Magnetic monopoles from antisymmetric tensor gauge fields

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(Received 17 December 1984)

We consider the generalization of the Dirac monopole solution of Maxwell theory to theories of antisymmetric tensor gauge fields of arbitrary rank. A Dirac quantization condition is derived. These generalized monopoles are expected to be relevant for models of strings and higher-dimensional extended objects, and perhaps also for Kaluza-Klein theories.

INTRODUCTION

Antisymmetric tensor gauge fields play a prominent role in theories of closed strings, and hence also in the corresponding ($\alpha' \to 0$) gravity theories.[1–3] Moreover, such gauge fields can trigger interesting compactifications in Kaluza-Klein models,[4] and may even be responsible for the smallness of the cosmological constant.[5]

Here we investigate rank-n antisymmetric-tensor-gauge-field configurations which are direct generalizations of the Dirac magnetic monopole. Such generalized magnetic monopoles are solutions to the Kalb-Ramond equations, and, in fact, were first considered by Freund.[6] (We became aware of Ref. 6 as our own investigation was nearing completion. Since there are some differences in the two approaches, and the former work is no longer readily available, we deemed it useful to present our work as well.) For clarity of presentation, we first exhibit the (static) spherically symmetric solutions, corresponding to point magnetic monopoles; we then study solutions which correspond to extended (i.e., string, bag,...) magnetic charges. The extended monopoles are the objects that necessarily appear in the electric-magnetic dual cases (i.e., those cases for which the electric and magnetic fields have the same number of components), except of course for the four-dimensional case.

That extended objects enter the picture should come as no surprise: whereas vector gauge fields couple to point electric charges, higher-rank antisymmetric tensor gauge fields couple to "electrically" charged extended objects. This coupling contributes to the amplitude for transporting such an object along a path.[3] In the latter part of this paper, we examine parallel transport in the presence of our generalized magnetic monopoles. The analysis is analogous to the familiar Maxwell theory case of point particles; in particular, a Dirac quantization condition is derived.

POINT MAGNETIC MONOPOLES ($D=n+3$)

Let B denote the n-form corresponding to the rank-n antisymmetric tensor gauge field $B_{\mu_1\cdots\mu_n}(x)$,

$$B \equiv B_{\mu_1\cdots\mu_n}(x)dx^{\mu_1}\cdots dx^{\mu_n}, \quad \mu_i=0,1,\ldots,D-1 .$$

Here, x^μ are Cartesian coordinates on D-dimensional Minkowski spacetime; as will soon become evident, we require $D=n+3$ for constructing static spherically symmetric field configurations, corresponding to point magnetic monopoles. Furthermore, we choose the gauge

$$B_{0a_1\cdots a_{n-1}}(x)=0, \quad a_i=1,\ldots,D-1 , \tag{1}$$

and take B to be time independent.

To exhibit the spherically symmetric configuration, it is convenient to change to spherical coordinates:

$$x_{n+2-k}=\rho_k\cos\theta_{n-k}, \quad 0\le k\le n-1$$
$$x_2=\rho_n\sin\phi ,$$
$$x_1=\rho_n\cos\phi ,$$

with

$$\rho_0\equiv r\equiv\left[\sum_{a=1}^{n+2}x^a x^a\right]^{1/2} ,$$
$$\rho_k\equiv r\prod_{m=1}^{k}\sin\theta_{n+1-m}, \quad 1\le k\le n .$$

The field configuration B_+ (B_-) on the upper (lower) hemispheres of S^{n+1}, corresponding to a point magnetic monopole at the center of the sphere, is

$$B_\pm=k_n[\pm c_n+f_n(\theta_n)]d\Omega_n , \tag{2}$$

where c_n and k_n are constants, the function $f_n(\theta)$ satisfies

$$\frac{df_n(\theta)}{d\theta}=\sin^n\theta ,$$

and $d\Omega_n$ is the n-form,

$$d\Omega_n\equiv\left[\prod_{m=1}^{n-1}\sin^m\theta_m d\theta_m\right]d\phi ;$$

that is, $d\Omega_n$ is the area element on a sphere S^n, which is the equator ($\theta_n=\pi/2$) of S^{n+1}.

