Time variation of coupling constants in Kaluza-Klein cosmologies

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We present a cosmological analysis of a six-dimensional Einstein-Yang-Mills-Higgs model with SO(3) invariance to which is added a matter term. An analytical expression for the time variation of the internal radius is obtained together with the predictions for the variations of the coupling "constants" involved. Their variation is limited to the very early stages of the cosmological evolution since the internal radius quickly becomes constant. It is shown that, once the asymptotic behavior sets in, the four-dimensional space-time expands linearly in time with a zero effective cosmological constant, after fine tuning.

I. INTRODUCTION

Over the past ten years or so, the construction of a unified theory of gravity and the other fundamental gauge interactions has received renewed attention. Generalized Kaluza-Klein theories[1,2] offer the possibility of achieving this unification via a geometrization of the gauge interactions in which the space-time has $4+D$ dimensions with the extra D dimensions forming a compact manifold whose isometry group is related to the invariance group of gauge interactions (see Ref. 2, and references therein). In this way, internal quantum numbers are associated with translations and rotational invariance in the extra dimensions much in the same way as energy and momentum are associated with invariance properties of the four-dimensional space-time.

Although this idea is extremely attractive for its formal beauty, there are still a number of questions to be answered and we are still far from obtaining a proper unified model. At the quantum level, there is the problem of anomalies[3] which impose strong restrictions on the possible internal spaces that can be used, whereas at the classical level there is the problem of obtaining fermions with the correct chirality once the theory is reduced. In this connection, it has been suggested that the inclusion of magnetic monopoles in the models may play an important role in solving the chirality question.[4]

Several models have been constructed in which a magnetic monopole is defined in the internal space.[5] The inclusion of the two-index (or three and four for supersymmetric theories) tensor field in the action allows solutions of the Einstein equations where the $(4+D)$-dimensional space-time "spontaneously compactifies" into the product form $M^{4+D}=M^4\times S^D$ with M^4 being our four-dimensional space-time (usually a Minkowski or anti—de Sitter space) and S^D being the compact D-dimensional space mentioned above.[6,2] Physical fields are then obtained by harmonically expanding the $(4+D)$-dimensional fields in the internal space.

At this point we might pause to ask to what extent the extra dimensions should be taken seriously. Are there any detectable effects that can prove or disprove the existence of extra dimensions?

It is generally believed that cosmology can provide the best way to look for evidence.[7] Because of the intimate relation between the coupling constants and the radius of the internal space in these models, we expect that a time variation of this radius would imply a time variation of the coupling constants. It has also been suggested that shrinking of the extra dimensions can generate enough entropy to support an inflationary epoch and even explain the origin of the cosmic background radiation.[8] Although these questions are obviously related, in this paper we wish to concentrate on the former, the main reason being that it is possible, as will be shown, to obtain a solution of the field equations that produce a shrinking of the internal radius without taking temperature into account. However, the physical consequences of our model can only be fully understood when both aspects are considered and we propose to present these results elsewhere.

In a recent paper, we obtained a number of power-law solutions for the D-dimensional Einstein-Maxwell, $D=11$, $N=1$ supergravity and $D=10$, $N=2$ supergravity theories.[9] A careful examination of these solutions shows that the inclusion of a scalar field may be the key to obtaining more attractive solutions, in particular if we compare the Einstein-Maxwell with the $D=10$ supergravity actions. In the former, a time-varying internal radius implies a time variation of the cosmological constant which is forbidden by the Bianchi identities. Thus, only static internal spaces are possible.

For this reason, we decided to study the six-dimensional Einstein-Yang-Mills-Higgs model proposed by Cremmer and Scherk[10] which features an SO(3) monopole taking values in the two-sphere. We have also added an extra matter term which has proved to be of fundamental importance in obtaining the solutions. As the analysis is restricted to the broken-symmetry phase, we may conjecture that this matter term is representing fermionic matter (in a way not very different from the linear σ model[11] or as a fermionic condensate).[11]

As is usual in these models, we have two free parameters involved: the initial radius, R_{20}, and the "compactification time," t_0. Whether t_0 is the initial singularity or the time of a dynamical compactification process is irrelevant for our solutions, so that for economy it was set to zero. More will be said about R_{20} later. The natural scale for Kaluza-Klein theories seems to be somewhere between the grand unification and the Planck scales. This implies that if a completely isotropic phase existed before, it can only be understood with quantum gravity effects being included.[12] Nevertheless, it is reasonable to adopt a classical treatment if the product-space structure has already been achieved, as in the present case.

