Restoration of spontaneously broken continuous symmetries in de Sitter spacetime

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We formulate a functional approach to scalar quantum field theory in $(n+1)$-dimensional de Sitter spacetime and solve the functional Schrödinger equation for the conformally and minimally coupled scalar fields in both the $k=0$ and $k=1$ gauges. We show that there is a natural initial condition, the requirement that the field energy remain finite as the scale factor a becomes small, which specifies a unique, time-dependent, de Sitter vacuum state. This initial condition is closely related to Hawking's prescription of including in the functional integral only those field configurations which are regular on the Euclidean section. The Green's functions constructed using this initial condition are explicitly shown to be the analytic continuation of those derived using the Euclidean path-integral formalism and the regularity (boundary) condition. These Green's functions are used to study the Hawking effect and the restoration of continuous symmetries. In particular we study the restoration of a broken O(2) symmetry of a Φ^4 theory. We argue that spontaneously broken continuous symmetries are always dynamically restored in de Sitter spacetime.

I. INTRODUCTION

Quantum field theory in nontrivial backgrounds has served as a particularly useful semiclassical approximation to the quantum theory of gravity. Scalar field theory in de Sitter space is a system that has come under much scrutiny, not only because de Sitter space is a space of high symmetry, and hence exact solutions for the free field theory can be written down, but also because it is a space of constant nonzero curvature, and thus field theory in a de Sitter background is not a trivial rewriting of Minkowski field theory. In this paper we will study scalar quantum field theory in a de Sitter background in some detail. We shall be particularly interested in the vacuum state.

The main purpose of this paper is to clarify issues relevant to scalar field theory in de Sitter spacetime by constructing and studying in some detail the functional Schrödinger equation. A major part of our discussion will be devoted to the question of the correct boundary conditions for quantum fields in de Sitter space. The specification of these boundary conditions is far from trivial.[1] In fact, some authors have suggested that the vacuum of field theory in de Sitter space depends on a parameter whose value is determined by an "extra" requirement.[2] We will show that this is not the case, in both de Sitter and Minkowski space if a sensible boundary/initial condition is used to specify the state. We suggest the following physically quite natural initial condition: In spatially flat coordinates, the wave function should tend to the wave function of the Minkowski vacuum as $t\to-\infty$. Remarkably, this boundary condition is equivalent to the condition of regularity on the Euclidean section introduced by Hawking.[1] Much of our analysis will be devoted to an explicit demonstration of this equivalence.

A recurring theme in attempts to study the quantization of gravity (particularly at the semiclassical level) has been the connection between quantum field theory in certain nontrivial gravitational backgrounds and at finite temperature. This connection is suggested by the periodicity in imaginary time of the Green's function in the gravitational background, or by the relation, usually ascribed to systems in thermal equilibrium, satisfied by the Bogoliubov coefficients between basis states at different times. The archetypical example of this phenomenon is the thermal spectrum of Hawking radiation found when scalar quantum field theory is studied in the background of a Schwarzschild black hole.[3] Field theory in de Sitter space is another system which exhibits similar behavior.[4] Of course, any semiclassical theory—i.e., a quantum field interacting with a classical source—will have inconsistencies, and the ultimate explanation of this effect will probably require some understanding of the quantum theory of gravity. However, its importance should not be understated, as this has led Hawking to suggest that quantum mechanics might need to be modified if we want to quantize gravity.[5,6] As an application of our formalism, we will study some aspects of the Hawking effect in de Sitter space and extend the DeWitt-Unruh construct of a particle detector. Motivated by the analogy between nontrivial backgrounds and finite temperature, we study symmetry restoration in de Sitter space. Surprisingly, we find that spontaneously broken continuous symmetries are dynamically restored, *in any number of spacetime dimensions,* leading us to believe that this analogy may not be quite complete. We will show that this symmetry restoration actually does not depend on the boundary conditions; for this demonstration our Schrödinger-picture formalism is a necessity.

De Sitter space has recently figured prominently in the application of field theory to the early universe.[7] Banks, Fischler, and Susskind[8] have perturbatively solved the Wheeler-DeWitt equation for the inflationary universe. They have found that to the lowest order in which the matter (scalar) field enters the calculation, the wave function of the universe factorizes into a part that describes the gravitational dynamics, and a part that describes the matter dynamics; the matter part is exactly the same as the wave function of a scalar field propagating in a de Sitter background. So, at least to this order, the semiclassical approximation of quantum field theory in a nontrivial background seems to be a good approximation to the complete theory.

In Sec. II, we review the de Sitter solution of the $(n+1)$-dimensional Einstein equations. In Sec. III, we develop the functional Schrödinger approach to field theory by analyzing the conformally coupled scalar field in spatially flat ($\mathrm{k}=0$) coordinates; our analysis is semiclassical in that the dynamics of the background metric are predetermined. We calculate the Feynman Green's function and use it in Sec. IV to analyze the Hawking effect in $(n+1)$-dimensional de Sitter spacetime by considering a conformally coupled scalar field interacting with a comoving detector. We establish a criterion by which one can decide if a given transition probability is thermal. In Sec. V, we discuss the minimally coupled scalar field in $\mathrm{k}=0$ de Sitter spacetime. We solve the functional Schrödinger equation in $n+1$ dimensions for the vacuum wave functional and calculate the Green's function. We analyze the $(3+1)$-dimensional case in some detail. We note that the equal-time Green's function is time dependent; in particular, the coincidence limit of the massless Green's function depends linearly on time. We show that the massless Green's function, *in any number of dimensions,* depends logarithmically on the separation (for large physical separation). This behavior is analogous to that of the scalar-field Green's function in flat spacetime in $1+1$ dimensions;[9] thus it suggests that it is impossible to break a continuous symmetry globally in $(n+1)$-dimensional de Sitter spacetime. In Secs. VI and VII, we repeat the above analysis for a scalar field in $\mathrm{k}=+1$ de Sitter coordinates. In Sec. VIII, we evaluate the Green's functions using the path-integral formalism with the boundary condition of Hawking, i.e., integrating over those field configurations which are regular on the Euclidean section of $(n+1)$-dimensional de Sitter space (in $\mathrm{k}=+1$ coordinates), an $(n+1)$-dimensional sphere. In Sec. IX we discuss how the requirements of finiteness of the field energy as the scale factor $a \to 0$ and that of regularity on the Euclidean section might be considered to be different aspects of the same "boundary" condition that uniquely specifies the vacuum wave functional, in $\mathrm{k}=+1$ coordinates. We analyze symmetry restoration in more detail in Sec. X, where we compute the Gaussian fluctuations about a state of broken U(1) symmetry and show that these fluctuations restore the symmetry. However, the correlations die out very slowly, as an inverse power of proper distance $ra(t)$, where r is coordinate distance. Physically, this calculation suggests that the scalar field expectation value wanders slowly as a function of position on a scale set by the scale factor. A local observer would always claim to be in a broken-symmetry phase of the theory. Although we only exhibit explicit solutions in $\mathrm{k}=+1$ and $\mathrm{k}=0$ coordinates, we expect this phenomenon to be coordinate independent.

The appendices contain technical details of the calculation. In Appendix E we examine the behavior of the equal-time Green's function of the minimally coupled scalar field for large and small spatial separation. We see that as the mass of the field goes to zero, there is an infinite contribution which appears both in the infrared and the ultraviolet and can be interpreted as being the zero mode on the n-sphere.

II. TECHNICAL PRELIMINARIES

De Sitter spacetime is the unique, maximally symmetric, negative-spacetime-curvature (i.e., positive Ricci scalar) solution to Einstein's equations with a cosmological constant and without matter. To solve Einstein's equations we need to make a choice of gauge. The conventional choice is the synchronous gauge where the metric is taken to be of the form

$$g_{\mu\nu}=\begin{bmatrix} 1 & 0 \\ 0 & -a^2(t)\tilde{g}_{ij}(x^i) \end{bmatrix} . \tag{2.1}$$

(Greek indices assume values from 0 to n, latin indices from 1 to n.) The requirements of spatial homogeneity and isotropy restrict $\tilde{g}_{ij}(x^i)$ to be the metric for an n-dimensional maximally symmetric space. We can then reduce Einstein's equations to an equation of evolution for the scale factor $a(t)$, which for an empty, spatially homogeneous and isotropic universe with a cosmological constant becomes

$$(\dot{a})^2=-\mathrm{k}-\kappa a^2 , \tag{2.2}$$

where k is the sign of the spatial curvature, which can assume the values ± 1 or 0, and κ is the constant spacetime curvature, to which we might assign the values $\pm h^2$ or 0; h is a real constant. κ is related to the Ricci scalar by $\kappa=-R/n(n+1)$. De Sitter spacetime is the (essentially unique) $\kappa=-h^2$ solution of this evolution equation.[10] It is conventionally viewed as being an $(n+1)$-dimensional hyperboloid embedded in $(n+2)$-dimensional Minkowski space.[11] As is well known, there is still some "gauge" freedom—i.e., different ways of aligning the de Sitter time axis with the embedding Minkowski spacetime axis. This exhibits itself in three different "de Sitter" solutions which correspond to three different ways of laying a coordinate system on the hyperboloid (i.e., three different ways of slicing spacetime into space and time).[10,11] These form one-parameter families. There is also a static de Sitter coordinate system. We list the solutions of (2.2) which have real Lorentzian sections:

	$\kappa=-h^2$	$\kappa=0$	$\kappa=+h^2$
k=0	$e^{\pm ht}$	1	
	de Sitter	Minkowski	
k=+1	$\cosh(ht)/h$		
	de Sitter-Lanczos		
k=−1	$\sinh(ht)/h$	t	$\sin(ht)/h$
	de Sitter-hyperbolic	Minkowski-hyperbolic	anti-de Sitter

We first consider the mathematically simplest case: the solution corresponding to k=0 in which the spatial hypersurfaces are flat. This allows us to use a Fourier expansion as opposed to an expansion in spherical harmonics. The scale factor is then of the form $a(t)=e^{\pm ht}$. We examine the expanding solution; some remarks about the contracting solution will be made later on. In these coordinates, the metric becomes

$$\mathrm{diag}(1,-e^{2ht}\delta_{ij}) \tag{2.3}$$

which is not static; also, the coordinate system has a horizon and only covers half the hyperboloid.[10,11] The Hubble constant $H=\dot{a}/a=h$. We can introduce a new time variable, conformal time

$$\tilde{t}=-\frac{e^{-ht}}{h}=-\frac{1}{ha(t)} \tag{2.4}$$

which puts the metric in a conformally flat form: $g_{\mu\nu}=(1/h^2\tilde{t}^2)\eta_{\mu\nu}$ ($\tilde{t}$ runs from $-\infty$ in the far past to 0 in the far future). For the contracting solution we define

$$\tilde{t}_c=\frac{e^{ht}}{h}=\frac{1}{ha(t)} \tag{2.5}$$

so $\tilde{t}_c\in[0,\infty]$ and $H=\dot{a}/a=-h$.

When the metric is of the form (2.3), the embedding space coordinates are

$$\begin{aligned} z_0&=\frac{\sinh(ht)}{h}+\frac{h}{2}e^{ht}|\mathbf{x}|^2 , \\ z_i&=e^{ht}x_i , \\ z_{n+1}&=\frac{\cosh(ht)}{h}-\frac{h}{2}e^{ht}|\mathbf{x}|^2 . \end{aligned} \tag{2.6}$$

The distance between two points on the hyperboloid is the square root of

$$\begin{aligned} \sigma^2&=\frac{e^{h(t+t')}}{h^2}[(e^{-ht}-e^{-ht'})^2-h^2(\mathbf{x}-\mathbf{x}')^2] \\ &=\frac{1}{h^2\tilde{t}\,\tilde{t}'}[(\tilde{t}-\tilde{t}')^2-(\mathbf{x}-\mathbf{x}')^2] . \end{aligned} \tag{2.7}$$

The geodesics in de Sitter spacetime are the intersections of the hyperboloid with planes through the origin. The de Sitter group in $n+1$ dimensions is just the homogeneous Lorentz group in $n+2$ dimensions, SO($n+1,1$), i.e., those Lorentz transformations in the ($n+2$)-dimensional Minkowski embedding space which do not move the hyperboloid around. The group SO($n+1,1$) has $(n+2)(n+1)/2$ generators which correspond to the following symmetries of the line element: n spatial translations, 1 dilatation, $n(n-1)/2$ spatial rotations, and n boosts.

We then consider the de Sitter solution in the gauge corresponding to k=+1 (Lanczos). The spatial hypersurfaces are now n-spheres; hence we will have to expand in generalized spherical harmonics. The scale factor is of the form $a(t)=\cosh(ht)/h$, so de Sitter space is an n-sphere, of radius $a(t)$, that first contracts and then expands; the Hubble "constant" $H=\dot{a}/a=h\tanh(ht)$. In these coordinates the metric is

$$\mathrm{diag}\left[1,-\frac{\cosh^2(ht)}{h^2}(1,\sin^2\theta_n(1,\sin^2\theta_{n-1}(\cdots)))\right] , \tag{2.8}$$

where $\theta_1\in[0,2\pi),\theta_i\in[0,\pi),i\neq1$. This coordinate system covers the hyperboloid.[9,10] Conformal time can be defined by

$$\sec^2\tilde{t}=\cosh^2(ht)=h^2a^2 ; \tag{2.9}$$

it assumes values from $-\pi/2$ to $\pi/2$. The metric is then in the conformally flat form

$$\frac{\sec^2\tilde{t}}{h^2}\mathrm{diag}(1,-(1,\sin^2\theta_n(1,\sin^2\theta_{n-1}(\cdots)))) . \tag{2.10}$$

When the metric is of the form (2.8), the embedding space coordinates are given by

$$\begin{aligned} z_0&=\frac{1}{h}\sinh(ht) , \\ z_1&=\frac{1}{h}\cosh(ht)\cos(\theta_n) , \\ &\cdots \\ z_i&=\frac{1}{h}\cosh(ht)\sin(\theta_n)\sin(\theta_{n-1})\cdots\sin(\theta_{n+2-i}) \\ &\quad\times\cos(\theta_{n+1-i}) , \\ &\cdots \\ z_{n+1}&=\frac{1}{h}\cosh(ht)\sin(\theta_n)\sin(\theta_{n-1})\cdots\sin(\theta_2)\sin(\theta_1) . \end{aligned} \tag{2.11}$$

The distance between two points Ω,Ω' is the square root of

$$\begin{aligned} \sigma^2&=\frac{2\cosh(ht)\cosh(ht')}{h^2}\left[-\frac{1+\sinh(ht)\sinh(ht')}{\cosh(ht)\cosh(ht')}+\cos\gamma\right] \\ &=\frac{2}{h^2\cos(\tilde{t})\cos(\tilde{t}')}[-\cos(\tilde{t}-\tilde{t}')+\cos\gamma] , \end{aligned} \tag{2.12}$$

where γ is the angle between Ω and Ω', given in 2 + 1 dimensions, for example, by the familiar formula

$$\cos\gamma = \cos\theta_2 \cos\theta_2' + \sin\theta_2 \sin\theta_2' \cos(\theta_1 - \theta_1') \ . \quad (2.13)$$

In this (k = +1) gauge, de Sitter space has a Euclidean extension in which the metric is definite. It is convenient to implement this analytic continuation by introducing the periodic real coordinate θ_{n+1}, with period π (see Fig. 3), defined by

$$\theta_{n+1} \equiv t_E = iht + \frac{\pi}{2} + m\pi \ , \quad (2.14)$$

where t_E is Euclidean "time" and m an integer. In these coordinates the metric is

$$-\frac{1}{h^2}\text{diag}(1, \sin^2\theta_{n+1}(1, \sin^2\theta_n(1, \sin^2\theta_{n-1}(\cdots))))$$

$$= -\frac{1}{h^2} S_{\mu\nu}^{(n+1)} \ , \quad (2.15)$$

where $S_{\mu\nu}^{(n+1)}$ is the metric on $S^{(n+1)}$, the $(n+1)$-dimensional unit sphere. The embedding space coordinates $z_{\mu E}$ are given by (2.11) with $(\sinh(ht), \cosh(ht))$ replaced by $(i\cos\theta_{n+1}, \sin\theta_{n+1})$ and $z_0 = iz_{0E}$. The square of the distance between two points Ω_E, Ω_E' becomes

$$\sigma_E{}^2 = -\frac{2}{h^2}(1 - \cos\gamma_{n+1}) \ , \quad (2.16)$$

where

$$\cos\gamma_{n+1} = \cos\theta_{n+1}\cos\theta_{n+1}' + \sin\theta_{n+1}\sin\theta_{n+1}'\cos\gamma \ . \quad (2.17)$$

An implicit assumption of all of our calculations is that the scalar field's contribution to the stress energy can be neglected as compared to the contribution from the cosmological constant. In other words, we assume that the addition of a scalar field (to de Sitter spacetime) does not radically modify the background geometry.