The constant c_n is determined by a simple consistency argument. Observe that the field strength $H=H_{\mu_1\cdots\mu_{n+1}}dx^{\mu_1}\cdots dx^{\mu_{n+1}}$ is given by

$$H\equiv dB_\pm=k_n d\Omega_{n+1} ; \tag{3}$$

hence the magnetic charge is

$$\int_{S^{n+1}} H = k_n \int_{S^{n+1}} d\Omega_{n+1} = k_n \frac{2\pi^{(n+2)/2}}{\Gamma(\frac{1}{2}(n+2))} .$$

Alternatively,

$$\begin{aligned}\int_{S^{n+1}} H &= \int_{(S^{n+1})_+} dB_+ + \int_{(S^{n+1})_-} dB_- \\ &= \int_{S^n} (B_+ - B_-) = 2k_n c_n \int_{S^n} d\Omega_n \\ &= 2k_n c_n \frac{2\pi^{(n+1)/2}}{\Gamma(\frac{1}{2}(n+1))} .\end{aligned}$$

Comparing the two expressions, we see that

$$c_n = \frac{\sqrt{\pi}}{2} \frac{\Gamma(\frac{1}{2}(n+1))}{\Gamma(\frac{1}{2}(n+1))} = \frac{2^{n-1}}{n!} [\Gamma(\tfrac{1}{2}(n+1))]^2 . \quad (4)$$

Finally, by fixing the normalization

$$\int_{S^{n+1}} H = 4\pi g , \quad (5)$$

we see that k_n is given by

$$k_n = g\, 2\pi^{-n/2} \Gamma(\tfrac{1}{2}(n+2)) . \quad (6)$$

Clearly, our construction is a direct generalization of the four-dimensional Dirac monopole. By smoothly covering the sphere with two patches, we have avoided the use of the Dirac string.

It is not difficult to see that (2) is a solution of the Kalb-Ramond theory. Recall that the action S is[7]

$$S = -(H,H) , \quad (7)$$

which is gauge invariant, since H (which locally is equal to dB) is itself invariant under the gauge transformation $B \to B + d\Lambda$. The corresponding field equations are

$$dH = 0 ,$$
$$\delta H = 0 .$$

Although H is *a priori* a form on a D-dimensional ($D = n+3$) spacetime with Lorentz metric, the gauge condition (1) and the requirement that H be static imply that we can treat H as a form on a $(D-1)$-dimensional space with Euclidean metric. It then follows[8] that such an H is a solution to the field equations if and only if H is harmonic; i.e., $\Delta H \equiv (d\delta + \delta d) H = 0$. In fact, the area element (3) is harmonic; indeed, a fundamental result of deRahm cohomology is that $d\Omega_{n+1}$ is the unique nontrivial harmonic $(n+1)$-form on S^{n+1}. (See, e.g., Ref. 9.)

The magnetic monopole field strength (3) can readily be re-expressed in Cartesian coordinates. Using the relation

$$d\Omega_{n+1} = \frac{1}{r^{n+2}} \epsilon_{a_1 \cdots a_{n+1} a_{n+2}} x^{a_{n+2}} dx^{a_1} \cdots dx^{a_{n+1}} ,$$
$$a_i = 1, \ldots, n+2, \quad r^2 \equiv \sum_{a=1}^{n+2} x^a x^a > 0 ,$$

it follows that

$$H_{a_1 \cdots a_{n+1}} = \frac{k_n}{r^{n+2}} \epsilon_{a_1 \cdots a_{n+1} a_{n+2}} x^{a_{n+2}} , \quad (8)$$

with all other components of $H_{\mu_1 \cdots \mu_{n+1}}$ vanishing.

We close this section with the following observation: from the action (7), it is evident that the field strength $H_{\mu_1 \cdots \mu_{n+1}}$ has dimensions (length)$^{-D/2}$. It follows from (8) that the magnetic charge g (to which k_n is proportional) has dimensions (length)$^{(n+1)-D/2}$.

EXTENDED MAGNETIC MONOPOLES ($D > n+3$)

For $D > n+3$, static spherically symmetric magnetic monopole solutions are no longer possible; indeed, the monopoles are then extended objects. One way to see this is to observe that the magnetic current $\delta * H$ is a $(D-n-2)$-form, corresponding to an extended object of dimension $D-n-3$.

In particular, consider the case $D = 2(n+1)$, for which both H and $*H$ are $(n+1)$-forms, and for which also the action is conformally invariant. (See, e.g., Ref. 10.) This is the so-called electric-magnetic dual case. By the previous counting, the monopoles of the electric-magnetic dual theories are extended objects for $n > 1$.

