Matter and radiation in equilibrium

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The Rayleigh-Jeans law and the Jüttner (relativistic Maxwell-Boltzmann) distribution are shown to be compatible in equilibrium, to order $\beta^2=(v/c)^2$, for the case of the Einstein-Hopf oscillator. The derivation of the matter distribution from the radiation law is thus consistent with the well-defined formulation of relativistic theories of interacting particles, in this approximation. One may formally define a temperature transformation law (admissible according to general Lorentz transformation requirements) such that the Rayleigh-Jeans law holds; however, a consistent classical relativistic statistical mechanics, and hence relativistic thermodynamics, of interacting particles is presently lacking.

I. INTRODUCTION

The question of the appropriate matter and radiation distributions for systems in equilibrium still appears to be unresolved, despite the early progress in blackbody radiation theory. Thus, for an assumed Maxwell-Boltzmann (nonrelativistic) distribution, the Rayleigh-Jeans radiation law is consistent with it, in equilibrium.[1] On the other hand, the same type of classical electrodynamic calculation with random radiation background does not seem to lead to compatible results when at least one of the systems has a relativistic distribution; that is, the Jüttner distribution (relativistic Maxwell-Boltzmann) and the Rayleigh-Jeans or Planck radiation law do not seem to be able to coexist in equilibrium.[2] Of course, one must bear in mind that such a situation would only arise strictly with (asymptotically) free particles (with and without mass), so the questions might be deferred until a suitable relativistic statistical mechanics of interacting particles is devised.[3] In response to Boyer, Blanco, Pesquera, and Santos[4] argue that for multiperiodic systems in which one considers the random radiation as a perturbation, there is no disagreement between the co-existence of the Rayleigh-Jeans law and the Jüttner distribution in equilibrium. Earlier work[5] has taken exception to the possibility of incompatibility between the Planck distribution and the relativistic particle distribution. Recent papers by Boyer,[6] invoking the equivalence principle, support the idea of a specific acceleration-dependent Planckian distribution in equilibrium with relativistic matter, when zero-point radiation and accelerated frames are involved.

In the present paper we show that to order $(v/c)^2$ there is no disagreement between Rayleigh-Jeans law and the Jüttner distribution. Thus, at least to an approximation for which a relativistic mechanics and relativistic statistical mechanics of interacting particles are well defined,[7–12] the expected result follows. This does not mean that a statistical mechanics of interacting relativistic matter and radiation can or cannot be thought to exist rigorously in equilibrium. Present ambiguity with regard to relativistic transformation of thermodynamic quantities is noted, with particular attention paid to the temperature[13] occurring in the Rayleigh-Jeans law.

II. EQUILIBRIUM FOR MATTER AND RADIATION

The system chosen is that of Einstein and Hopf,[14] who associated a nonrelativistic particle distribution with the Rayleigh-Jeans law; the addition of zero-point energy[15] shows that the Planck radiation law can be made compatible with a nonrelativistic particle model, but such may not be the case for a relativistic particle distribution. However, it is not to be expected that the Rayleigh-Jeans distribution will be exactly compatible with a relativistic particle since equipartition does not hold for relativistic particles.

From the virial[16]

$$\left\langle x_i \frac{\partial H}{\partial x_j}\right\rangle = kT\delta_{ij}, \quad i,j=1,\ldots,2N; i\neq j \; . \tag{1}$$

Thus, already from the assumed form

$$H'=\sum_i (p_i{}^2c^2+m_{0i}{}^2c^4)^{1/2}-\sum_i m_{0i}c^2 \equiv \sum_i E_{k_i} \; , \tag{2}$$

$$\left\langle \frac{p_i{}^2c^2}{(p_i{}^2c^2+m_{0i}{}^2c^4)^{1/2}}\right\rangle = kT \neq 2\langle E_{k_i}\rangle \; . \tag{3}$$

Since the spectral energy density associated with a frequency ω is

$$\rho(\omega)=(\omega^2/\pi^2c^3)\langle E_i\rangle \; , \tag{4}$$

where $\langle E_i\rangle$ is the average energy per normal mode, the Rayleigh-Jeans law does not follow strictly. We shall see later that there is an approximate sense in which agreement can be said to exist.