III. THE SPATIALLY FLAT METRIC: THE CONFORMALLY COUPLED FIELD

De Sitter spacetime is conformally flat, so a suitably rescaled, conformally coupled scalar field does not recognize as special the length scale set by the curvature of the spacetime in which it lives (this rescaling symmetry is actually broken by the conformal anomaly but this is not relevant at the level to which we calculate). It is thus a trivial matter of rescaling variables to get the de Sitter two-point functions from the corresponding Minkowski two-point functions. Because this case is simple, though, it is instructive to use it as a first example of our more generally applicable methods. In this section we will solve the functional Schrödinger equation for the evolution of a conformally coupled field; the resulting wave function will provide the Green's function of this field.

We shall deal with the massless scalar field so that the equation of motion is conformally invariant. The solution of the massive conformally coupled case can be obtained from that of the massive minimally coupled case, solved in Sec. V, by means of a suitable redefinition of the mass.

The action for the massless case is

$$S = \int dt\, d^n x\, \mathcal{L}(x)$$
$$= \int dt\, d^n x \sqrt{|g|}\, \tfrac{1}{2}(g^{\mu\nu}\partial_\mu\phi^*\partial_\nu\phi - \xi R\, |\phi|^2) \ , \quad (3.1)$$

where $\mathcal{L}(x)$ is the Lagrangian density, $\phi(x)$ is a complex scalar field, $g_{\mu\nu} = \text{diag}(1, -a^2(t)\delta_{ij})$, $a(t) = e^{ht}$, and

$$\xi = \frac{(n-1)}{4n} \quad (3.2)$$

to make the resulting Klein-Gordon equation conformally invariant. In de Sitter space we have $R = n(n+1)h^2$. Then the action becomes

$$S = \int dt\, d^n x \left[\frac{a^n}{2} |\dot{\phi}|^2 - \frac{a^{n-2}}{2} |\nabla\phi|^2 - \frac{a^n}{8}(n-1)(n+1)h^2 |\phi|^2 \right] \ . \quad (3.3)$$

We can rewrite this in terms of a dimensionless field $\chi = a^{(n-1)/2}\phi$:

$$S = \int dt\, d^n x \left[\frac{a|\dot{\chi}|^2}{2} - \frac{|\nabla\chi|^2}{2a} \right] , \quad (3.4)$$

where we have integrated by parts once and dropped a surface term (this affects only the phase of the resulting Schrödinger wave function). Let us now introduce conformal time $\tilde{t}$ [see (2.4)], by $d\tilde{t} = a^{-1}(t)dt$; then

$$S = \int d\tilde{t}\, d^n x \left[\frac{|\dot{\chi}|^2}{2} - \frac{|\nabla\chi|^2}{2} \right] , \quad (3.5)$$

where the dot now means a derivative with respect to conformal time.

Fourier expanding,

$$\chi(\mathbf{x}) = \int \frac{d^n k}{(2\pi)^n} \chi(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}} \ , \quad (3.6)$$

we can rewrite the action as

$$S = \int d\tilde{t} \frac{d^n k}{(2\pi)^n} \tilde{\mathcal{L}}_k$$
$$= \int d\tilde{t} \frac{d^n k}{(2\pi)^n} \left[\frac{\dot{\chi}(\mathbf{k})\dot{\chi}(-\mathbf{k})}{2} - \frac{k^2\chi(\mathbf{k})\chi(-\mathbf{k})}{2} \right] \quad (3.7)$$

or in terms of the real and imaginary parts of χ $(= \chi_1 + i\chi_2)$,

$$S = \int d\tilde{t} \frac{d^n k}{(2\pi)^n} \left[\frac{\dot{\chi}_i(\mathbf{k})\dot{\chi}_i(\mathbf{k})}{2} - \frac{k^2\chi_i(\mathbf{k})\chi_i(\mathbf{k})}{2} \right] , \quad (3.8)$$

where i runs over 1,2.

In the following development we will treat the real and imaginary parts of χ as independent real variables, which we denote generically as χ. The action for χ is

$$S = \int d\tilde{t} \frac{d^n k}{(2\pi)^n} \tilde{\mathcal{L}}_k = \int d\tilde{t} \frac{d^n k}{(2\pi)^n} \left[\frac{(\dot{\chi})^2}{2} - \frac{k^2\chi^2}{2} \right] . \quad (3.9)$$

We recognize that χ is a quantum-mechanical variable with the Hamiltonian density

$$\tilde{\mathscr{H}}_k=\frac{p^2}{2}+\frac{k^2\chi^2}{2}\ , \tag{3.10}$$

where p is conjugate to χ. The functional Schrödinger equation

$$\tilde{\mathscr{H}}_k\Psi_k[\chi,\tilde{t}]=i\frac{\partial}{\partial\tilde{t}}\Psi_k[\chi,\tilde{t}] \tag{3.11}$$

becomes

$$\left[-i\frac{\partial}{\partial\tilde{t}}-\frac{1}{2}\frac{\partial^2}{\partial\chi^2}+\frac{k^2\chi^2}{2}\right]\Psi_k=0\ . \tag{3.12}$$

We will look for solutions of the form

$$\Psi_k[\chi,\tilde{t}]=g(\tilde{t})\exp[-\tfrac{1}{2}f(\tilde{t})\chi^2]\ . \tag{3.13}$$

Equating coefficients of χ^0 and χ^2 to 0, we see that we need to solve the pair of equations:

$$-i\frac{\dot{g}}{g}+\frac{f}{2}=0\ , \tag{3.14}$$

$$i\dot{f}-f^2+k^2=0\ . \tag{3.15}$$

The first equation determines g in terms of f. To find f, substitute $f=-i\dot{R}/R$ into the second equation to reduce it to

$$\ddot{R}+k^2R=0\ , \tag{3.16}$$

which has as solution

$$R(\tilde{t})=c_1e^{ik\tilde{t}}+c_2e^{-ik\tilde{t}}\ . \tag{3.17}$$

Hence f is given by

$$f(\tilde{t})=k\left[\frac{c_1e^{ik\tilde{t}}-c_2e^{-ik\tilde{t}}}{c_1e^{ik\tilde{t}}+c_2e^{-ik\tilde{t}}}\right]\ , \tag{3.18}$$

so the vacuum wave functional depends on infinitely many undetermined constants, one for each mode (demanding de Sitter invariance effectively makes these constants mode independent). This is the same as what we would have found in flat spacetime. This problem is not noticed in the conventional method for determining f (separation of variables) because separating variables (even at one time) effectively imposes an initial condition by requiring that $\dot{f}$ vanish at a particular time. One can then show that all higher temporal derivatives of f need vanish at this point; hence f is a constant. The requirement of normalizability of the wave function then fixes the sign.

If this wave function is to describe a harmonic oscillator with a time-independent frequency, as it must, and is to be normalizable, then we need to choose $c_2=0$. We may impose an initial condition by requiring that far in the past ($\tilde{t}\rightarrow-\infty$ or $a\rightarrow 0$) the wave function be in the harmonic-oscillator (Gaussian) ground state (we cannot determine the constant for the zero mode since f vanishes; however, we may choose it to have the same value as for the other modes; this will also be done for the other examples we consider). Then we have for the vacuum wave function for mode k

$$\Psi_{k_0}(\chi,\tilde{t})=\langle\chi\,|\,0_k\rangle=\left[\frac{k}{\pi}\right]^{1/4}e^{-ik\tilde{t}/2}e^{-k\chi^2/2}\ . \tag{3.19}$$

This is normalized so $\langle 0_k\,|\,0_k\rangle=\int\Psi^*_{k_0}\Psi_{k_0}d\chi=1$. The complete vacuum wave functional is given by

$$\Psi_0=\langle\chi\,|\,0\rangle=\prod_k\langle\chi\,|\,0_k\rangle=\prod_k\Psi_{k_0}\ . \tag{3.20}$$

We can easily evaluate the equal-time Green's function in momentum space:

$$\langle 0_k\,|\,\chi^2(k,\tilde{t})\,|\,0_k\rangle=\frac{1}{2k}\ . \tag{3.21}$$

This is time independent, so the wave functional does not spread in field space [in fact,

$$\langle 0\,|\,\phi^2\,|\,0\rangle=\frac{\kappa_u{}^{n-1}}{(4\pi)^{n/2}\Gamma(n/2)(n-1)}\ , \tag{3.22}$$

where κ_u is an ultraviolet proper momentum cutoff]. Returning to position space we have

$$\begin{aligned}\langle 0\,|\,\phi(\mathbf{x})\phi(\mathbf{x}')\,|\,0\rangle&=\frac{\langle 0\,|\,\chi(\mathbf{x})\chi(\mathbf{x}')\,|\,0\rangle}{a^{n-1}}\\&=\frac{1}{a^{n-1}}\int\frac{d^nk}{(2\pi)^n}\frac{e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}}{2k}\\&=\frac{\Gamma((n-1)/2)}{4\pi^{(n+1)/2}}\frac{1}{[a(t)\,|\,\mathbf{x}-\mathbf{x}'\,|\,]^{n-1}}\ .\end{aligned} \tag{3.23}$$

Here $a(t)\,|\,\mathbf{x}-\mathbf{x}'\,|$ is just the proper distance, so we have found the Minkowskian Green's function suitably modified to take account of the conformal factor relating the de Sitter and Minkowski line elements [the $(1+1)$-dimensional massless case needs to be treated more carefully since it is logarithmically infrared divergent, as in flat spacetime[9]]. To find the Green's function for "nonequal" times, we need the propagator of the functional Schrödinger equation. We can write

$$\langle T\chi(\tilde{t},k)\chi(\tilde{t}',k)\rangle=-\int d\chi\,d\chi'\chi\chi'\Psi^*_{k_0}(\chi,\tilde{t})\times G^s_k(\chi,\chi';\tilde{t},\tilde{t}')\Psi_{k_0}(\chi',\tilde{t}')\ , \tag{3.24}$$

where G^s_k is the Schrödinger propagator. The Schrödinger equation is just that of a harmonic oscillator; hence the propagator is[12]

$$G_k^s(\chi,\chi';\tilde{t},\tilde{t}') = \left[\frac{k}{2\pi i \sin k\tilde{T}}\right]^{1/2} \exp\left[\frac{ik}{2\sin k\tilde{T}}[\cos k\tilde{T}(\chi^2+\chi'^2)-2\chi\chi']\right], \tag{3.25}$$

where $\tilde{T}=\tilde{t}-\tilde{t}'$. Performing the Gaussian integrations in (3.24) we find

$$\langle 0_k \mid T\chi(\tilde{t},k)\chi(\tilde{t}',k) \mid 0_k \rangle = \frac{1}{2k} e^{-ik\tilde{T}}, \tag{3.26}$$

which again shows that the system is conformally trivial. We can now Fourier transform to recover

$$\begin{aligned}\langle T\phi(\tilde{t},\mathbf{x})\phi(\tilde{t}',\mathbf{x}')\rangle &= \frac{1}{[a(\tilde{t})a(\tilde{t}')]^{(n-1)/2}} \int \frac{d^n k}{(2\pi)^n} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')} \langle T\chi(\tilde{t},k)\chi(\tilde{t}',k)\rangle \\ &= (h^2\tilde{t}\,\tilde{t}')^{(n-1)/2} \frac{\Gamma((n-1)/2)}{4\pi^{(n+1)/2}} \frac{1}{[\,|\mathbf{x}-\mathbf{x}'|^2-(\tilde{T}-i\epsilon)^2]^{(n-1)/2}}\;;\end{aligned} \tag{3.27}$$

or in our original coordinates (2.3)

$$\langle T\phi(t,\mathbf{x})\phi(t',\mathbf{x}')\rangle = \frac{\Gamma((n-1)/2)}{4\pi^{(n+1)/2}(e^{h(t+t')})^{(n-1)/2}[\,|\mathbf{x}-\mathbf{x}'|^2-(1/h^2)(e^{-ht}-e^{-ht'}-i\epsilon)^2]^{(n-1)/2}}\;. \tag{3.28}$$

In the flat-space limit ($h\to 0$) we recover the usual Green's function:

$$\langle T\phi(t,\mathbf{x})\phi(t',\mathbf{x}')\rangle = \frac{\Gamma((n-1)/2)}{4\pi^{(n+1)/2}[\,|\mathbf{x}-\mathbf{x}'|^2-(t-t_0-i\epsilon)^2]^{(n-1)/2}}\;. \tag{3.29}$$

For the exponentially contracting case, f is again given by (3.18). We can require that the wave-function approach a harmonic oscillator as $a\to 0$ (or $\tilde{t}_c\to\infty$, which is in the far future); then it is given by (3.19).

IV. THE SPATIALLY FLAT METRIC: THE HAWKING EFFECT FOR THE CONFORMALLY COUPLED SCALAR FIELD

Now that we have the Green's functions for a conformally coupled scalar field in a de Sitter background we can analyze what an idealized, comoving DeWitt-Unruh detector,[13] interacting with this field, will see. Following DeWitt,[13] we assume a coupling of the form $\mathcal{L}_{\text{int}}=m(\tau)\phi(x(\tau))$ between the detector and the scalar field along the detector's trajectory, where $m(\tau)$ is the monopole moment operator of the detector and $x(\tau)$ is the detector's trajectory. First-order perturbation theory then gives the transition probability per unit time for the detector to go from an energy level E_i to an energy level E_j as

$$\begin{aligned}P_{i\to j} = |m(0)_{ji}|^2 \int_{-\infty}^{+\infty} d(t-t') e^{-i(E_j-E_i)(t-t')} \\ \times\langle 0 \mid \phi(x(t))\phi(x(t')) \mid 0\rangle\,,\end{aligned} \tag{4.1}$$

where

$$\langle 0 \mid \phi(x(t))\phi(x(t')) \mid 0\rangle = \langle 0 \mid T\phi(t,\mathbf{x})\phi(t',\mathbf{x}) \mid 0\rangle$$

for $t>t'$. To be able to talk about equilibrium thermodynamics we need to work in a coordinate system in which g_{00} is time independent (so we have a time-independent scale of energy); hence, we use the coordinates (2.3). Defining $\Delta E=E_j-E_i$, $\tau=t-t'$, we have

$$\begin{aligned}P_{i\to j} &= \frac{|m(0)_{ji}|^2 h^{n+1}\Gamma((n-1)/2)}{2^{n+1}\pi^{(n+1)/2}(-1)^{(3n-3)/2}} \\ &\times \int_{-\infty}^{+\infty} d\tau \frac{e^{-i(\Delta E)\tau}}{[\sinh(h\tau/2-i\epsilon)]^{n-1}}\;.\end{aligned} \tag{4.2}$$

To evaluate

$$I(\Delta E) = \int_{-\infty}^{+\infty} d\tau \frac{e^{-i(\Delta E)\tau}}{[\sinh(h\tau/2-i\epsilon)]^{n-1}}\,, \tag{4.3}$$

we note that the integrand has poles on the imaginary τ axis at $\tau=i2n\pi/h+i\epsilon$, where n is an integer. So we can choose a contour C, as illustrated in Fig. 1, and use the method of residues to get

$$I(\Delta E) = \frac{2\pi(-1)^n(i)^{n+1}(\Delta E)^{n-2}e^{-\Delta E 2\pi/h}}{1+(-1)^n e^{-\Delta E 2\pi/h}}\,, \tag{4.4}$$

so

$$P_{i\to j} = P(\Delta E) = c(n)\frac{(\Delta E)^{n-2}e^{-\Delta E 2\pi/h}}{1+(-1)^n e^{-\Delta E 2\pi/h}} |m(0)_{ji}|^2\,, \tag{4.5}$$

where

$$c(n) = \frac{h^{n+1}\Gamma((n-1)/2)}{2^n \pi^{(n-1)/2}}\;. \tag{4.6}$$

We would now like to show that this transition probability is exactly the same as would have been gotten if the scalar field were in equilibrium with a thermal bath at some temperature T in flat spacetime. In the usual examples treated, e.g., the massless scalar field in Rindler coordinates (or interacting with a uniformly accelerated detector) in (3 + 1) dimensions,[14] the appearance of a "Planck factor" in the transition probability is taken to mean that the detector is in thermal equilibrium at some temperature T. This argument is, however, incomplete, as can easily be seen either by looking at the massive scalar field (in an arbitrary number of dimensions), where the transition probability is a Bessel function, or by looking at the massless case in some other number of dimensions. For both

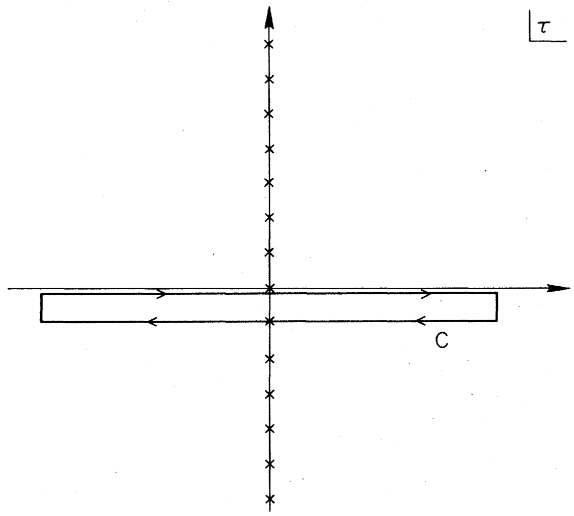

FIG. 1. The contour C for integral (4.3).

of these cases one can show that the transition probability is thermal. We now establish a criterion by which one can decide whether a transition probability is thermal or not. The states of a system in thermal equilibrium satisfy the principle of detailed balance, i.e., if the probability of being in the ith state is n_i, then $dn_i/dt=0\forall i$. Now we can relate dn_i/dt to the transition probability per unit time between states by

$$\frac{dn_i}{dt}=\sum_j{}' P_{j\to i}n_j-\sum_j{}' P_{i\to j}n_i \ .$$

A thermally populated set of states also satisfies $n_j=n_i e^{-\beta(E_j-E_i)}$, where β is the inverse temperature. Hence a system in thermal equilibrium with a heat bath will satisfy

$$\sum_j{}' (P_{j\to i}-e^{\beta(E_j-E_i)}P_{i\to j})=0 \tag{4.7}$$

and vice versa.