Monopole solutions for $D > n+3$ can be obtained trivially by imbedding the spherically symmetric solutions presented above. Suppose that the monopole has coordinates $0 = x^1 = x^2 = \cdots = x^{n+2}$; i.e., it lies on the spacelike $(D-n-3)$-hyperplane spanned by $x^{n+3}, x^{n+4}, \ldots, x^{D-1}$.[11] One can verify that a solution to the field equations is given by

$$H_{a_1 \cdots a_{n+1}} = \frac{k_n}{r^{n+2}} \epsilon_{a_1 \cdots a_{n+1} a_{n+2}} x^{a_{n+2}} ,$$
$$a_i = 1, \ldots, n+2, \quad r^2 \equiv \sum_{a=1}^{n+2} x^a x^a > 0 , \quad (9)$$

with all other components of $H_{\mu_1 \cdots \mu_{n+1}}$ (where $\mu_i = 1, \ldots, D-1$) vanishing. [Compare with (8).] Taking k_n to be given by (6), one can see that

$$\int_{S^{n+1}} H = \int_{S^{n+1}} \frac{k_n}{r^{n+2}} \epsilon_{a_1 \cdots a_{n+1} a_{n+2}} \times x^{a_{n+2}} dx^{a_1} \cdots dx^{a_{n+1}} = 4\pi g , \quad (10)$$

where S^{n+1} is the sphere $r^2 = \text{const}$, which is imbedded in the $(n+2)$-hyperplane orthogonal to $x^{n+3}, x^{n+4}, \ldots, x^{D-1}$. We say that this S^{n+1} "surrounds" the monopole, although this does not coincide with the usual notion of the word. For the electric-magnetic dual case, the constant g is dimensionless. Solutions of this type were first considered in Ref. 6.

EXTENDED ELECTRIC CHARGES AND PARALLEL TRANSPORT

The physical interest in antisymmetric tensor gauge fields lies primarily in their coupling to extended electric charges. Consider an $(n-1)$-dimensional closed extended object having the topology of S^{n-1}. (For instance, $n=1$ corresponds to a point particle, $n=2$ to a closed string, etc.) Let $x^\mu(\sigma_1, \sigma_2, \ldots, \sigma_n)$ be the coordinates of the

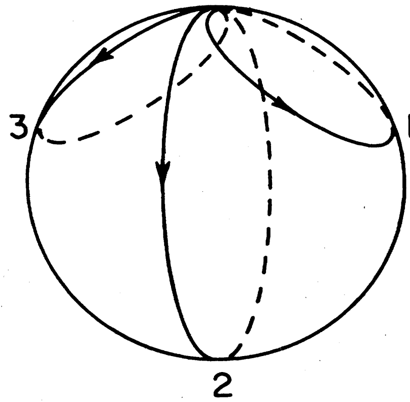

FIG. 1. A loop of loops (keeping one point, the north pole, fixed) is a sphere S^2. Similarly, a loop of S^{n-1} spheres is S^n.

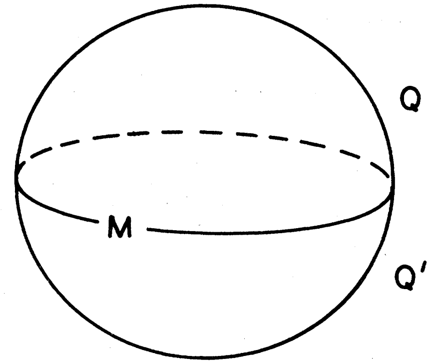

FIG. 2. The manifold $M \sim S^n$ has two capping surfaces Q and Q', which together form the sphere S^{n+1}.

world surface M swept out by this object. The coupling to the rank-n antisymmetric tensor gauge field $B_{\mu_1 \cdots \mu_n}(y)$ is

$$S_{\text{int}} = \int d^D y \, B_{\mu_1 \cdots \mu_n}(y) J^{\mu_1 \cdots \mu_n}(y) \ , \qquad (11)$$

where the current $J^{\mu_1 \cdots \mu_n}(y)$ is given by

$$J^{\mu_1 \cdots \mu_n}(y) = e \int d^n \sigma \, \delta^{(D)}(y-x) \frac{\partial(x^{\mu_1}, \ldots, x^{\mu_n})}{\partial(\sigma_1, \ldots, \sigma_n)} \ ,$$

and e is the electric charge of the extended object. The charge e has dimensions $(\text{length})^{D/2-(n+1)}$.

