 = -\bar{q}_0 , \tag{31b}$$

where the constant of integration vanishes because of initial conditions. Consistency between (27a) and (28b) along with (31b) implies

$$r_2 = -\bar{q}_1 . \tag{31c}$$

Equations (31) are the requirements then for the second equality to hold in Eq. (26). But these are satisfied identically in (25) and $U_2{}^{-1}(\lambda) = U_2^{\dagger}(\lambda)$ and U_2 is unitary. This result is independent of ϵ, that is, whether the group is SU(1,1) or SU(2) and whether the representations are finite or infinite dimensional.

Upon reflection it is a remarkable fact that, for SU(1,1), the same BCH formula can be computed utilizing any faithful 2×2 matrix realization of its Lie algebra. Although matrix methods may have some computational advantages over solving a system of first-order, nonlinear differential equations, the former leaves unanswered, in certain instances, analytical questions such as unitarity for infinite-dimensional representations. The technique of differential equations is ideally suited for checking such requirements and can be applied to any Lie group. For higher-dimensional groups, the nonlinear superposition principles of Anderson *et al.*[10] will play an important role in writing down the general solution to the system of first-order ordinary differential equations obtained in the analysis.

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