In the conformally coupled case we had $P(\Delta E)$ given by (4.5) and it is easy to see that this form satisfies

$$P(-\Delta E)=P(\Delta E)e^{\Delta E 2\pi/h}$$

so $\beta=2\pi/h$ or $T=h/2\pi$. So it seems that a conformally coupled scalar field in the $(n+1)$-dimensional de Sitter spacetime vacuum behaves like a scalar field in Minkowski spacetime at a temperature $T=h/2\pi$ where $h=H$ is the Hubble constant, in these coordinates.

V. THE SPATIALLY FLAT METRIC: THE MINIMALLY COUPLED FIELD

We repeat the analysis of the previous section for the minimally coupled, massive scalar field in $(n+1)$-dimensional de Sitter spacetime. We have

$$\begin{aligned} S&=\int dt\, d^n x\, \mathscr{L}(x) \\ &=\int dt\, d^n x \sqrt{|g|}\,\tfrac{1}{2}(g^{\mu\nu}\partial_\mu\phi^*\partial_\nu\phi-m^2|\phi|^2) \ . \end{aligned} \tag{5.1}$$

Following the manipulations of the previous section, we can put this in the form

$$\begin{aligned} S&=\int d\tilde{t}\frac{d^n k}{(2\pi)^n}\tilde{\mathscr{L}}_k \\ &=\int d\tilde{t}\frac{d^n k}{(2\pi)^n}\left[\frac{(\dot{\chi})^2}{2}+\frac{1}{2}\left[\frac{n^2-1}{4\tilde{t}^2}-\frac{m^2}{h^2\tilde{t}^2}-k^2\right]\chi^2\right] , \end{aligned} \tag{5.2}$$

where the dot now means a derivative with respect to conformal time. From this equation, we see that the Hamiltonian for χ is

$$\tilde{\mathscr{H}}_k=\frac{p^2}{2}+\frac{\chi^2}{2}\left[k^2+\frac{1}{\tilde{t}^2}\left[\frac{m^2}{h^2}-\frac{n^2-1}{4}\right]\right] . \tag{5.3}$$

Notice that the Hamiltonian is explicitly time dependent; hence, the Schrödinger equation will not separate; also, far in the past it reduces to that of a harmonic oscillator.

The Schrödinger equation is

$$\left[-i\frac{\partial}{\partial\tilde{t}}-\frac{1}{2}\frac{\partial^2}{\partial\chi^2}-\frac{1}{2}\left[\frac{n^2-1}{4\tilde{t}^2}-\frac{m^2}{h^2\tilde{t}^2}-k^2\right]\chi^2\right]\Psi_k=0 \ . \tag{5.4}$$

Again we look for solutions of the form

$$\Psi_k[\chi,\tilde{t}]=g(\tilde{t})\exp[-\tfrac{1}{2}f(\tilde{t})\chi^2] \ . \tag{5.5}$$

Equating coefficients of χ^0 and χ^2 to 0, we see that we need to solve the pair of equations

$$-i\frac{\dot{g}}{g}+\frac{f}{2}=0 , \tag{5.6}$$

$$i\dot{f}-f^2-\left[\frac{n^2-1}{4\tilde{t}^2}-\frac{m^2}{h^2\tilde{t}^2}-k^2\right]=0 \ . \tag{5.7}$$

As before, we substitute $f=-i\dot{R}/R$ into the second equation to reduce it to

$$\ddot{R}+\left[k^2+\frac{m^2}{h^2\tilde{t}^2}-\frac{n^2-1}{4\tilde{t}^2}\right]R=0 \ . \tag{5.8}$$

This is just a form of Bessel's equation; defining $\nu=(n^2/4-m^2/h^2)^{1/2}$ we get

$$R(\tilde{t})=c_1\tilde{t}^{1/2}H_\nu^{(1)}(k\tilde{t})+c_2\tilde{t}^{1/2}H_\nu^{(2)}(k\tilde{t}) , \tag{5.9}$$

where $H_\nu^{(1)}$ and $H_\nu^{(2)}$ are Hankel functions, and hence we can solve for f and g. In general,

$$f(\tilde{t})=-i\left[\frac{1-2\nu}{2\tilde{t}}+k\frac{c_1H_{\nu-1}^{(1)}(k\tilde{t})+c_2H_{\nu-1}^{(2)}(k\tilde{t})}{c_1H_\nu^{(1)}(k\tilde{t})+c_2H_\nu^{(2)}(k\tilde{t})}\right] \tag{5.10}$$

and

$$g(\tilde{t})=\frac{C}{R^{1/2}}=C\exp\left[-\frac{i}{2}\int_{-\infty}^{\tilde{t}}f(t)dt\right] , \tag{5.11}$$

where C is a normalization constant. As in the conformally coupled case we see that the wave function for each mode k depends on an undetermined parameter. If we impose as our boundary condition the requirement that the vacuum wave functional tend to the harmonic-oscillator (Gaussian) ground state in the far past ($\tilde{t}\rightarrow-\infty$ or $a\rightarrow 0$), then it can be written as

$$\langle \mathcal{X} \mid 0_k \rangle = \Psi_k[\mathcal{X},\tilde{t}] = C \exp\left[-\frac{i}{2}\int_{-\infty}^{\tilde{t}} f(t)dt - \tfrac{1}{2}f(\tilde{t})\mathcal{X}^2\right] \quad (5.12)$$

$$= \left[\frac{\mathrm{Re}f}{\pi}\right]^{1/4}\left[\frac{|R|}{R}\right]^{1/2} e^{-f\mathcal{X}^2/2}, \quad (5.13)$$

where

$$f(\tilde{t}) = -i\left[\frac{1-2\nu}{2\tilde{t}} + \frac{kH^{(1)}_{\nu-1}(k\tilde{t})}{H^{(1)}_{\nu}(k\tilde{t})}\right] \quad (5.14)$$

and

$$R(\tilde{t}) = c_1\tilde{t}^{1/2}H^{(1)}_{\nu}(k\tilde{t}) . \quad (5.15)$$

C was chosen by requiring $\langle 0_k \mid 0_k \rangle = 1$. Of course, if we did not make use of an initial condition, any R satisfying (5.8) would be allowed and the vacuum wave functional, of each mode, would form a one-parameter family. Requiring that the wave functional be de Sitter invariant is not enough to remove this degeneracy, although it does reduce it tremendously by effectively making the constants mode independent. Equation (5.14) is also the wave function in the exponentially contracting coordinates if a boundary condition is used in the far future.

The equal-time Green's function in momentum space is just

$$\langle \mathcal{X}^2(k,\tilde{t})\rangle = \frac{1}{2\,\mathrm{Re}f(\tilde{t})} = \frac{\pi\tilde{t}}{4}[J_\nu{}^2(k\tilde{t}) + Y_\nu{}^2(k\tilde{t})] , \quad (5.16)$$

where J_ν and Y_ν are Bessel functions. Here, unlike the conformally coupled case, the Green's function is time dependent.

If we restrict ourselves to (3 + 1) dimensions and look at the massless scalar field, we see that the wave functional is

$$\langle \mathcal{X} \mid 0_k \rangle = \Psi_k[\mathcal{X},\tilde{t}] = \left[\frac{k}{\pi}\right]^{1/4}\left[\frac{k\tilde{t}}{i+k\tilde{t}}\right]^{1/2} \times \exp\left[-\frac{i}{2}k\tilde{t} - \frac{1}{2\tilde{t}}\left[\frac{i+k^3\tilde{t}^3}{1+k^2\tilde{t}^2}\right]\mathcal{X}^2\right] \quad (5.17)$$

up to a (formally infinite) phase in the exponent. So, we see that a minimally coupled scalar field in de Sitter spacetime looks like a collection of harmonic oscillators with time-dependent frequencies. The flat-spacetime limit of this wave functional is just the harmonic-oscillator wave functional we found for the conformal field, times an infinite phase which cancels the phase alluded to above.

This wave function gives

$$\langle \mathcal{X}^2(k,\tilde{t})\rangle = \frac{1}{2k}\left[1+\frac{1}{k^2\tilde{t}^2}\right] = \frac{1}{2k}\left[1+\frac{h^2a^2}{k^2}\right], \quad (5.18)$$

which differs from the conformally coupled Green's function by a piece which grows in time (remember that conformal time $\tilde{t}\rightarrow 0$ corresponds to the far future). This expression is also valid in the exponentially contracting coordinates.

Brandenberger[15] has also derived this expression for the real part of the coefficient of $\mathcal{X}^2$ in the exponent. There are, however, differences between our wave functions; primarily the time-dependent normalization, which is of some importance, and the imaginary part of the coefficient of $\mathcal{X}^2$. These do not affect the two-point function in momentum space $\langle \mathcal{X}^2(k,\tilde{t})\rangle$. It should be pointed out that Brandenberger has exactly the same $\langle \mathcal{X}^2(k,\tilde{t})\rangle$ as we have obtained for de Sitter space; not, as he claims, something that is valid only in a de Sitter phase of a Friedmann-Robertson-Walker (FRW) cosmology. The result he attributes to Hawking[16] is applicable only for very long wavelengths—much outside the horizon—where the k^{-3} term in (5.18) dominates the k^{-1} term.

To study the spreading of the wave functional in field space we need to look at this Green's function in position space:

$$\langle \phi^2 \rangle = \frac{1}{2\pi^2 a^2}\int k^2 dk \langle \mathcal{X}^2(k,\tilde{t})\rangle . \quad (5.19)$$

This integral is logarithmically infrared divergent in (3 + 1) dimensions; in fact, it is in any number of dimensions. The infrared structure of the scalar field propagator in de Sitter space is very similar to that of the propagator in (1 + 1)-dimensional flat spacetime (see Ma and Rajaraman[9]). As we will discuss in detail in Sec. X, the logarithmic infrared divergence in the de Sitter scalar field propagator leads one to the same conclusion about symmetry restoration as in the low-dimensional flat-spacetime examples.

Evaluating this integral with suitable infrared and ultraviolet fixed proper momentum cutoffs, we find in (3 + 1) dimensions

$$\langle \phi^2 \rangle = \frac{1}{4\pi^2}\left[h^3(t-t_i) + h^2\ln\left[\frac{\kappa_u}{\kappa_i}\right]\right], \quad (5.20)$$

and in an odd number (n) of spatial dimensions:

$$\langle \phi^2 \rangle = \frac{\Gamma(n/2)}{2\pi^{(n+2)/2}}\left[h^n(t-t_i) + h^{n-1}\ln\left[\frac{\kappa_u}{\kappa_i}\right]\right]. \quad (5.21)$$

Here we have retained some of the cutoff-dependent terms [all terms discarded either depend on the ultraviolet cutoff or disappear when the infrared cutoff is removed ($\kappa_i=0$ or $t_i=-\infty$)]; κ_u and κ_i are the ultraviolet and infrared proper momentum cutoffs and t and t_i are the times at which the momentum scales κ_u and κ_i were the size of the horizon. The (1 + 1) de Sitter propagator exactly reproduces the flat-spacetime result in the limit $h=0$. The extra piece (for $h\neq 0$) arises from the red-shifting of the proper momentum cutoff in de Sitter space.

We could view (5.21) as describing field theory in a universe which at time t_i went from a Minkowski to a de Sitter phase; the infrared cutoff would then correspond to eliminating all information outside the initial de Sitter horizon which could not influence the scalar field's evolution. The symmetry group of such a spacetime would not be as big as the de Sitter group; equivalently we may say that the momentum cutoffs do not preserve the de Sitter symmetry. This is the reason that the form of $\langle\phi^2\rangle$ seems to be inconsistent with the fact that de Sitter spacetime is a maximally symmetric space. This expression is probably also valid for even n. The time dependence may be interpreted as the wave function spreading linearly with time in field space. Massive scalar field theory in $(1+1)$-dimensional flat spacetime with a time-dependent mass that goes from a constant nonzero value to zero abruptly also has a time-dependent $\langle\phi^2\rangle$. The $(3+1)$-dimensional result has been noted previously by Linde.[17] Hawking and Moss[17] have noticed that the propagator is logarithmically infrared divergent in $(3+1)$ dimensions. Their result is of interest particularly because they use the Euclidean version of the coordinate system which covers the whole de Sitter hyperboloid, i.e., the coordinate system in which de Sitter space is a contracting and then expanding compact three-sphere (we shall look at scalar field theory in this coordinate system in more detail in the next three sections). This checks that the infrared properties of the propagator are independent of the coordinatization of the hyperboloid. Field theory in the contracting metric will also be infrared divergent.

We make a short digression to discuss how to recover the massive conformally coupled case from the massive minimally coupled case. Looking at the Schrödinger equation for the minimally coupled case we see that for a particular value of the mass, $m^2/h^2=(n^2-1)/4$, we recover the conformally coupled Schrödinger equation; for this value of the mass, $\nu=\frac{1}{2}$ and $f(\tilde{t})=k$; thus we have the conformally coupled wave functional (up to an unimportant phase). In fact, if we had an arbitrary coupling to the curvature of the background geometry—i.e., a term of the form $-\frac{1}{2}\sqrt{|g|}\xi R\phi^2$ in the Lagrangian with ξ arbitrary—we would just need to make the replacement $m^2\to m^2+\xi R$ [i.e., $m^2\to m^2+\xi n(n+1)h^2$] to obtain the wave functional.