This paper is organized as follows: In Sec. II we give a brief description of the model and obtain the field equations. In Sec. III we derive the solution for $R_2(t)$ and analyze the asymptotic behavior of the solutions. We conclude by discussing our results in Sec. IV.

II. DESCRIPTION OF THE MODEL

As usual, we take the metric to be of a generalized Robertson-Walker form.[7–9] This ensures the homogeneity and isotropy of each space separately, stressing the product structure of the $(4+D)$-dimensional space-time. It can be written as

$$g_{MN}(z^P)=\begin{bmatrix} -1 & 0 & 0 \\ 0 & R_3{}^2(t)\tilde{g}_{ij}(x^l) & 0 \\ 0 & 0 & R_2{}^2(t)\tilde{g}_{mm}(h^P) \end{bmatrix}, \tag{1}$$

where

$$M,N,\ldots=0,1,2,3,5,6\;,$$
$$\mu,\nu,\ldots=0,1,2,3\;,$$
$$i,j,\ldots=1,2,3\;,$$
$$m,n,\ldots=5,6\;,$$

so that the index splitting is $z^P=(x^\mu,Y^m)$. We also take $c=\hbar=1$.

Here, $\tilde{g}_{ij}(x^l)$ is the maximally symmetric metric of the three-dimensional spacelike hypersurface, whereas

$$\tilde{g}_{mn}(\theta,\psi)=d\theta^2+\sin^2\theta\, d\psi^2 \tag{2}$$

is the metric of the two-sphere written in terms of the polar coordinates $y^5=\theta$, $y^6=\psi$. $R_3(t)$ and $R_2(t)$ are the scale factors (radii) of the three- and two-dimensional spaces, respectively.

With this choice of metric, the nonvanishing Ricci tensor components are

$$R_{00}=\frac{3\ddot{R}_3}{R_3}+\frac{2\ddot{R}_2}{R_2}\;, \tag{3a}$$

$$R_{ij}=-\left[\frac{2K_3}{R_3{}^2}+\frac{d}{dt}\left[\frac{\dot{R}_3}{R_3}\right]+\left[\frac{3\dot{R}_3}{R_3{}^2}+\frac{2\dot{R}_2}{R_2}\right]\frac{\dot{R}_3}{R_3}\right]g_{ij}\;, \tag{3b}$$

$$R_{mn}=-\left[\frac{K_2}{R_2{}^2}+\frac{d}{dt}\left[\frac{\dot{R}_2}{R_2}\right]+\left[\frac{3\dot{R}_3}{R_3}+\frac{2\dot{R}_2}{R_2}\right]\frac{\dot{R}_2}{R_2}\right]g_{mn}\;, \tag{3c}$$

where K_3 and K_2 are the curvature constants for the three- and two-dimensional spaces, respectively.

In view of Eq. (1), the most general form of the energy-momentum tensor for matter is[9]

$$T_{MN}^{\text{matter}}=(\epsilon P_m+\epsilon' P'_m)g_{MN}+(\rho_m+P_m)U_M U_N\;, \tag{4}$$

where

$$\epsilon'=\begin{cases}0, & M,N=\mu,\nu\\ 1, & M,N=m,n\end{cases} \quad\text{and}\quad \epsilon=\begin{cases}0, & M,N=m,n\\ 1, & M,N=\mu,\nu\end{cases}$$

ρ_m, P_m, and P'_m are the matter density, the external, and internal pressures, respectively.

The six-dimensional Einstein-Yang-Mills-Higgs action is given by[10]

$$S_6=-\int d^6z(-g^{(6)})^{1/2}\left[\frac{R}{16\pi G}+\frac{1}{4}F^a_{MN}F_a^{MN}+\frac{1}{2}D_M\phi^a D^M\phi_a+\frac{1}{2}V(\phi^a\phi_a)\right], \tag{5}$$

where

$$F^a_{MN}=\partial_M A^a_N-\partial_N A^a_M+e\epsilon^a{}_{bc}A^b_M A^c_N\;. \tag{6}$$