To see more explicitly the problem posed by the relativistic case, we ask whether a "temperature" can be defined consistently, such that a gas of "free" particles can coexist in equilibrium with radiation. Define a kinetic temperature T_k by

$$T_k=(h\nu/k)[\exp(h\nu/kT)-1]^{-1} \; , \tag{5}$$

where also, for a particle distribution of matter,[17]

$$\tfrac{3}{2}kT_k=3kT+mc^2\frac{K_1(mc^2/kT)}{K_2(mc^2/kT)}-mc^2 \; . \tag{6}$$

The right-hand side of Eq. (6) follows from the analysis of the average energy for a particle in a Jüttner distribution. The K functions are modified Bessel functions. However, in the relativistic regime, Eqs. (5) and (6) lead to inconsistency, although in the nonrelativistic regime $kT \ll mc^2$ and $kT \ll h\nu$, the equipartition limit is defined and the equilibrium temperature T and the kinetic temperature T_k are the same.

The incompatibility of the Jüttner and radiation distributions may have roots in other directions as well. The assumption of a Lorentz-invariant zero-point energy[2] should be compared with the requirement[16,18]

$$n'/\nu'^2 = n/\nu^2 \, , \tag{7}$$

which may be cast in the form

$$\nu'/T' = \nu/T \, . \tag{8}$$

Above, the n, ν, and T are the photon density, frequency, and temperature, respectively, and the primed quantities refer to a transformed frame of reference. Thus the frame-independent isotropy of zero-point radiation should be examined with the temperature form required by (8), and by the (anisotropic) Doppler shift,[18] namely,

$$T'(\theta') = T(1-v^2/c^2)^{1/2}[1-(v/c)\cos\theta']^{-1} \, . \tag{9}$$

While zero-point motion is permitted in Boyer's formulation,[15] observable effects due to velocity-dependent forces are not, since otherwise an anisotropy could be induced at absolute zero. Note that the walls and their effects on isotropy, though ingeniously treated by Boyer, should really have more explicit justification. Also, more general considerations, allowing other relativistic thermodynamic formulations, as derived from statistical mechanics[13,19,20] should be examined here. There will be a discussion of the latter later in the paper.

III. EQUILIBRIUM WITH FOKKER-PLANCK EQUATION

Further, thus, we follow the procedure of Boyer,[2] to see whether consistency may be obtained in the case where distributions are involved in a kinetic equation context. The *convergent* β^2-relativistic kinetic equation which is exact to order in the coupling constant squared, for a Hamiltonian of the form

$$H_{\text{int}} = \sum_{i<j} H_{ij}(\mathbf{p}_i, \mathbf{p}_j, \mathbf{r}_{ij}) \, , \tag{10}$$

has been given in Ref. 21. (For a charged gas the explicit forms for the coefficients in the kinetic equation have been investigated[21] and the divergences typically associated with the long-range Coulomb force and the point-particle limit are cancelled; it is important to recognize that the cutoffs traditionally associated with the Fokker-Planck equation do not give any formal difficulty in this case.)

Starting with the equilibrium condition for the Fokker-Planck equation (see the Appendix)

$$-P(p)F_x(p)\tau + \frac{1}{2}\frac{\partial}{\partial p}[P(p)\langle \Delta^2(p)\rangle] = 0 \, , \tag{11}$$

where[2]

$$F_x(p) = \frac{3}{8}\frac{\Gamma\omega_0^2}{c\gamma}kT\left[\frac{2}{3\beta} + \frac{2}{\beta}\left[1+\frac{1}{\beta^2}\right] - \frac{1}{\beta^2}\left[1+\frac{1}{\beta^2}\right]\ln\left[\frac{1+\beta}{1-\beta}\right]\right] \tag{12}$$

and

$$\langle \Delta^2(p)\rangle = \frac{9}{32}\frac{\Gamma\omega_0^2}{c^2\gamma}\tau(kT)^2\left[-\frac{2}{\beta^2} + \frac{1}{\beta}\left[1+\frac{1}{\beta^2}\right]\ln\left[\frac{1+\beta}{1-\beta}\right]\right]\left[-\frac{2}{3\beta^2} - \frac{2}{\beta^2}\left[1+\frac{1}{\beta^2}\right] + \frac{1}{\beta^3}\left[1+\frac{1}{\beta^2}\right]\ln\left[\frac{1+\beta}{1-\beta}\right]\right] \, . \tag{13}$$