Let us now compute the full equal-time Green's function, taking some care as regards the divergences. To remove the trivial ultraviolet divergence we evaluate the two-point function at finite separation; to handle the infrared divergence we study the case of a nonzero mass. Then

$$\langle\phi(\mathbf{x})\phi(\mathbf{x}')\rangle=\frac{\pi\tilde{t}}{4(2\pi)^n a^{n-1}}\int d^nk e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}[J_\nu{}^2(k\tilde{t})+Y_\nu{}^2(k\tilde{t})] \tag{5.22}$$

$$=\frac{\tilde{t}}{2^{(n+4)/2}\pi^{(n-2)/2}a^{n-1}r^{(n-2)/2}}\int dk\, k^{n/2}J_{(n-2)/2}(kr)[J_\nu{}^2(k\tilde{t})+Y_\nu{}^2(k\tilde{t})] \tag{5.23}$$

$$=\frac{h^{n-1}}{(4\pi)^{(n+1)/2}}\frac{\Gamma(n/2-\nu)\Gamma(n/2+\nu)}{\Gamma((n+1)/2)}F\left[\frac{n}{2}-\nu,\frac{n}{2}+\nu;\left[\frac{n+1}{2}\right];1-\frac{r^2}{4\tilde{t}^2}\right] \tag{5.24}$$

$$=\frac{h^{n-1}}{(4\pi)^{(n+1)/2}}\frac{\Gamma(\delta)\Gamma(n-\delta)}{\Gamma((n+1)/2)}F\left[\delta,n-\delta;\left[\frac{n+1}{2}\right];1-\frac{r^2}{4\tilde{t}^2}\right], \tag{5.25}$$

where $\delta=m^2/nh^2$, $r=|\mathbf{x}-\mathbf{x}'|$, and F is a hypergeometric function. The integral in (5.23) has been evaluated in Appendix A. The massless limit of this two-point function, in $(3+1)$ dimensions, is exhibited in Fig. 2 (we have suppressed the zero mode). We will consider the case $\delta\ll 1$. To study the infrared properties of the integrand in (5.23) we need to look at its behavior at low momenta; consequently we can use the limiting form of the Bessel function for small arguments,

$$J_\mu(z)\approx\left[\frac{z}{2}\right]^\mu\frac{1}{\Gamma(\mu+1)}$$

and

$$Y_\mu(z)\approx-\frac{1}{\pi}\Gamma(\mu)\left[\frac{2}{z}\right]^\mu .$$

Clearly the infrared divergence comes from the second term in the integrand. At low momenta the momentum integral goes like

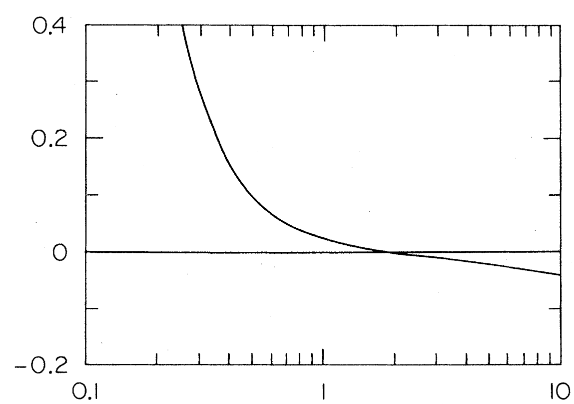

FIG. 2. The massless limit of the two-point function, in $3+1$ dimensions (with the zero mode removed) $[\langle\phi(\mathbf{x})\phi(\mathbf{x}')\rangle-h^2/8\pi^2\delta]/h^2$; as a function of proper distance $ha\,|\mathbf{x}-\mathbf{x}'|$, in Hubble units. All contributions in the ultraviolet from scattering off of the background have been suppressed.

$$\int dk\, k^{-2\nu+n-1} = \int dk\, k^{-(n^2-4m^2/h^2)^{1/2}+n-1} = \int \frac{dk}{k} k^{2\delta+O(\delta^2)} . \tag{5.26}$$

This is logarithmically infrared divergent for the massless case. For fixed n an infinitesimal positive mass cures this divergence (even with a fixed positive mass the integral is still divergent in the limit $n \to \infty$).

In spite of a claim to the contrary,[18] this infrared divergence is a real physical divergence—it is not an artifact of a wrong choice of initial condition but is present in *the* de Sitter invariant vacuum state (we will elaborate on this later). It is easy to see that even if we start out with a state which has no infrared divergences initially, time evolution will generate them, provided the state is allowed to evolve for a sufficiently long (formally infinite) period of time. Consider the following initial condition—let the [(3 + 1)-dimensional massless] wave function go to that of a harmonic oscillator at some finite time $\tilde{t}_0$ in the past (not at $-\infty$ as before). Then we obtain

$$\frac{c_1}{c_2} = \frac{\cos(k\tilde{t}_0) + [ik\tilde{t}_0 - (k\tilde{t}_0)^2]\exp(-ik\tilde{t}_0)}{\sin(k\tilde{t}_0) + i[ik\tilde{t}_0 - (k\tilde{t}_0)^2]\exp(-ik\tilde{t}_0)} \tag{5.27}$$

[the coefficient of χ^2 approaches that in (5.17) in the limit $\tilde{t}_0 \to -\infty$], and

$$\frac{1}{\mathrm{Re} f(\tilde{t})} = \frac{1}{k(k\tilde{t}_0)^4(k\tilde{t})^2}\{(k\tilde{t}_0)^4[(k\tilde{t})^2+1] + [(k\tilde{t}_0)^2(k\tilde{t})^2 - (k\tilde{t}_0)^2 + 2(k\tilde{t}_0)(k\tilde{t})](\sin^2 k\tilde{T} - \cos^2 k\tilde{T})$$
$$+ \sin^2 k\tilde{T} + (k\tilde{t})^2\cos^2 k\tilde{T} + [4(k\tilde{t}_0)^2(k\tilde{t}) + 2(k\tilde{t}_0) - 2(k\tilde{t}_0)(k\tilde{t})^2 - 2k\tilde{t}]\cos k\tilde{T}\sin k\tilde{T}\} , \tag{5.28}$$

where $\tilde{T} = \tilde{t} - \tilde{t}_0$. Clearly,

$$\mathrm{Re} f(\tilde{t}_0) = k \tag{5.29}$$

and so this state describes a harmonic oscillator with a time-independent frequency. There are no infrared divergences. Consider a much later time, $\tilde{T} \to \infty$; then $\sin^2 k\tilde{T} \sim \cos^2 k\tilde{T} \sim \frac{1}{2}$ and $\sin k\tilde{T} \sim \cos k\tilde{T} \sim 0$; we obtain

$$\frac{1}{\mathrm{Re} f(\tilde{t})} = \frac{1+(k\tilde{t})^2}{k^3\tilde{t}^2} + \frac{1+(k\tilde{t})^2}{2k(k\tilde{t})^2(k\tilde{t}_0)^4} . \tag{5.30}$$

The first term is what was present when we imposed the initial condition at $\tilde{t}_0 = -\infty$; it has a logarithmic infrared divergence. The second term has an even more infrared divergent structure, but we need not consider it if we are interested in finite $\tilde{t}$, i.e., $\tilde{t}_0 \to -\infty$. This exercise suggests that the conclusions we draw about symmetry restoration (in Sec. X) probably do not depend too strongly on our choice of initial/boundary condition. One can check explicitly, from (5.28), that for $\tilde{T} \ll k^{-1}$, these infrared divergences disappear.

In the coordinate system (2.3), the equal-time two-point function is

$$\frac{h^{n-1}}{(4\pi)^{(n+1)/2}} \frac{\Gamma(n/2-\nu)\Gamma(n/2+\nu)}{\Gamma((n+1)/2)} \times F\left[\frac{n}{2}-\nu, \frac{n}{2}+\nu; \left[\frac{n+1}{2}\right]; 1-\frac{r^2h^2e^{2ht}}{4}\right] . \tag{5.31}$$

The flat-spacetime limit may be obtained by letting $h \to 0$ in (5.23). In this limit ν becomes imaginary. We make use of the asymptotic formula valid for large real b and x:

$$J_{ib}^2(x) + Y_{ib}^2(x) \approx \frac{2}{\pi(b^2+x^2)^{1/2}} , \tag{5.32}$$

which gives us

$$\lim_{h\to 0}\langle\phi(\mathbf{x})\phi(\mathbf{x}')\rangle = \frac{1}{2}\int \frac{d^nk}{(2\pi)^n}\frac{e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}}{(\mathbf{k}^2+m^2)^{1/2}} = \oint_{C^+}\frac{dk^0 d^nk}{(2\pi)^{n+1}}\frac{e^{ik\cdot(x-x')}}{(k^2-m^2)} \tag{5.33}$$

(where C^+ is a contour in the complex k_0 plane enclosing the positive pole), the Minkowskian result.

At this point we could repeat the analysis of the previous section by finding the path integral of the Schrödinger equation and the "nonequal" times Green's function for the minimally coupled scalar field. However, we do not have to go through this exercise as there is an easy way to obtain the Feynman Green's function; looking at the conformally coupled case we see that we just need to make the substitution:

$$e^{2ht}r^2 \to e^{h(t+t')}\left[r^2 - \frac{1}{h^2}(e^{-ht} - e^{-ht'} - i\epsilon)^2\right] = -\sigma^2 \tag{5.34}$$

(notice that in the equal-time limit the right-hand side reduces to the left-hand side). In conformal coordinates we are making the replacement

$$a^2r^2 \to \frac{1}{(h\tilde{t})(h\tilde{t}')}[r^2 - (-\tilde{t} + \tilde{t}' - i\epsilon)^2] \tag{5.35}$$

which is the unique object, up to a multiplicative factor in front, that is de Sitter invariant, and that reduces to $[r^2 - (t-t')^2]$ in the flat-spacetime limit. So we have

$$\langle T\phi(t,\mathbf{x})\phi(t',\mathbf{x}')\rangle = \frac{h^{n-1}}{(4\pi)^{(n+1)/2}}\frac{\Gamma(n/2-\nu)\Gamma(n/2+\nu)}{\Gamma((n+1)/2)} \times F\left[\frac{n}{2}-\nu, \frac{n}{2}+\nu; \left[\frac{n+1}{2}\right]; 1+\frac{h^2\sigma^2}{4}\right] . \tag{5.36}$$

The Minkowskian limit may be obtained as before by letting $h \to 0$.

VI. THE LANCZOS METRIC: THE CONFORMALLY COUPLED FIELD

In this section we again consider the massless conformally coupled scalar field, but in the k = +1 background. The action is given by Eqs. (3.1) and (3.2). Using Eq. (2.8) and integrating, spatially by parts, it becomes

$$S=\int dt\, d^n x\, a^n |g_{ij}|^{1/2}\left[\frac{|\dot{\phi}|^2}{2}+\frac{1}{2a^2}\phi^*\mathscr{L}^2_{(n)}\phi-\frac{(n-1)(n+1)h^2}{8}\phi^*\phi\right], \tag{6.1}$$

where $\mathscr{L}^2_{(n)}$ is the Laplacian on the unit sphere S^n (see Appendix B). As before, we rewrite this in terms of a dimensionless field $\chi=a^{(n-1)/2}\phi$:

$$S=\int d\tilde{t}\, d^n x\, |g_{ij}|^{1/2}\left[\frac{|\dot{\chi}|^2}{2}-\frac{|\chi|^2}{2}\left[\frac{n-1}{2}\right]^2+\tfrac{1}{2}\chi^*\mathscr{L}^2_{(n)}\chi\right], \tag{6.2}$$

where the dot now means a derivative with respect to conformal time (we have integrated by parts and dropped a surface term).

The eigenfunctions of $\mathscr{L}^2_{(n)}$ are generalized spherical harmonics Y_{AW} with n indices (W represents the $n-1$ magnetic indices), which we shall generically denote by k; these are discussed in Appendix B. Now

$$\mathscr{L}^2_{(n)}Y_{AW}=-A(A+n-1)Y_{AW}, \tag{6.3}$$

so

$$S=\sum_k\int d\tilde{t}\left[\frac{|\dot{\chi}_k|^2}{2}-\frac{|\chi_k|^2}{2}\left[A+\frac{n-1}{2}\right]^2\right], \tag{6.4}$$

or, treating the real and imaginary parts of χ_k as independent real variables, which we generically denote as χ, we have

$$S=\sum_k\int d\tilde{t}\left[\frac{\dot{\chi}^2}{2}-\frac{\chi^2}{2}\left[A+\frac{n-1}{2}\right]^2\right], \tag{6.5}$$

or

$$\tilde{\mathscr{H}}_A=\frac{p^2}{2}+\frac{\chi^2}{2}\left[A+\frac{n-1}{2}\right]^2. \tag{6.6}$$

An analysis similar to that performed in Sec. III then allows us to write the functions R and f as

$$R(\tilde{t})=c_1e^{i(\mu+1/2)\tilde{t}}+c_2e^{-i(\mu+1/2)\tilde{t}} \tag{6.7}$$

and

$$f(\tilde{t})=(\mu+\tfrac{1}{2})\left[\frac{c_1e^{i(\mu+1/2)\tilde{t}}-c_2e^{-i(\mu+1/2)\tilde{t}}}{c_1e^{i(\mu+1/2)\tilde{t}}+c_2e^{-i(\mu+1/2)\tilde{t}}}\right], \tag{6.8}$$

where $\mu=A+n/2-1$. As we are considering a conformally coupled scalar field we need to have a time-independent f which means that we need to choose $c_2=0$ (the choice $c_1=0$ gives us an unnormalizable wave function). This choice will be discussed in detail later on. So the vacuum wave function for mode A is

$$\Psi_{A_0}(\chi,\tilde{t})=\langle\chi|0_A\rangle$$
$$=\left[\frac{2\mu+1}{2\pi}\right]^{1/4}\exp[-\tfrac{1}{2}(\mu+\tfrac{1}{2})(i\tilde{t}+\chi^2)]. \tag{6.9}$$

The equal-time two-point function in momentum space is

$$\langle\chi^2\rangle=\frac{1}{2\,\mathrm{Re}f(\tilde{t})}=\frac{1}{2A+n-1}; \tag{6.10}$$

it is time independent. Transforming back to position space we have

$$\langle\phi(\Omega)\phi(\Omega')\rangle=\frac{1}{a^{n-1}}\sum_k\frac{Y_k^*(\Omega)Y_k(\Omega')}{(2A+n-1)}. \tag{6.11}$$

The addition formula (B13) simplifies this to

$$\langle\phi(\mathbf{x})\phi(\mathbf{x}')\rangle=\frac{\Gamma((n-1)/2)}{4\pi^{(n+1)/2}(-\sigma^2)^{(n-1)/2}}. \tag{6.12}$$

This is the same as the expression we had for the $k=0$ metric (3.28).

VII. THE LANCZOS METRIC: THE MINIMALLY COUPLED FIELD

We extend the analysis of the previous section to the minimally coupled scalar field, where the action is given by Eq. (5.1). We can write the conformal Hamiltonian as

$$\tilde{\mathscr{H}}_A=\frac{p^2}{2}+\frac{\chi^2}{2}\left[\left[A+\frac{n-1}{2}\right]^2+\sec^2\tilde{t}\left[\frac{m^2}{h^2}-\frac{n^2-1}{4}\right]\right], \tag{7.1}$$

where

$$\sec^2\tilde{t}=h^2a^2. \tag{7.2}$$

Unlike the Hamiltonian for field theory in the $k=0$ background, this Hamiltonian does not have a classically allowed asymptotic region in which it approaches that of a harmonic oscillator with a time-independent frequency. The formal similarity to (5.3) should, however, be noted. Also, at $\tilde{t}=0$, $\tilde{\mathscr{H}}_A$ has no explicit time dependence and hence the $\tilde{t}=0$ hypersurface might be a good surface on which an initial condition can be prescribed (this is not what we do). Notice that as $a\to 0$ (this limit does not lie on the Lorentzian section of de Sitter space) this Hamiltonian approaches the conformally coupled Hamiltonian; this fact shall be used to impose a boundary condition on the Schrödinger equation.

The functions R and f appearing in the wave functional have the form

$$R(\tilde{t})=\cos^{1/2}\tilde{t}[c_1 R^{(1)\nu}_{\mu}(\sin\tilde{t})+c_2 R^{(2)\nu}_{\mu}(\sin\tilde{t})]\ , \tag{7.3}$$

$$f(\tilde{t})=i\left[\frac{1-2\nu}{2}\right]\tan\tilde{t}-i(\mu+\nu)(\mu-\nu+1)\left[\frac{c_1 R^{(1)\nu-1}_{\mu}(\sin\tilde{t})+c_2 R^{(2)\nu-1}_{\mu}(\sin\tilde{t})}{c_1 R^{(1)\nu}_{\mu}(\sin\tilde{t})+c_2 R^{(2)\nu}_{\mu}(\sin\tilde{t})}\right]\ , \tag{7.4}$$

where $\mu=A+n/2-1$, $\nu=(n^2/4-m^2/h^2)^{1/2}$, and $R^{(1)\beta}_{\alpha}, R^{(2)\beta}_{\alpha}$ are related to the Legendre functions $P^{\beta}_{\alpha}, Q^{\beta}_{\alpha}$ by

$$R^{(i)\beta}_{\alpha}(z)=P^{\beta}_{\alpha}(z)\pm\frac{i2}{\pi}Q^{\beta}_{\alpha}(z)\ . \tag{7.5}$$

This is similar to the relation between the Hankel and Bessel functions. The asymptotic analysis is more easily understood if one uses the $R^{(i)\beta}_{\alpha}$ instead of the P^{β}_{α} and Q^{β}_{α}. Although these functions do not seem to have been studied before, the formulas that we shall use may be derived using the Legendre functions.