Performing the integration over spacetime y^μ, (11) reduces to an integral over the world surface M,

$$S_{\text{int}} = e \int_M B \ . \qquad (12)$$

Thus the amplitude for transporting the extended electric charge along M is proportional to $\exp(ie \int_M B)$.[3]

Let us suppose that the $(n-1)$-dimensional charge $(\sim S^{n-1})$ is transported, keeping one of its points fixed, slowly along a closed path, so that it sweeps out the surface $M \sim S^n$. (See Fig. 1.) The corresponding amplitude is

$$e^{ie \int_M B} = e^{ie \int_Q H} \ ,$$

where $H = dB$, and Q is an $(n+1)$-manifold such that $\partial Q = M \sim S^n$. However, there is an ambiguity in the choice of capping surface for M; one could equally well have chosen Q' (see Fig. 2), in which case the amplitude is

$$e^{ie \int_M B} = e^{-ie \int_{Q'} H} \ .$$

Consistency requires

$$e \int_{Q+Q'} H = e \int_{S^{n+1}} H = 2k\pi \ , \qquad (13)$$

where k is an integer. If the sphere S^{n+1} surrounds a magnetic monopole of the type considered earlier, then this and (5) and (10) imply the quantization condition

$$eg = \frac{k}{2} \ . \qquad (14)$$

This result is independent of the dimension D of spacetime.

In passing, we observe that the recent cohomological discussions[12] of covariant translation operators in the background field of a magnetic monopole can now also be generalized. In particular, following Ref. 12, a magnetic monopole for a rank-n antisymmetric tensor gauge field leads to an $(n+1)$-cochain (namely, the integrated field strength) and a trivial $(n+2)$-cocycle.

We have seen that the Dirac magnetic monopole solution has a natural generalization for antisymmetric tensor gauge fields of arbitrary rank. These solutions are expected to play a role in models of extended objects, to which these gauge fields naturally couple. Other situations in which these solutions might arise can also be imagined. For instance, Kaluza-Klein theories involving antisymmetric tensor fields may compactify on such monopoles. Interestingly, the $D=10$, $N=1$ supergravity theory contains the rank-two gauge field $B_{\mu\nu}$, whose "preferred" dimension (electric-magnetic dual) is six.

Note added in proof. (1) It is easy to see that generalized magnetic monopoles are precisely Freund-Rubin[4] configurations. Indeed, recall that $r^n d\Omega_n = d^n y \sqrt{g}$, where y^μ are coordinates on S^n. (In the text, we used Cartesian coordinates on $\mathbb{R}^{n+1}$ instead.) It follows that $H = H_{\mu_1 \cdots \mu_n} dy^{\mu_1} \cdots dy^{\mu_n} = k \, d\Omega_n$ implies

$$H_{\mu_1 \cdots \mu_n} = \frac{k}{r^n} \frac{1}{n!} \epsilon_{\mu_1 \cdots \mu_n} \sqrt{g} \ .$$

Clearly, this is a Freund-Rubin configuration. We emphasize that if there is an extended object which couples to $B_{\mu_1 \cdots \mu_{n-1}}$ with strength e, then the product ek is quantized. (2) Topological objects similar to those considered here have been discussed in a different context by R. Savit [Phys. Rev. Lett. **39**, 55 (1977)] and P. Orland (Nucl. Phys. **B205** [FS5], 107 (1982)).

ACKNOWLEDGMENTS

It is a pleasure to thank P. G. O. Freund, Y.-S. Wu, and A. Zee for very useful discussions. This work was supported in part by the U.S. Department of Energy, Contract No. DE-AC06-81ER-40048.

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$$*(dx^{\mu_1}\cdots dx^{\mu_p})=\frac{1}{(n-p)!}\epsilon^{\mu_1\cdots\mu_p}{}_{\mu_{p+1}\cdots\mu_d}\,dx^{\mu_{p+1}}\cdots dx^{\mu_d}\;.$$

See, e.g., Ref. 9.

[8]Clearly, $\Delta\alpha=0$ implies $0=(\alpha,\Delta\alpha)=(\alpha,(d\delta+\delta d)\alpha)=(d\alpha,d\alpha)+(\delta\alpha,\delta\alpha)$. If the inner product is positive-definite, this implies $d\alpha=0=\delta\alpha$.

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