Equation (11) may be written in the form

$$P(p)F_x(p) + \frac{\partial}{\partial p}\left\{\frac{kT}{c}P(p)F_x(p)\left[\frac{mc\gamma}{p} - \frac{F_x(p)}{CkT}\right]\right\} = 0 \, , \tag{14}$$

where $p \equiv mc\gamma\beta$ and C is defined as $\Gamma\omega_0{}^2/c$. The formal solution of (14) is given as

$$P(p)F_x(p)\left[\frac{mc\gamma}{p} - \frac{F_x(p)}{CkT}\right] = K\exp\left\{-\int \frac{c\,dp}{\left[\frac{mc\gamma}{p} - \frac{F_x(p)}{CkT}\right]kT}\right\} \, . \tag{15}$$

In (14) and (15) it is important to note that in fact the distribution is independent of the damping constant.

In the event that

$$|F_x(p)| \ll CkTmc\gamma/p \, , \tag{16}$$

then, ignoring the linear $F_x(p)$ terms in both brackets,

$$P(p) = N\exp\{-mc^2[1+(p/mc)^2]^{1/2}/kT\} \, . \tag{17}$$

Thus the Jüttner result follows under these circumstances.

The inconsistency between the Rayleigh-Jeans law and the Jüttner distribution is, to the extent of the above approximation, resolved. However, to establish this, we must investigate further Eq. (12), in terms of a velocity expansion.

Expanding (12) and retaining terms of order β^2, since relativistic corrections are at least of this order,

$$F_x(p) = -\tfrac{2}{5}CkT(\beta+\tfrac{1}{7}\beta^3+\cdots)$$
$$\cong -\tfrac{2}{5}CkT\beta\ . \qquad (18)$$

In dropping the latter term, Eq. (15) becomes

$$-P(p)\tfrac{2}{5}CkT = K'\exp\left[-\int\frac{p\,dp/kT}{[1+(p/mc)^2]^{1/2}}\right], \qquad (19)$$

which leads to (17). The neglect of terms of order β^3 and retention of terms of order β^2 is consistent with the procedure with relativistic theories, both quantal[7-9] and classical,[10-12] considered elsewhere. Of course, on the right-hand side of (19) one would consistently expand the relativistic kinetic energy, retaining terms of order β^2. [It would not be incorrect *formally* to use (18) to all orders for free particles, but then, (15) will give a $P(p)$ different from the Jüttner distribution. It is important also to note that, while time-delay effects enter in order β^3, Čerenkov poles are of lower order (β^2) and so these radiation effects are subsumed (see the first paper in Ref. 12).]

IV. TEMPERATURE TRANSFORMATION

Finally we ask the question as to whether a redefinition of temperature under Lorentz transformation might circumvent the problem of consistency, in general, between rigorously relativistic distributions for matter and radiation, in equilibrium. We reject the idea of different transformations for matter and radiation, since, although it is possible to do this in a Lorentz-invariant way in each case, the concept of equilibrium (zeroth law) becomes frame-dependent.[13]

For instance, if one sets, from Eq. (12),

$$F'_x(p') = -\frac{2}{5}\frac{\Gamma\omega_0{}^2}{c\gamma}\beta kT'\ , \qquad (20)$$

where

$$T' = -\frac{15}{16}\frac{T}{\beta}\left[\frac{2}{3\beta}+\frac{2}{\beta}\left[1+\frac{1}{\beta^2}\right] - \frac{1}{\beta^2}\left[1+\frac{1}{\beta^2}\right]\ln\left[\frac{1+\beta}{1-\beta}\right]\right], \qquad (21)$$

then

$$F'\to F = -\frac{2}{5}\frac{\Gamma\omega_0^2}{c\gamma}\beta kT \qquad (22)$$

as $\beta\to 0$ [note agreement with Eq. (18)]. Therefore, the Rayleigh-Jeans law emerges in the nonrelativistic limit; Eq. (20) also sustains a generalized Rayleigh-Jeans form.