Burges[2] has attempted to use de Sitter invariance to obtain an expression for $f(0)$. He has considered a massless scalar field with, presumably, no coupling to the background geometry. Our results, when restricted to $\tilde{t}=0$ and $m=0$ (and assuming that the ratio c_1/c_2 is the same for each mode) only agree with his expressions in 1 + 1 dimensions. We shall elaborate on this discrepancy later.

First we present a heuristic argument for the "correct" initial condition. This is more of a self-consistency requirement, that the minimally coupled solution should reproduce the conformally coupled solution for some particular value of ν, rather than the stronger requirement concerning the behavior of the wave function as $a\to 0$, which probably depends to some extent on the quantum theory of gravity. For $\nu=\frac{1}{2}$ we have

$$f(\tilde{t})=-i(\mu+\tfrac{1}{2})^2\left[\frac{c_1 R^{(1)-1/2}_{\mu}(\sin\tilde{t})+c_2 R^{(2)-1/2}_{\mu}(\sin\tilde{t})}{c_1 R^{(1)1/2}_{\mu}(\sin\tilde{t})+c_2 R^{(2)1/2}_{\mu}(\sin\tilde{t})}\right]\ . \tag{7.6}$$

Using the following relations,

$$R^{(1)-1/2}_{\mu}(\sin\theta)=\left[\frac{2}{\pi\cos\theta}\right]^{1/2}\frac{2i}{2\mu+1}\exp\left[i\left[\theta-\frac{\pi}{2}\right](\mu+\tfrac{1}{2})\right]\ ,$$
$$R^{(1)1/2}_{\mu}(\sin\theta)=\left[\frac{2}{\pi\cos\theta}\right]^{1/2}\exp\left[i\left[\theta-\frac{\pi}{2}\right](\mu+\tfrac{1}{2})\right]\ , \tag{7.7}$$

and

$$[R^{(1)\nu}_{\mu}(x)]^{*}=R^{(2)\nu}_{\mu}(x)\ , \tag{7.8}$$

we obtain

$$f(\tilde{t})=(\mu+\tfrac{1}{2})\left[\frac{c_1 e^{i(\tilde{t}-\pi/2)(\mu+1/2)}-c_2 e^{-i(\tilde{t}-\pi/2)(\mu+1/2)}}{c_1 e^{i(\tilde{t}-\pi/2)(\mu+1/2)}+c_2 e^{-i(\tilde{t}-\pi/2)(\mu+1/2)}}\right]\ . \tag{7.9}$$

So if this f is to describe the conformally coupled scalar field, then we need to choose $c_2=0$. With this choice we have

$$R(\tilde{t})=c_1\left[\frac{2}{\pi}\right]^{1/2}e^{i(\tilde{t}-\pi/2)(\mu+1/2)} \tag{7.10}$$

and

$$f(\tilde{t})=(\mu+\tfrac{1}{2})\ , \tag{7.11}$$

as in the conformally coupled case. A more general argument will be presented later on to show that this is indeed the correct initial condition. We can temporarily accept this as an ansatz.

This ansatz then gives us

$$\mathrm{Re}f(\tilde{t})=\frac{2}{\pi\cos\tilde{t}}\frac{1}{[P^{\nu}_{\mu}(\sin\tilde{t})]^2+\frac{4}{\pi^2}[Q^{\nu}_{\mu}(\sin\tilde{t})]^2}\frac{\Gamma(1+\mu+\nu)}{\Gamma(1+\mu-\nu)}\ . \tag{7.12}$$

The coincidence limit of the massless scalar (rescaled) field's two-point function in (3 + 1) dimensions is then given by

$$\langle\chi^2\rangle=\frac{1}{2\,\mathrm{Re}f(\tilde{t})}=\frac{1}{2(A+1)}\left[1+\frac{h^2a^2}{A(A+2)}\right]\ , \tag{7.13}$$

where the prefactor is the conformally coupled scalar's two-point function. This is very similar to (5.18), the expression in $k=0$ coordinates. As before, the second term will lead to an infrared divergence in the propagator.

The position space two-point function can be expressed as

$$\langle\phi(\mathbf{x})\phi(\mathbf{x}')\rangle=\frac{1}{a^{n-1}}\sum_k\frac{1}{2\,\mathrm{Re}f(\tilde{t})}Y_k^*(\Omega)Y_k(\Omega') \tag{7.14}$$

$$=\frac{\Gamma((n-1)/2)}{4\pi^{(n-1)/2}a^{n-1}}\frac{\cos\tilde{t}}{2}\sum_A C_A^{(n-1)/2}(\cos\gamma)\left[A+\frac{n-1}{2}\right]\frac{\Gamma\left[A+\frac{n}{2}-\nu\right]}{\Gamma\left[A+\frac{n}{2}+\nu\right]}$$

$$\times\{[P^{\nu}_{A+n/2-1}(\sin\tilde{t})]^2+(4/\pi^2)[Q^{\nu}_{A+n/2-1}(\sin\tilde{t})]^2\}\ , \tag{7.15}$$

where $C_A^{(n-1)/2}$ is a Gegenbauer polynomial, and we have used (B12).

For the massless conformal case $\nu=\frac{1}{2}$ and so using (B8), the expression above reduces to

$$\langle\phi(\mathbf{x})\phi(\mathbf{x}')\rangle=\frac{\Gamma((n-1)/2)}{4\pi^{(n+1)/2}a^{n-1}}\sum_A C_A^{(n-1)/2}(\cos\gamma)=\frac{\Gamma((n-1)/2)}{4\pi^{(n+1)/2}(-\sigma^2)^{(n-1)/2}} \tag{7.16}$$

which is exactly what we had before.

We can evaluate the general expression (see Appendix C) to obtain

$$\langle\phi(\mathbf{x})\phi(\mathbf{x}')\rangle=\frac{h^{n-1}}{(4\pi)^{(n+1)/2}}\frac{\Gamma(n/2-\nu)\Gamma(n/2+\nu)}{\Gamma((n+1)/2)}F\left[\frac{n}{2}+\nu,\frac{n}{2}-\nu;\frac{n+1}{2};1+\frac{h^2\sigma^2}{4}\right]\ , \tag{7.17}$$

which is the same as (5.24) and also agrees with the earlier analysis of Ref. 19.

VIII. EUCLIDEAN GREEN'S FUNCTIONS

In this section we use the Euclidean path-integral representation of the generating functional to evaluate the Green's function. The generating functional, for any of the theories we have considered, in the presence of an external source J, is given by

$$Z[J]=N\int\mathscr{D}\phi\exp\left[iS+i\int d^nx\,d(ht)\sqrt{|g|}J\phi\right]\ . \tag{8.1}$$

Here S could be any of the actions which we have considered, N is a normalization constant chosen in such a manner that $Z[0]=1$, and we have chosen to work with the variable ht instead of t.

As discussed in Sec. II, the analytic continuation of de Sitter space is an $(n+1)$-dimensional sphere embedded in $(n+2)$-dimensional Euclidean space. Then

$$|g|^{1/2}d^nx\,d(ht)=-\frac{i}{h^{n+1}}|S^{(n+1)}|^{1/2}d^{n+1}x_E\ , \tag{8.2}$$

where

$$d^{n+1}x_E\equiv\prod_{i=1}^{n+1}d\theta_i\ , \tag{8.3}$$

and

$$g^{\mu\nu}\partial_\mu\phi^*\partial_\nu\phi=-h^2S^{(n+1)\mu\nu}\partial^E_\mu\phi^*\partial^E_\nu\phi\ , \tag{8.4}$$

where

$$\partial^E_\mu\equiv\frac{\partial}{\partial\theta^\mu}\ . \tag{8.5}$$

We first discuss the conformally coupled case. In Euclidean space, using the Euclideanized version of (3.1) we may write the exponent in (8.1) as

$$-\tfrac{1}{2}\int d^{n+1}x_E\frac{|S^{(n+1)}|^{1/2}}{h^{n-1}}\times\left[S^{(n+1)\mu\nu}\partial^E_\mu\phi^*\partial^E_\nu\phi+\frac{(n+1)(n-1)}{4}|\phi|^2-\frac{J^*\phi}{h^2}-\frac{J\phi^*}{h^2}\right]\ . \tag{8.6}$$

Integrating by parts, using the generalized spherical harmonics defined in Appendix B and

$$\mathscr{L}^2_{(n+1)}Y_{AW}=-A(A+n)Y_{AW}\ , \tag{8.7}$$

where

$$\mathscr{L}^2_{(n+1)}=\frac{1}{|S^{(n+1)}|^{1/2}}\partial^E_\nu|S^{(n+1)}|^{1/2}S^{(n+1)\mu\nu}\partial^E_\mu\ , \tag{8.8}$$

we obtain for the momentum-space representation of the exponent in the functional integral

$$-\frac{1}{2h^{n-1}}\sum_k\left[\phi_k^*\left[A+\frac{n-1}{2}\right]\left[A+\frac{n+1}{2}\right]\phi_k-\frac{J_k^*\phi_k}{h^2}-\frac{J_k\phi_k^*}{h^2}\right]\ . \tag{8.9}$$

By using an expansion in spherical harmonics, we effectively exclude from the functional integral those field configurations which are not regular on the Euclidean section. We may introduce a shifted field

$$\phi'_k=\phi_k-\frac{J_k}{h^2\left[A+\frac{n-1}{2}\right]\left[A+\frac{n+1}{2}\right]} . \tag{8.10}$$

The path-integral measure does not change under this transformation, so we obtain

$$Z[J]=Z[0]\exp\left[\frac{1}{2h^{n-1}}\sum_k \frac{J_k^*}{h^2}\frac{1}{(\mu+\frac{1}{2})(\mu+\frac{3}{2})}\frac{J_k}{h^2}\right] , \tag{8.11}$$

or rewriting the Green's function in the position representation

$$Z[J]=\exp\left[\tfrac{1}{2}\int d^{n+1}x_E\frac{|S^{(n+1)}|^{1/2}}{h^{n+1}}\int d^{n+1}x'_E\frac{|s^{(n+1)}|^{1/2}}{h^{n+1}}J^*(x)\sum_k h^{n-1}\frac{Y_k(\Omega)Y_k^*(\Omega')}{(\mu+\frac{1}{2})(\mu+\frac{3}{2})}J(x')\right] . \tag{8.12}$$

So the Euclidean Feynman Green's function is

$$\langle T\phi(\Omega)\phi(\Omega')\rangle_E=h^{n-1}\sum_k \frac{Y_k(\Omega)Y_k^*(\Omega')}{(\mu+\frac{1}{2})(\mu+\frac{3}{2})} . \tag{8.13}$$

Using the equivalent of (B12) in $n+1$ dimensions, we find

$$\langle T\phi(\Omega)\phi(\Omega')\rangle_E=\frac{h^{n-1}\Gamma(n/2)}{4\pi^{(n+2)/2}}\sum_A \frac{(2A+n)}{\left[A+\frac{n-1}{2}\right]\left[A+\frac{n+1}{2}\right]}C_A^{n/2}(\cos\gamma_{n+1}) , \tag{8.14}$$

which, from Appendix D, is equal to

$$\frac{h^{n-1}}{2(2\pi)^{(n+1)/2}}\frac{\Gamma((n-1)/2)}{(1-\cos\gamma_{n+1})^{(n-1)/2}}=\frac{1}{4\pi^{(n+1)/2}}\frac{\Gamma((n-1)/2)}{(-\sigma_E{}^2)^{(n-1)/2}} . \tag{8.15}$$

This is the Euclidean extension of the Feynman Green's function that we had calculated in Secs. III and VI.

The minimally coupled case is analyzed in exactly the same way. An appropriate definition of the shifted field ϕ'_k allows us to express the generating functional as

$$Z[J]=\exp\left[\frac{1}{2h^{n-1}}\sum_k \frac{J_k^*}{h^2}\frac{1}{(A+n/2+\nu)(A+n/2-\nu)}\frac{J_k}{h^2}\right] . \tag{8.16}$$

We can therefore write the Euclidean Green's function as

$$\langle T\phi(\Omega)\phi(\Omega')\rangle_E=\frac{h^{n-1}\Gamma(n/2)}{4\pi^{(n+2)/2}}\sum_A \frac{(2A+n)}{(A+n/2+\nu)(A+n/2-\nu)}C_A^{n/2}(\cos\gamma_{n+1}) , \tag{8.17}$$

which is (see Appendix D) equal to

$$\frac{h^{n-1}}{(4\pi)^{(n+1)/2}}\frac{\Gamma(n/2-\nu)\Gamma(n/2+\nu)}{\Gamma((n+1)/2)}F\left[\frac{n}{2}+\nu,\frac{n}{2}-\nu;\frac{n+1}{2};\frac{1+\cos\gamma_{n+1}}{2}\right] , \tag{8.18}$$

the Euclidean continuation of (5.36).

IX. THE INITIAL/BOUNDARY CONDITION AND DE SITTER INVARIANCE

There is a widespread belief that the de Sitter vacuum state belongs to a one parameter family and that some extra criterion must be used to pick a suitable vacuum state.[2] We have shown that if the wave function for each mode is taken to be the general solution of the functional Schrödinger equation without imposing an initial condition [on the Lorentzian section, or, equivalently a boundary (regularity) condition on the Euclidean section], then each wave function forms a one-parameter family. We now elaborate on this statement and present an argument for the initial/boundary condition; we then relate this condition to Hawking's prescription for quantum gravity.[1] We may rephrase the initial condition which we have been using in the following manner: instead of requiring that the wave-functional (as a functional of the dimensionless field χ) approach that of a harmonic-oscillator ground state in some limit, we may equivalently require that the energy of this state not diverge in the same limit.[20]

The expectation value of the scalar field (ϕ_p) Hamiltonian, $\mathcal{H}_p$, for the mode p, is related to that of $\tilde{\mathcal{H}}_p$ as follows:

$$E_p = \langle 0_p \,|\, \mathscr{H}_p \,|\, 0_p \rangle$$
$$= \frac{1}{a} \langle 0_p \,|\, \tilde{\mathscr{H}}_p \,|\, 0_p \rangle = \frac{i}{a} \left\langle 0_p \left| \frac{\partial}{\partial \tilde{t}} \right| 0_p \right\rangle \tag{9.1}$$
$$= \frac{1}{2a(f+f^*)}(f^2 - i\dot{f} + ff^*) \,. \tag{9.2}$$

We have used the standard form for the wave function (3.13) and eliminated $\dot{g}$ by using the equation of motion (3.14). We may now eliminate $f^2 - i\dot{f}$ by using the other equation of motion; this will result in a different expression for each of the cases we consider. As we will be working with the wave functional for a particular mode, we will refer to c_1 and c_2 as constants; in reality, they could be different for different modes. We first consider the k=0 conformally coupled scalar field. Using (3.15) we obtain

$$E_k = \frac{1}{2a(f+f^*)}(k^2 + ff^*) \,. \tag{9.3}$$

Then from (3.18) we find

$$E_k = \frac{k}{2a} \frac{c_1^2 + c_2^2}{c_1^2 - c_2^2} \,. \tag{9.4}$$

We have assumed that the ratio c_1/c_2 is real; in general it could be complex. However, the independent argument concerning regularity on the Euclidean section justifies this choice. So, if E_k is to remain finite as $a \to 0$ we require

$$\frac{c_1^2 + c_2^2}{c_1^2 - c_2^2} = 1 \text{ or } c_2 = 0 \tag{9.5}$$

which means that E_k is just $k/2a$, or all excited states of the harmonic oscillator are unoccupied. The ground-state energy just leads to a shift in the zero of energy and may be taken care of by appropriately normal ordering the Hamiltonian. Similarly, for the minimally coupled scalar field in the same coordinate system we may use (5.7) to rewrite (9.2) as

$$E_k = \frac{1}{2a(f+f^*)}[k^2 + ff^* + h^2a^2(\tfrac{1}{4} - \nu^2)] \,. \tag{9.6}$$

Using (5.10) and the relevant asymptotic forms of the Hankel functions, we see that in the limit $a \to 0$ this becomes

$$E_k = \frac{k}{2a} \frac{c_1^2 + c_2^2}{c_1^2 - c_2^2} \tag{9.7}$$

exactly as in the previous case (again we assume c_1/c_2 is real). This should not come as a surprise as this Hamiltonian approaches the conformally coupled scalar field's Hamiltonian in the limit $a \to 0$. So finiteness of energy as $a \to 0$ again requires $c_2 = 0$. Notice that de Sitter space in k=0 coordinates has no real Euclidean section.

For the conformally coupled scalar field in k=+1 coordinates, we obtain

$$E_A = \frac{1}{2a(f+f^*)}[(\mu + \tfrac{1}{2})^2 + ff^*] \,. \tag{9.8}$$

In these coordinates it is not clear, *a priori*, what the $a \to 0$ limit means. In fact, $a = 0$ does not lie in the Lorentzian section of de Sitter space. It is easy to show that $a = 0$ corresponds to two points, the North and South poles of the Euclidean sphere in this coordinate system (actually the Euclidean section consists of an infinite number of spheres, one on top of the other, with the contiguous North and South poles identified, see Fig. 3. From (2.9) we can write

$$\sin\tilde{t} = \frac{(h^2a^2 - 1)^{1/2}}{ha} \,. \tag{9.9}$$

So the $a \to 0$ limit is clearly equivalent to $\tilde{t} \to \pm i\infty$, one limit corresponding to the North pole and the other to the South pole of the Euclidean section. We need to satisfy the condition of finiteness of energy at only one of these points, and it will be automatically satisfied at the other because these points are identified on contiguous spheres (so, effectively we use only one boundary condition). Let us write $\tilde{t} = iT$, this places us on the Euclidean section; then (9.8) becomes

$$E_A = \frac{(\mu + \frac{1}{2})}{2a} \frac{c_1^2 e^{-(2\mu+1)T} + c_2^2 e^{(2\mu+1)T}}{c_1^2 e^{-(2\mu+1)T} - c_2^2 e^{(2\mu+1)T}} \,. \tag{9.10}$$

Now $\frac{1}{2}(\mu + \frac{1}{2})$ is the harmonic-oscillator ground-state energy, so to satisfy the requirement that the energy remain finite at either $T = +\infty$ or $-\infty$ we need to choose $c_2 = 0$ (as before, we have chosen c_1/c_2 to be real). Alternatively, we could have evaluated the energy on the Lorentzian

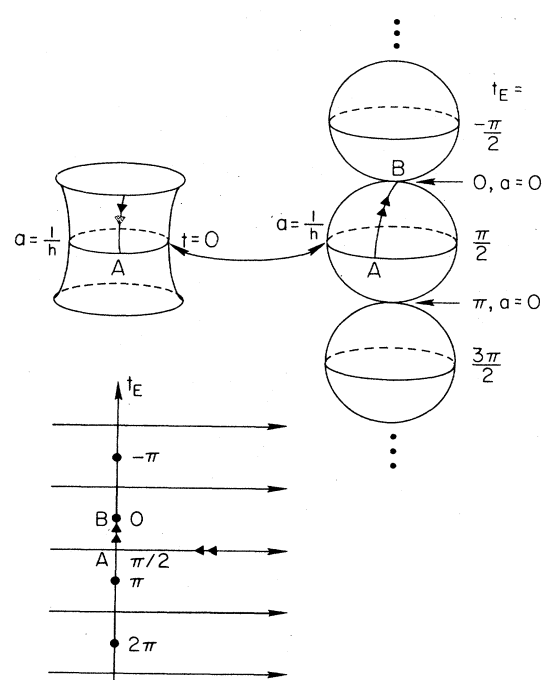

FIG. 3. The Lorentzian and Euclidean sections of de Sitter spacetime. The set of axes represent the complex time plane; the vertical axis is imaginary "time" t_E, the horizontal axes are real time.

section and then required it be finite as $a \to 0$. The energy on the Lorentzian section is

$$E_A = \frac{(\mu+\frac{1}{2})}{2a}\frac{c_1^2+c_2^2}{c_1^2-c_2^2}, \tag{9.11}$$

and so, as before, we need $c_2=0$.

The minimally coupled case in $k=+1$ coordinates can be analyzed in a similar manner. The expressions for the $R^{(i)\beta}_{\alpha}$ need to be analytically continued from $(-1,1)$ to $(-\infty,\infty)$. Evaluating the energy on the Lorentzian section, we see that $c_2=0$ keeps the field energy finite as $a \to 0$. Equivalently, with $c_2=0$, $f(\tilde{t})$ given by (7.4) approaches the conformally coupled scalar field's $f(\tilde{t})=(\mu+\frac{1}{2})$ in this limit.

We now argue that the initial condition we have described above reduces to the boundary condition of regularity on the Euclidean section proposed by Hawking. The analytic continuation (2.14) leads to what may be considered to be an infinite set of $(n+1)$-dimensional Euclidean de Sitter spheres with contiguous North and South poles identified; see Fig. 3 (it should also be possible to identify the spheres and hence replace the infinite set by one sphere; this is not important). The waist of the hyperboloid is the equator of the sphere.

Consider a trajectory which comes in from $\tilde{t}=\pi/2$ $(a=\infty)$ on the hyperboloid and goes to $\tilde{t}=0$ $(a=1/h)$—this point, $\tilde{t}=0$, will be at the intersection of a real time axis and the Euclidean time axis in the complex time plane. If we now analytically continue to Euclidean time, this is equivalent to the trajectory moving off the equator on the Euclidean sphere towards either the North or South pole, depending on which way we move along the Euclidean time axis. The requirement that the energy be finite as $a \to 0$ then corresponds to including in the functional integral only those field configurations which are regular on the Euclidean section (in particular we discard field configurations which are singular at the poles).

It is instructive to discuss the approaches of Ref. 2 to field theory in de Sitter spacetime. Chernikov and Tagirov have studied the conformally coupled scalar field in $k=+1$ coordinates. They use the Heisenberg representation and exhibit normal mode expansions for the field operators. Since they have not used an initial condition when solving the equation of motion, they find a one parameter set of vacuums, which they show are invariant under the de Sitter group. They then use the correspondence principle to argue that particles with large momenta must travel on geodesics and so choose a particular vacuum in which particles behave appropriately in this limit.

Burges, on the other hand, argues that the massless minimally coupled scalar field's vacuum wave functional must be de Sitter invariant and proceeds to construct generators which should annihilate it. His arguments seem incomplete, for reasons which we now discuss. We can write

$$\tilde{\mathscr{H}}_{MC}=\tilde{\mathscr{H}}_{CC}-\frac{\chi^2 h^2 a^2}{2}(\nu^2-\tfrac{1}{4}), \tag{9.12}$$

so we see that the minimally coupled Hamiltonian $\tilde{\mathscr{H}}_{MC}$, for a particular mode, describes the quantum mechanics of a particle in a time-dependent potential. In fact, as the expansion proceeds, the scale factor a grows and the time-dependent term soon dominates the χ^2 term in $\tilde{\mathscr{H}}_{CC}$; because of the relative minus sign, the time-dependent term corresponds to an inverted harmonic-oscillator potential. The time evolution of this system is easily visualized; the equivalent quantum-mechanical particle oscillates in a harmonic oscillator well which starts flattening out. Eventually the potential turns over and the particle is now in a position of unstable equilibrium. In a time-dependent potential like this, the wave function does not factorize into a part that depends only on time and a part that depends only on the field—clearly the frequency of the equivalent harmonic-oscillator ground state is time dependent.

Burges requires that the symmetry generators annihilate the vacuum state wave functional on the $\tilde{t}=0$ hypersurface. If the wave function describes a system of harmonic oscillators, with a time-independent frequency, this assumes the existence of a normal-ordering prescription. For example, consider the total Hamiltonian on ψ_0; using the Schrödinger equation, we may reduce this to a time-independent problem by replacing $i\,\partial/\partial t$ with the total ground-state energy (which is infinite). We may then consider this equivalent to requiring that the normal-ordered Hamiltonian annihilate the time-independent part of the wave function. However, if the wave function describes a system of harmonic oscillators with a time-dependent frequency, we cannot reduce the problem to a time-independent one and therefore do not have a normal-ordering prescription. Even if we only consider the equations at $\tilde{t}=0$ we need to be able to normal order. Notice that in $(1+1)$ dimensions the massless minimally coupled case has exactly the same Hamiltonian as the conformally coupled case, i.e., $\nu=\frac{1}{2}$; hence the wave function describes a system of harmonic oscillators with a time-independent frequency and so it can be separated.

X. THE RESTORATION OF CONTINUOUS SYMMETRIES

We study the restoration of continuous symmetry in de Sitter spacetime by considering an interacting scalar field theory which has a broken-symmetry phase; Goldstone's original example,[21] a complex scalar field Φ in a Φ^4 potential, lends itself readily to analysis. The Lagrangian density is

$$\mathscr{L}=\tfrac{1}{2}\partial_\mu\Phi\partial^\mu\Phi^* - V(\Phi\Phi^*), \tag{10.1}$$

where the potential

$$V(\Phi\Phi^*)=\frac{\mu_0^2}{2}(\Phi\Phi^*)+\frac{\lambda_0}{4}(\Phi\Phi^*)^2$$

has an O(2) symmetry. For $\lambda_0>0$, $\mu_0^2<0$ we find the conventional symmetry-breaking potential. The Euler-Lagrange equation

$$(\partial_\mu\partial^\mu+\mu_0^2)\Phi+\lambda_0\Phi^3=0 \tag{10.2}$$

then has stable minima at

$$|\Phi|=\rho=\left[\frac{-\mu_0^2}{\lambda_0}\right]^{1/2}. \tag{10.3}$$

The Goldstone modes of this theory are the massless excitations along the circle $|\Phi|=\rho$. These are spin-wave excitations which do not cost energy (which is proportional to gradients) since only the direction, and not the magnitude, of the field Φ changes. If we are interested in the low-energy behavior of this theory we need only consider these modes. We can, hence, approximate $\Phi(x)=\rho(x)e^{i\theta(x)}$ by $\Phi(x)=\rho e^{i\theta(x)}$ where $\theta\in(-\infty,\infty)$. From the previous Lagrangian we get the new Lagrangian that determines the equation of motion of the real field $\theta(x)$ which lives on the circle:

$$\mathscr{L}=\frac{\rho^2}{2}\partial_\mu\theta\partial^\mu\theta \; . \tag{10.4}$$

Thus the field $\phi(x)\equiv\rho\theta(x)$, which is essentially the field on the circle, satisfies a minimally coupled Klein-Gordon equation, as it must; any other term would break the U(1) symmetry (translational invariance in θ space). To study symmetry restoration we need to look at correlation functions such as

$$\langle\Phi(\mathbf{x})\Phi^*(\mathbf{x}')\rangle=\rho^2\left\langle\exp\left[\frac{i\phi(\mathbf{x})}{\rho}\right]\exp\left[\frac{-i\phi(\mathbf{x}')}{\rho}\right]\right\rangle . \tag{10.5}$$

If this correlation function asymptotically tends to zero for very large physical separation the theory is in a symmetric phase; if it asymptotically tends to a constant (>0) then the theory is in a Goldstone phase.

Now

$$\langle\Phi(\mathbf{x})\Phi^*(\mathbf{x}')\rangle=Z\rho^2\exp\left[\frac{\langle\phi(\mathbf{x})\phi(\mathbf{x}')\rangle}{\rho^2}-\frac{\langle\phi(\epsilon)\phi(0)\rangle}{\rho^2}\right], \tag{10.6}$$

where we have regulated the object $\langle\phi^2(0)\rangle$ by point splitting at equal time. This expression is ultraviolet singular; hence we need to renormalize it: Z is a renormalization constant chosen in such a manner that the correlation function $\langle\Phi(\mathbf{x})\Phi^*(\mathbf{x}')\rangle=1$ at a physical separation $|\mathbf{x}-\mathbf{x}'|a(t)=l$, where l is much less than a Hubble radius. With this definition

$$\langle\Phi(\mathbf{x})\Phi^*(\mathbf{x}')\rangle =\exp\left[\frac{\langle\Phi(\mathbf{x})\phi(\mathbf{x}')\rangle}{\rho^2}-\left.\frac{\langle\phi(\mathbf{x})\phi(\mathbf{x}')\rangle}{\rho^2}\right|_{|\mathbf{x}-\mathbf{x}'|a(t)=l}\right] . \tag{10.7}$$

To study symmetry restoration, we would like to evaluate this expression in the limit $|\mathbf{x}-\mathbf{x}'|a(t)\to\infty$. Now from Eqs. (E6) and (E12) we see that the part of the exponent which depends on $X=(har/2)^2$ in the appropriate limit is just

$$-\frac{h^{n-1}}{(4\pi)^{(n+1)/2}\rho^2}\frac{\Gamma(n)}{\Gamma((n+1)/2)}\ln X \; . \tag{10.8}$$

Therefore,

$$\lim_{|\mathbf{x}-\mathbf{x}'|a(t)\to\infty}\langle\Phi(\mathbf{x})\Phi^*(\mathbf{x}')\rangle\approx\left[\frac{2}{h\,|\mathbf{x}-\mathbf{x}'|a(t)}\right]^\alpha , \tag{10.9}$$

where

$$\alpha=\frac{h^{n-1}\Gamma(n/2)}{2\pi^{(n+2)/2}\rho^2} \; . \tag{10.10}$$

So, the correlation function asymptotically approaches zero (as a power) for very large physical separations.

A related indication of symmetry restoration is[9]

$$\begin{aligned}\langle\Phi(\mathbf{x})\rangle&=\left\langle\exp\left[\frac{i\phi(\mathbf{x})}{\rho}\right]\right\rangle\\&=\exp\left[-\frac{1}{2\rho^2}\langle\phi^2(\mathbf{x})\rangle\right]\\&=\exp\left[-\frac{h^{n-1}\Gamma(n/2)}{4\pi^{(n+2)/2}\rho^2}\ln\left[\frac{\kappa_u a(t)}{\kappa_i a(t_i)}\right]\right]\end{aligned} \tag{10.11}$$

or $\langle\Phi(\mathbf{x})\rangle\to 0$ as $\kappa_i\to 0$, so as we remove the infrared cutoff, the expectation value of the field vanishes. For $n=1$ and $h=0$ we recover the massless $(1+1)$-dimensional flat-spacetime result[9]

$$\langle\Phi(\mathbf{x})\rangle=\exp\left[-\frac{1}{4\pi\rho^2}\ln\left[\frac{\kappa_u}{\kappa_i}\right]\right] . \tag{10.12}$$

The restoration of continuous global symmetries by anomalously large correlations in the infrared is a well-known phenomena in lower-dimensional field theories and spin systems in flat spacetime. For a nice discussion of the physics involved, see Ma and Rajaraman.[9] We briefly review some of the points discussed in their paper. It is clear that symmetry restoration is a quantum-mechanical phenomena. Quantum fluctuations (zero-point motion) usually lead to spreading of the wave function about a classically allowed trajectory; if they are large enough then no trace of the classical trajectory remains. Clearly this is what happens to the field θ which lives on the circle; a logarithmic infrared divergence in its two-point function just means that there are many paths in the space of θ's connecting two points; some of these will subtend an angle that is equal to the difference between the two points plus an integral multiple (which could even be infinite) of 2π. These correlations wipe out the classical minimum, which is at some fixed value of θ on the circle.

It must be stressed that the zero mode on the n-sphere, which is present both in the infrared and the ultraviolet (see Appendix E), is not responsible for the spreading of the wave functional in field space. Symmetry restoration is a direct consequence of the infrared logarithm in the propagator.

A few comments are in order; as we go up in dimension α decreases (for fixed h^{n-1}/ρ^2). This means that $\langle\Phi(\mathbf{x})\Phi^*(\mathbf{x}')\rangle$ for large separations dies more slowly in higher dimensions; which is what we expect. The logarithmic divergences present in scalar field theory in $(n+1)$-dimensional de Sitter spacetime are very similar to those in $2+1$ flat-space finite-temperature field theory or $1+1$ zero-temperature field theory. However, these divergences do not seem to be like finite-temperature divergences, because field theory at finite temperature can effectively be identified with zero-temperature field theory in the same total number of dimensions but with the time dimension curled up. So, as far as the infrared divergences of the theory are concerned, the number of dimensions has been effectively reduced by one, and not by $n-2$ as seems to be the case in de Sitter space.