Equation (21) satisfies the criterion of being an even function of β.[13] Thus it is admissible as a temperature transformation law, although it is possibly an inelegant candidate when one is considering the foundations of a rigorously relativistic statistical mechanics; however, in comparison to the present problem, Marshall, for example, argues for the transformation[22]

$$T' = \frac{T}{2\gamma\beta}\ln\left[\frac{1+\beta}{1-\beta}\right]. \qquad (23)$$

From Eqs. (7) or (8) it is not explicit how the temperature of the relativistic distribution transforms. For example, an assumed isotropy of the cosmic blackbody background radiation would result in T transforming [see Eq. (8)] according to[23,13]

$$T' = T[(1-\beta)/(1+\beta)]^{1/2}\ . \qquad (24)$$

An analogy of the zero-point blackbody radiation with the cosmic blackbody background radiation suggests itself in the present context. On the other hand, the transformation (24) is not even,[13] although neither is the form which determines *anisotropy* in the blackbody radiation, namely, Eq. (9).

If T is assumed to be a Lorentz scalar (and ν is not) then the reason for the disagreement of the Rayleigh-Jeans and Jüttner distributions is clear [from Eq. (8)]. On the other hand, Eq. (20) is even in β and will give the formal appearance of the Rayleigh-Jeans law in every Lorentz frame, but the transformation law is complicated. Of course, the last statement is predicated on the establishment of a consistent relativistic thermodynamics and statistical mechanics of interacting particles.[3,13,19]

V. CONCLUSION

The view is taken that there is still disagreement between radiation and matter distributions in equilibrium, from relativistic analysis of the Einstein-Hopf oscillator. Assuming the equilibrium condition leads to consistency to order β^2, but, for higher orders, this may not be the case. Analyses of multiperiodic systems, as given recently,[4] possibly do not reflect properly the sensitivity of the Lorentz and Lorentz-Dirac[20] equations to initial conditions, since the system may respond in a chaotic way even under the effect of a small stochastic perturbation. In addition, a fundamental basis for the relativistic statistical mechanics of interacting particles has not been given.

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APPENDIX

Assuming a function $R(p')$, which goes to zero sufficiently rapidly as $p\to\pm\infty$, but is otherwise arbitrary, define the integral

$$\int dp' R(p') \frac{\partial P(p\,|\,p',t)}{\partial t} = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \int dp' R(p') [P(p\,|\,p',t+\Delta t) - P(p\,|\,p',t)]$$

$$= \lim_{\Delta t \to 0} \frac{1}{\Delta t} \left[\int dp' R(p') \int dp'' P(p\,|\,p'',t) P(p''\,|\,,p',\Delta t) - \int dp'' R(p'') P(p\,|\,p'',t) \right], \quad \text{(A1)}$$

where the Markoff assumption has been made. Expanding in a Taylor's series about p'', and assuming that only first and second moments are proportional to Δt,

$$\int dp' R(p'') \frac{\partial P}{\partial t} = \int dp'' P(p\,|\,p'',t) [R'(p'') A(p'') + \tfrac{1}{2} R''(p'') B(p'')] \,. \quad \text{(A2)}$$

Partial integration then leads to

$$\int dp' R(p') \left[\frac{\partial P}{\partial t} + \frac{\partial}{\partial p'}(AP) - \frac{1}{2} \frac{\partial^2}{\partial p'^2}(BP) \right] = 0 \,. \quad \text{(A3)}$$

The quantity in brackets vanishes and, in equilibrium, the condition (11) follows. Explicit forms, for a charged gas, have been given for A and B.[12]

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