XI. DISCUSSION

The functional Schrödinger approach to field theory has proved to be both intuitively and technically useful for analyzing quantum field theory in curved spacetime. Although we have only considered scalar field theory, our results may easily be extended to allow analysis of nonzero-spin fields in de Sitter space. Similar analysis may also prove useful for understanding field theory in other backgrounds of cosmological and astrophysical interest. We are now investigating scalar field theory in matter- and radiation-dominated FRW cosmologies using these methods.

We have seen that spontaneously broken symmetries are dynamically restored in de Sitter space. Although it is clear that this is caused by an infrared divergence in the propagator, it is not obvious why the propagator diverges logarithmically for large physical separation—independent of the number of dimensions. It is tempting to try to identify the Hawking effect as the cause for this symmetry restoration, but this identification does not seem to be correct. This is primarily because from finite-temperature field theory, we know that the infrared properties of an $(n+1)$-dimensional finite-temperature field theory are the same as the n-dimensional zero-temperature version of the theory. Here it seems that the $(n+1)$-dimensional field theory in a gravitational background is very similar to $(1+1)$-dimensional Minkowski-space zero-temperature field theory. Furthermore, we know, from the analysis of Shore,[27] that discrete symmetries in de Sitter space do not seem to be as drastically affected. Finally, similar analysis in other metrics, in particular the Schwarzschild metric, do not seem to reinforce this interpretation; in fact, this phenomena may be peculiar to de Sitter spacetime. Alternatively, this could be interpreted as being inconsistent with the conventional identification of field theory in nontrivial backgrounds and at finite temperature.

Whether this effect has any consequences for the inflationary scenario remains to be seen. In the inflationary scenario one can conceive of an earlier FRW phase effectively acting as an infrared cutoff. However, as we have seen, the de Sitter evolution will generate infrared divergences, on a characteristic time scale of the order of the Hubble time. It would be interesting to see if familons[28] would be affected by this phenomena and if so, whether these effects would survive reheating.

To show that a broken continuous symmetry is restored we have considered the simplest possible case, a broken U(1) symmetry. In flat spacetime $[(2+1)$ dimensions, finite temperature$]$ McBryan and Spencer[29] have shown that the two-point correlation function for the field Φ [with a U(1) symmetry] can be used as a bound for two-point functions of O(N) nonlinear σ models, and so if a U(1) symmetry is restored so will an O(N) symmetry. We expect that the behavior of the U(1) will also bound the O(N) case here.

The functional Schrödinger formalism readily permits an analysis of the uniqueness of the vacuum wave functional. We have shown that the coefficient $f(\tilde{t})$ of χ^2, in the exponent of the wave functional, satisfies a first-order nonlinear differential equation which can be transformed into a second-order linear differential equation. This has two linearly independent solutions, but the transformation connecting f to the general solution is such that only the ratio of the constants is important; hence, f depends on one constant (this is because the Schrödinger equation is first order in time) whose value we must determine.

We find no substantial difference between the uniqueness of this wave functional and the equivalent one in Minkowski spacetime. The main difference between these two wave functionals lies in the interpretation of the initial conditions imposed. In Minkowski space one can but does not have to invoke regularity on the Euclidean section. In de Sitter space, in $k=0$ coordinates, a real Euclidean section does not exist; however, we may impose as the initial condition the requirement that the field energy remain finite as $a \to 0$ ($\tilde{t} \to -\infty$). When we try to do a similar thing in $k=+1$ coordinates, we find that we end up with Hawking's[1] prescription because $a \to 0$ (in fact, all $a < 1/h$) lies in the Euclidean section of the manifold. Thus Hawking's prescription for the semiclassical case may be interpreted, physically, as a special case of the requirement that the field energy remain finite as $a \to 0$. This interpretation could, perhaps, be extended to the fully quantum-mechanical case; certainly it is correct if we consider the metric fluctuation as just another quantum field propagating in the background metric.

It seems conceivable that this formalism (along with the initial/boundary condition prescription) can be used to resolve the problem of the correct vacuum state (mode expansion)[30] for those spacetimes to which it is applicable. Particle production manifests itself in the time dependence of the vacuum wave functional.[31] Perhaps the major advantage of such an approach is that it allows one to utilize physical intuition developed solving quantum-mechanical problems. Also one need solve a first-order (in time) differential equation instead of the Klein-Gordon equation; hence, we require only one initial condition.

We have also succeeded in finding a creation operator that allows us to explicitly construct the excited state wave functionals from the ground-state wave functional. We hope to discuss this and some other topics, in particular, the behavior of nonlinear σ models and discrete symmetries in de Sitter spacetime, in the future.

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APPENDIX A: EVALUATION OF THE INTEGRAL (5.23)

We use

$$J_\nu{}^2(k\tilde{t})+Y_\nu{}^2(k\tilde{t})=\frac{2}{\pi^2}\int_0^\infty \frac{dy}{y}\exp\left[\frac{-k^2}{2y}+\tilde{t}^2 y\right]K_\nu(\tilde{t}^2 y) \tag{A1}$$

(Ref. 23, p. 94), to rewrite the integral in (5.23) as a double integral and interchange orders of integration (all integrals are convergent) to obtain

$$\frac{2}{\pi^2}\int_0^\infty \frac{dy}{y}e^{\tilde{t}^2 y}K_\nu(\tilde{t}^2 y)\int_0^\infty dk\, k^{n/2}e^{-k^2/2y}J_{(n-2)/2}(kr) \; . \tag{A2}$$

We can evaluate the second integral, using Eq. (5.9) of Ref. 24. We find

$$\frac{2}{\pi^2}r^{n/2-1}\int_0^\infty dy\, y^{n/2-1}K_\nu(\tilde{t}^2 y)\exp\left[\tilde{t}^2 y-\frac{r^2}{2}y\right] . \tag{A3}$$

Then Eq. (3.31) of Ref. 24 gives us

$$\left[\frac{2}{\pi^3\tilde{t}^2 r}\right]^{1/2}\Gamma\left[\frac{n}{2}-\nu\right]\Gamma\left[\frac{n}{2}+\nu\right]\left[\tilde{t}^2-\frac{r^2}{4}\right]^{(1-n)/4} \times P_{\nu-1/2}^{(1-n)/2}\left[\frac{r^2}{2\tilde{t}^2}-1\right] . \tag{A4}$$

So

$$\langle\phi(\mathbf{x})\phi(\mathbf{x}')\rangle=\frac{1}{4\pi^{(n+1)/2}a^{n-1}r^{(n-1)/2}} \times\frac{\Gamma(n/2-\nu)\Gamma(n/2+\nu)}{(4\tilde{t}^2-r^2)^{(n-1)/4}} \times P_{\nu-1/2}^{(1-n)/2}\left[\frac{r^2}{2\tilde{t}^2}-1\right] . \tag{A5}$$

Or using Eq. (6) on p. 143 of Ref. 25, we find

$$\langle\phi(\mathbf{x})\phi(\mathbf{x}')\rangle=\frac{h^{n-1}}{(4\pi)^{(n+1)/2}}\frac{\Gamma(n/2-\nu)\Gamma(n/2+\nu)}{\Gamma((n+1)/2)} \times F\left[-\nu+\frac{n}{2},\nu+\frac{n}{2};\left[\frac{n+1}{2}\right];1-\frac{r^2}{4\tilde{t}^2}\right] . \tag{A6}$$

APPENDIX B: GENERALIZED SPHERICAL HARMONICS

The n-dimensional spherical harmonics are the eigenfunctions of $\mathscr{L}^2_{(n)}$, the Laplacian on the unit sphere S^n [for coordinatization and metric see (2.15)]:

$$\begin{aligned}\mathscr{L}^2_{(n)}=\frac{1}{|S^{(n)}|^{1/2}}\partial_i\,|S^{(n)}|^{1/2}S^{(n)ij}\partial_j&=\frac{1}{\sin^{n-1}\theta_n}\frac{\partial}{\partial\theta_n}\sin^{n-1}\theta_n\frac{\partial}{\partial\theta_n}\\&+\frac{1}{\sin^2\theta_n\sin^{n-2}\theta_{n-1}}\frac{\partial}{\partial\theta_{n-1}}\sin^{n-2}\theta_{n-1}\frac{\partial}{\partial\theta_{n-1}}+\cdots\\&+\frac{1}{\sin^2\theta_n\sin^2\theta_{n-1}\cdots\sin^2\theta_2}\frac{\partial^2}{\partial\theta_1{}^2} ,\end{aligned} \tag{B1}$$

where ∂_i stands for the derivative with respect to the coordinate θ_i. The n-indexed $Y_{AW}(\Omega)$ (W stands for the collection of "magnetic" indices $B,C,\ldots,$ which run over the integers $[-A,A],[-B,B],\ldots,$ respectively) are defined by the following equations:

$$\mathscr{L}^2_{(n)}Y_{AW}(\Omega)=\lambda_A Y_{AW}(\Omega) , \tag{B2}$$

where the O$(n+1)$ symmetry makes the eigenvalues independent of all but A; and

$$\int d\Omega\,|Y_{AW}(\Omega)|^2=1 . \tag{B3}$$

We can find λ_A by studying the case where $W=0$. Equation (B2) gives us

$$\left[\frac{1}{\sin^{n-1}\theta_n}\frac{\partial}{\partial\theta_n}\sin^{n-1}\theta_n\frac{\partial}{\partial\theta_n}\right]Y_{A0}(\Omega)=\lambda_A Y_{A0}(\Omega) . \tag{B4}$$

The substitution $x=\cos\theta_n$ reduces this to

$$\left[(1-x^2)\frac{\partial^2}{\partial x^2}-nx\frac{\partial}{\partial x}\right]Y_{A0}(x)=\lambda_A Y_{A0}(x) , \tag{B5}$$

which is just the Gegenbauer equation [Eq. (22.6.5) of Ref. 22]. So $\lambda_A=-A(A+n-1)$ and $Y_{A0}(x)=c_1C_A^{(n-1)/2}(x)$. To determine the constant c_1 we make use of the orthonormality of the Y's; using Eq. (22.2.3) of Ref. 22 we find

$$Y_{A0}(\Omega)=\left[\frac{A!(2A+n-1)\left[\Gamma\left[\frac{n-1}{2}\right]\right]^2\Gamma\left[\frac{n}{2}\right]}{2^{4-n}\pi^{(n+2)/2}\Gamma(A+n-1)}\right]^{1/2} \times C_A^{(n-1)/2}(\cos\theta_n) . \tag{B6}$$

Using Eq. (22.3.12) of Ref. 22 we obtain

$$Y_{00}(\Omega)=\left[\frac{\Gamma\left[\frac{n+1}{2}\right]}{2\pi^{(n+1)/2}}\right]^{1/2} . \tag{B7}$$

Explicit forms for the $C_m^{(\alpha)}(x)$ may be obtained from the generating function [Ref. 22, Eq. (22.9.3)]

$$(1-2xz+z^2)^{-\alpha}=\sum_{m=0}^{\infty} z^m C_m^{(\alpha)}(x) \ . \tag{B8}$$

The addition formula is

$$C_A^{(n-1)/2}(\cos\gamma)=c_2\sum_W Y_{AW}^*(\Omega)Y_{AW}(\Omega') \ , \tag{B9}$$

where γ is the angle between Ω and Ω' and c_2 is a constant which we must determine. The right-hand side is invariant under rotations, so we can rotate Ω' to the North pole:

$$Y_{AW}(\Omega'=\text{North pole})=\delta_{W,0}\left[\frac{A!(2A+n-1)\left[\Gamma\left[\frac{n-1}{2}\right]\right]^2\Gamma\left[\frac{n}{2}\right]}{2^{4-n}\pi^{(n+2)/2}\Gamma(A+n-1)}\right]^{1/2} C_A^{(n-1)/2}(1) \ . \tag{B10}$$

Using

$$C_A^{(n-1)/2}(1)=\frac{(A+n-2)!}{A!(n-2)!} \ , \tag{B11}$$

we eventually obtain

$$C_A^{(n-1)/2}(\cos\gamma)=\frac{4\pi^{(n+1)/2}}{(2A+n-1)\Gamma\left[\frac{n-1}{2}\right]}\sum_W Y_{AW}^*(\Omega)Y_{AW}(\Omega') \ . \tag{B12}$$

We are now in a position to expand $|\mathbf{x}-\mathbf{y}|^{-(n-1)}$ in spherical harmonics. Let $|\mathbf{x}|\geq|\mathbf{y}|$; then

$$\begin{aligned}\frac{1}{|\mathbf{x}-\mathbf{y}|^{n-1}}&=\frac{1}{x^{n-1}}\sum_{A=0}^{\infty}\left[\frac{y}{x}\right]^A C_A^{(n-1)/2}(\cos\gamma)\\ &=\frac{4\pi^{(n+1)/2}}{\Gamma\left[\frac{n-1}{2}\right]}\sum_{AW}\frac{y^A}{x^{A+n-1}}\frac{Y_{AW}^*(\Omega)Y_{AW}(\Omega')}{(2A+n-1)} \ .\end{aligned} \tag{B13}$$

APPENDIX C: EVALUATION OF THE SUM (7.14)

Using Eqs. (18), p. 144, (13), p. 141, and (2), p. 143 of Ref. 25, we can write

$$[P_{A+n/2-1}^{\nu}(\sin\tilde{t})]^2+\frac{4}{\pi^2}[Q_{A+n/2-1}^{\nu}(\sin\tilde{t})]^2=\frac{2}{\pi\cos\tilde{t}}\left[\Gamma\left[A+\frac{n}{2}+\nu\right]\right]^2 P_{\nu-1/2}^{-A-(n-1)/2}(i\tan\tilde{t})P_{\nu-1/2}^{-A-(n-1)/2}(-i\tan\tilde{t}) \ . \tag{C1}$$

Now Eq. (10) on p. 140 of Ref. 23 allows us to replace $P_{\nu-1/2}^{-A-(n-1)/2}(-i\tan\tilde{t})$ with a linear combination of $P_{\nu-1/2}^{-A-(n-1)/2}(i\tan\tilde{t})$ and $Q_{\nu-1/2}^{-A-(n-1)/2}(i\tan\tilde{t})$. Then using equations on p. 179 of Ref. 21 (the expansion for $Q_\nu^\mu=\mathscr{D}_\nu^\mu$, has an extra factor of $e^{-i\pi\mu}$ and of $e^{-i2\pi\mu}$, dropping both of these corrects it) reduces (7.15) to

$$\begin{aligned}\langle\phi(\mathbf{x})\phi(\mathbf{x}')\rangle=&\frac{h^{n-1}}{(4\pi)^{(n+1)/2}}\Gamma\left[\frac{n}{2}+\nu\right]\Gamma\left[\frac{n}{2}-\nu\right]\left[\frac{h^2\sigma^2}{4}\right]^{(1-n)/4}\\ &\times\left[1+\frac{h^2\sigma^2}{4}\right]^{(1-n)/4}\left[e^{-i(\nu-1/2)\pi}P_{\nu-1/2}^{(1-n)/2}\left[1+\frac{h^2\sigma^2}{2}\right]\right.\\ &\left.+2e^{i[(n-1)/2]\pi}\sin\pi\left[\frac{n}{2}-\nu\right]Q_{\nu-1/2}^{(1-n)/2}\left[1+\frac{h^2\sigma^2}{2}\right]\right] \ ,\end{aligned} \tag{C2}$$

where the sign of the phase of the coefficient of $P_{\nu-1/2}^{(1-n)/2}$ is determined by the fact that this is the equal-time limit of the Feynman propagator:

$$\mathrm{Im}\left[1+\frac{h^2\sigma^2}{2}\right]=\lim_{\tilde{t}\to\tilde{t}'}\mathrm{Im}\left[\frac{-1}{a(t)a(t')}[r^2-(-\tilde{t}+\tilde{t}'-i\epsilon)^2]\right]<0 \ .$$

Then Eq. (10) on p. 140 of Ref. 25 allows us to write this as

$$\langle \phi(\mathbf{x})\phi(\mathbf{x}')\rangle = \frac{h^{n-1}}{(4\pi)^{(n+1)/2}}\Gamma\left[\frac{n}{2}+\nu\right]\Gamma\left[\frac{n}{2}-\nu\right]\left[\frac{h^2\sigma^2}{4}\right]^{(1-n)/4}\left[1+\frac{h^2\sigma^2}{4}\right]^{(1-n)/4} \times P_{\nu-1/2}^{(1-n)/2}\left[-\left[1+\frac{h^2\sigma^2}{2}\right]\right] \tag{C3}$$

and finally, using Eq. (15.4.18) of Ref. 22, we have

$$\frac{h^{n-1}}{(4\pi)^{n+1}}\frac{\Gamma(n/2+\nu)\Gamma(n/2-\nu)}{\Gamma((n+1)/2)}F\left[\frac{n}{2}+\nu,\frac{n}{2}-\nu;\frac{n+1}{2};1+\frac{h^2\sigma^2}{4}\right]. \tag{C4}$$

APPENDIX D: EVALUATION OF THE SUM (8.18)

In this appendix we establish Eqs. (8.15) and (8.18). Let

$$x=\cos\gamma_{n+1}\,, \tag{D1}$$

$$F_1(x)=\frac{h^{n-1}\Gamma(n)f_1(x)}{(4\pi)^{(n+1)/2}\Gamma((n+1)/2)}=\frac{h^{n-1}\Gamma(n/2)}{4\pi^{(n+2)/2}}\sum_A\frac{(2A+n)}{(A+n/2+\nu)(A+n/2-\nu)}C_A^{(n/2)}(x)\,, \tag{D2}$$

and

$$F_2(x)=\frac{h^{n-1}\Gamma(n)f_2(x)}{(4\pi)^{(n+1)/2}\Gamma((n+1)/2)}=\frac{h^{n-1}\Gamma(n/2-\nu)\Gamma(n/2+\nu)}{(4\pi)^{(n+1)/2}\Gamma((n+1)/2)}F\left[\frac{n}{2}+\nu,\frac{n}{2}-\nu;\frac{n+1}{2};\frac{1+x}{2}\right]. \tag{D3}$$

We shall now establish that $F_1(x)$ and $F_2(x)$ satisfy the same linear second-order differential equation, and the same boundary conditions:

$$F_1(x_0)=F_2(x_0)$$

and

$$\left.\frac{d}{dx}F_1(x)\right|_{x=x_0}=\left.\frac{d}{dx}F_2(x)\right|_{x=x_0}. \tag{D4}$$

It then follows that $F_1(x)=F_2(x)$ (see, for instance, Whittaker and Watson,[26] Sec. 10.21).

Clearly, $F_2(x)$ satisfies the hypergeometric equation,

$$HF_2(x)=0\,, \tag{D5}$$

where

$$H\equiv(1+x)(1-x)\frac{d^2}{dx^2}-(n+1)x\frac{d}{dx}-\left[\frac{n}{2}+\nu\right]\left[\frac{n}{2}-\nu\right]. \tag{D6}$$

Now

$$HF_1(x)=-\frac{h^{n-1}\Gamma(n/2)}{4\pi^{(n+2)/2}}\sum_A(2A+n)C_A^{(n/2)}(x)\,, \tag{D7}$$

where we have made use of Gegenbauer's equation. We may use relations between Gegenbauer polynomials (Ref. 25, p. 178) to express (D7) as

$$HF_1(x)=\frac{nh^{n-1}\Gamma(n/2)}{4\pi^{(n+2)/2}}[C_{-2}^{(n+2)/2}(x)+C_{-1}^{(n+2)/2}(x)]\,. \tag{D8}$$

It is easy to show that Gegenbauer functions with negative integral subscripts vanish and, hence,

$$HF_1(x)=0\,. \tag{D9}$$

Now using Eq. (15.2.1) of Ref. 22, we have

$$\frac{d}{dx}F_2(x)_n=\frac{2\pi}{h^2}F_2(x)_{n+2}\,, \tag{D10}$$

where the second subscript on F indicates the value of n wherever it appears in the expression for $F_2(x)$, except in the $\cos\gamma_{n+1}$ term, which remains unchanged. Similarly [Eq. (30) on p. 178 of Ref. 25] we find

$$\frac{d}{dx}F_1(x)_n=\frac{2\pi}{h^2}F_1(x)_{n+2}\,. \tag{D11}$$

We shall now show that the two functions satisfy the same boundary conditions. It is convenient to work with $f_1(x)$ and $f_2(x)$. The series representation of $f_1(x)$ [see (D2)] becomes relatively simple at $x=\pm1$ and 0; we consider the case $x=-1$. Then Eq. (22.4.2) of Ref. 22 allows us to rewrite this as

$$f_1(-1)=\frac{1}{\Gamma(n)}\sum_A\left[\frac{1}{A+n/2+\nu}+\frac{1}{A+n/2-\nu}\right]\times\frac{\Gamma(A+n)}{A!}(-1)^A\,, \tag{D12}$$

and from Eq. (15.3.1) of Ref. 22 we find

$$f_2(-1)=\frac{1}{\Gamma(n)}\Gamma\left[\frac{n}{2}-\nu\right]\Gamma\left[\frac{n}{2}+\nu\right]. \tag{D13}$$

Using the integral representation of $\Gamma(A+n)$, and interchanging the order of integration and summation (the integral and sum are convergent) we may rewrite (D12) as

$$\frac{1}{\Gamma(n)}\int^{\infty} dy\, e^{-y} y^{n-1} \times \sum_{A=0}^{\infty}\left[\frac{1}{A+n/2+\nu}+\frac{1}{A+n/2-\nu}\right]\frac{(-y)^A}{A!}\ . \tag{D14}$$

Now, both of the series in (D14) are related to incomplete gamma functions [see Ref. 22, Eq. (6.5.29)], so we find

$$f_1(-1)=\frac{1}{\Gamma(n)}\int_0^{\infty} dy\, e^{-y} y^{n/2-1}\left[y^{-\nu}\gamma\left[\frac{n}{2}+\nu,y\right]+y^{\nu}\gamma\left[\frac{n}{2}-\nu,y\right]\right]\ . \tag{D15}$$

Using the integral representation of the incomplete gamma function we then obtain

$$f_1(-1)=\frac{1}{\Gamma(n)}\int_0^{\infty} dy\int_0^{y} dt\, e^{-y}e^{-t}y^{n/2-1}\times t^{n/2-1}(t^{\nu}y^{-\nu}+y^{\nu}t^{-\nu})\ . \tag{D16}$$

The integrand of this double integral is symmetric in y and t; hence, we can extend the upper limit on the second integral to ∞ while simultaneously dividing by 2:

$$f_1(-1)=\frac{1}{\Gamma(n)}\int_0^{\infty} dy\int_0^{\infty} dt\, e^{-y}e^{-t}y^{n/2-\nu-1}t^{n/2+\nu-1} =\frac{\Gamma(n/2+\nu)\Gamma(n/2-\nu)}{\Gamma(n)}=f_2(-1)\ . \tag{D17}$$

So, we have established $F_1(-1)=F_2(-1)$. Then from (D10) and (D11) we see

$$\left.\frac{d}{dx}F_1(x)\right|_{x=-1}=\left.\frac{d}{dx}F_2(x)\right|_{x=-1}\ . \tag{D18}$$

Thus we have established

$$F_1(x)=F_2(x)\ . \tag{D19}$$

The sum that we need to evaluate in the conformal case (8.15) may be obtained from the general result by considering the value $\nu=\frac{1}{2}$ and simplifying the hypergeometric function.

APPENDIX E: APPROXIMATE GREEN'S FUNCTIONS

In this appendix we develop two power-series expansions of the minimally coupled scalar field's equal time Green's function, one valid for large, the other for small, separations. For large separations we would find a power series in $(1/r)$ helpful. Using Eq. (15.3.8) of Ref. 22, we have

$$\frac{\Gamma(a)\Gamma(b)}{\Gamma(c)}F[a,b;c;1-z]=\frac{\Gamma(a)}{\Gamma(c-a)\Gamma(1-b+a)}\frac{\pi}{\sin\pi(b-a)}z^{-a}F[a,c-b;a-b+1;z^{-1}] +\frac{\Gamma(b)}{\Gamma(c-b)\Gamma(1-a+b)}\frac{\pi}{\sin\pi(a-b)}z^{-b}F[b,c-a;b-a+1;z^{-1}]\ . \tag{E1}$$

Defining $X=(h^2a^2r^2)/4$, we can convert Eq. (5.24) to the form

$$\langle\phi(\mathbf{x})\phi(\mathbf{x}')\rangle=\frac{h^{n-1}}{(4\pi)^{(n+1)/2}}\frac{\pi}{X^{n/2}}\left[\frac{\Gamma(n/2-\nu)X^{\nu}}{\Gamma(\frac{1}{2}+\nu)\Gamma(1-2\nu)\sin(2\pi\nu)}F\left[-\nu+\tfrac{1}{2},\frac{n}{2}-\nu;1-2\nu;X^{-1}\right] -\frac{\Gamma(n/2+\nu)X^{-\nu}}{\Gamma(\frac{1}{2}-\nu)\Gamma(1+2\nu)\sin(2\pi\nu)}F\left[\nu+\tfrac{1}{2},\frac{n}{2}+\nu;1+2\nu;X^{-1}\right]\right]\ . \tag{E2}$$

We then use the power-series expansion for the hypergeometric function to write this as

$$\frac{h^{n-1}}{(4\pi)^{n+1/2}}\frac{1}{2\sin(\pi\nu)X^{n/2}}\left[X^{\nu}\sum_{p=0}^{\infty}\frac{\Gamma(\frac{1}{2}-\nu+p)\Gamma(n/2-\nu+p)X^{-p}}{\Gamma(1-2\nu+p)p!}-X^{-\nu}\sum_{p=0}^{\infty}\frac{\Gamma(\frac{1}{2}+\nu+p)\Gamma(n/2+\nu+p)X^{-p}}{\Gamma(1+2\nu+p)p!}\right]\ . \tag{E3}$$

Now if we are interested in the $X\to\infty$ limit, the leading term will be the $p=0$ contribution from the first power series, which is

$$\frac{h^{n-1}}{(4\pi)^{(n+1)/2}}\frac{\Gamma(2\nu)}{\Gamma(\frac{1}{2}+\nu)}\left[X^{\nu-n/2}\Gamma\left[\frac{n}{2}-\nu\right]\right] \tag{E4}$$

or for a very small mass,

$$\lim_{X\to\infty}\langle\phi(\mathbf{x})\phi(\mathbf{x}')\rangle\approx\frac{h^{n-1}}{(4\pi)^{(n+1)/2}}\frac{\Gamma(n)}{\Gamma((n+1)/2)}[X^{-\delta}\Gamma(\delta)] \tag{E5}$$

$$\approx \frac{h^{n-1}}{(4\pi)^{(n+1)/2}} \frac{\Gamma(n)}{\Gamma((n+1)/2)} \left[\frac{1}{\delta} - \ln X \right] . \tag{E6}$$

For small separations a power series in r would be helpful; Eq. (15.3.6) of Ref. 22 gives

$$\frac{\Gamma(a)\Gamma(b)}{\Gamma(c)} F[a,b;c;1-z] = \frac{\pi}{\Gamma(c-a)\Gamma(c-b)\sin\pi(c-a-b)} \left[\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b-c+1)} F[a,b;a+b-c+1;z] - \frac{\Gamma(c-a)\Gamma(c-b)}{\Gamma(c-a-b+1)} z^{c-a-b} \times F[c-a,c-b;c-b-a+1;z] \right] \tag{E7}$$

so that

$$\langle \phi(\mathbf{x})\phi(\mathbf{x}') \rangle = \frac{h^{n-1}\pi}{(4\pi)^{(n+1)/2} X^{(n-1)/2} \Gamma(\frac{1}{2}-\nu)\Gamma(\frac{1}{2}+\nu)\sin\pi[(n-1)/2]} \times \left\{ \frac{\Gamma(\frac{1}{2}-\nu)\Gamma(\frac{1}{2}+\nu)}{\Gamma((3-n)/2)} F\left[\tfrac{1}{2}-\nu, \tfrac{1}{2}+\nu; \frac{3-n}{2}; X \right] - X^{(n-1)/2} \frac{\Gamma(n/2+\nu)\Gamma(n/2-\nu)}{\Gamma((n+1)/2)} F\left[\frac{n}{2}+\nu; \frac{n}{2}-\nu; \frac{1+n}{2}, X \right] \right\} . \tag{E8}$$

Using the power-series expansion for the hypergeometric function this becomes

$$\frac{h^{n-1}\pi}{(4\pi)^{(n+1)/2} \sin\pi[(n-1)/2] \Gamma(\frac{1}{2}+\nu)\Gamma(\frac{1}{2}-\nu) X^{(n-1)/2}} \left[\sum_{p=0}^{\infty} \frac{\Gamma(\frac{1}{2}-\nu+p)\Gamma(\frac{1}{2}+\nu+p) X^p}{\Gamma((3-n)/2+p) p!} - X^{(n-1)/2} \sum_{p=0}^{\infty} \frac{\Gamma(n/2+\nu+p)\Gamma(n/2-\nu+p) X^p}{\Gamma((n+1)/2+p) p!} \right] . \tag{E9}$$

We notice that the Green's function has an X-independent piece which comes from the $p=0$ term in the second series and is given by

$$-\frac{h^{n-1}}{(4\pi)^{(n+1)/2}} \frac{\Gamma\left[\frac{n-1}{2}\right]\Gamma\left[\frac{3-n}{2}\right]}{\Gamma(\frac{1}{2}+\nu)\Gamma(\frac{1}{2}-\nu)} \frac{\Gamma\left[\frac{n}{2}+\nu\right]\Gamma\left[\frac{n}{2}-\nu\right]}{\Gamma\left[\frac{n+1}{2}\right]} . \tag{E10}$$

In the limit of small mass, we can write this as

$$\frac{h^{n-1}}{(4\pi)^{(n+1)/2}} \frac{\Gamma(n)}{\Gamma\left[\frac{n+1}{2}\right]} \frac{1}{\delta} . \tag{E11}$$

However, if we keep all terms which contain negative powers of X we get a power series:

$$\lim_{X\to 0}\langle\phi(\mathbf{x})\phi(\mathbf{x}')\rangle=\frac{h^{n-1}}{(4\pi)^{(n+1)/2}}\left[\frac{\Gamma(n)}{\Gamma\left[\frac{n+1}{2}\right]}\frac{1}{\delta}+\frac{\Gamma\left[\frac{1+n}{2}\right]}{\left[\frac{n-1}{2}\right]}X^{-[(n-1)/2]}+\frac{\Gamma\left[\frac{3+n}{2}\right]}{\left[\frac{n-3}{2}\right]1!}X^{-[(n-3)/2]}\right.$$

$$\left.+\frac{\Gamma\left[\frac{5+n}{2}\right]}{\left[\frac{n-5}{2}\right]2!}X^{-[(n-5/2)]}+\cdots\right] . \tag{E12}$$

Notice that the first term is the only term that diverges as we let the mass go to zero (this can be interpreted as the zero mode on the n sphere, see below). The terms with negative powers of X presumably are the ultraviolet divergences of the theory; in fact, in (3 + 1) dimensions, the second term is just $1/4\pi^2a^2r^2$ which is the standard ultraviolet divergence in three dimensions.

The contribution of the zero mode on the n-sphere to the propagator is [from (7.14) and (B7)]

$$\langle\phi(\mathbf{x})\phi(\mathbf{x}')\rangle_0=\frac{\Gamma\left[\frac{n+1}{2}\right]}{4\pi^{(n+1)/2}a^{n-1}}\frac{1}{[\mathrm{Re}f(\tilde{\tau})]_{A=0}} . \tag{E13}$$

For a small mass we have [from (7.12) and (C1)]

$$\frac{1}{[\mathrm{Re}f(\tilde{\tau})]_{A=0}}=\frac{2}{\pi\cos\tilde{\tau}}[\Gamma(n)]^2P_{(n-1)/2}^{-(n-1)/2}(i\tan\tilde{\tau})P_{(n-1)/2}^{-(n-1)/2}(-i\tan\tilde{\tau}) . \tag{E14}$$

Then using Eq. (14) on p. 150 of Ref. 25, we can rewrite (E13) (for a small mass) as

$$\langle\phi(\mathbf{x})\phi(\mathbf{x}')\rangle_0=\frac{h^{n-1}}{(4\pi)^{(n+1)/2}}\frac{\Gamma(n)}{\Gamma((n+1)/2)}\frac{1}{\delta} , \tag{E15}$$

which is exactly the same as the first term in either (E6) or (E12). From (8.17) we see that this is also the zero mode on the $(n+1)$-sphere that is the Euclidean section of de Sitter spacetime.

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