R and G are the six-dimensional curvature scalar and gravitational constant, respectively. The gauge group is taken to be SO(3). Thus $a,b,c,\ldots=1,2,3$, and A^a_m and ϕ^a are triplets in SO(3). $\epsilon^a{}_{bc}$ is the usual Levi-Civita permutation symbol. The covariant derivative and the symmetry-breaking potential are defined as

$$D_M\phi^a=\partial_M\phi^a+e\epsilon^a{}_{bc}A^b_M\phi^c\;, \tag{7}$$

$$V(\phi^a\phi_a)=\frac{\lambda}{4}(\phi^a\phi_a-\phi_0{}^2)+\Lambda\;, \tag{8}$$

where e is the six-dimensional Yang-Mills coupling, $\lambda>0$, Λ is an adjustable constant. Also,

$$\phi_0{}^2=\langle 0\,|\,\phi^a\phi_a\,|\,0\rangle=2\mu^2/\lambda\;. \tag{9}$$

From the action (5) we obtain the following field equations:

(i) *Einstein equations:*

$$R_{MN}-\tfrac{1}{2}g_{MN}R=-8\pi G T_{MN} \tag{10}$$

and we add the matter contribution to T_{MN} which is given by

$$T_{MN}=T^{\text{gauge}}_{MN}+T^{\text{matter}}_{MN}\ , \tag{11a}$$

$$T^{\text{gauge}}_{MN}=D_M\phi^a D_N\phi_a+F^a_{MP}F_N{}^P{}_a -\tfrac{1}{2}g_{MN}[\tfrac{1}{2}F^a_{PQ}F^{PQ}_a+D_P\phi^a D^P\phi_a+V(\phi^a\phi_a)]\ . \tag{11b}$$

We have to satisfy the energy-momentum conservation:

$$T^{MN}{}_{;N}=(T^{MN}_{\text{gauge}}+T^{MN}_{\text{matter}})_{;N}=0\ . \tag{12}$$

(ii) *Higgs field equations:*

$$\frac{1}{(-g^{(6)})^{1/2}}D_M[(-g^{(6)})^{1/2}g^{MN}D_N\phi]^a=\frac{\partial V}{\partial(\phi^b\phi_b)}\phi^a\ . \tag{13}$$

(iii) *Yang-Mills field equations:*

$$\frac{1}{(-g^{(6)})^{1/2}}D_N[(-g^{(6)})^{1/2}F^{MNa}]=e\epsilon^a{}_{bc}(D^M\phi^b)\phi^c\ . \tag{14}$$

In order to obtain a solution of the field equation that exhibits spontaneous compactification, we must have an ansatz for the Yang-Mills and Higgs fields compatible with the SO(3) invariance of the internal space. We follow Cremmer and Scherk[10] to write

$$A^a_\mu\equiv 0\ , \tag{15a}$$

$$A^a_5=\frac{1}{e}(-\sin\psi,\cos\psi,0)\ , \tag{15b}$$

$$A^a_6=-\frac{1}{e}(-\cos\psi\cos\theta,-\sin\psi\cos\theta,\sin\theta)\sin\theta\ , \tag{15c}$$

$$\phi^a=p(t)(\cos\psi\sin\theta,\sin\psi\sin\theta,\cos\theta)\ . \tag{15d}$$

From Eq. (15d) we see that $\phi^a(t,\theta,\psi)$ is proportional to the normal vector to the two-sphere at each point but now the radius need not be constant. For a more detailed discussion of the symmetries of the model we refer the reader to Ref. 10. For us it is sufficient to note that our ansatz preserves the required symmetry together with the isotropy of the internal space.

If we now use Eqs. (1) and (15) in Eqs. (10)–(14) we obtain the following field equations:

(i) *Einstein equations:*

$$\frac{3\ddot R_3}{R_3}+\frac{2\ddot R_2}{R_2}=-8\pi G\left[\frac{1}{4e^2R_2{}^4}+\dot p^2-\frac{\lambda}{16}(p^2-\phi_0{}^2)^2 -\frac{\Lambda}{4}+\frac{3}{4}(\rho_m+P_m)+\frac{P'_m}{2}\right]\ , \tag{16a}$$

$$\frac{2K_3}{R_3{}^2}+\frac{\ddot R_3}{R_3}+\frac{2\dot R_3{}^2}{R_3{}^2}+\frac{2\dot R_3}{R_3}\frac{\dot R_2}{R_2} =-8\pi G\left[\frac{1}{4e^2R_2{}^4}-\frac{\lambda}{16}(p^2-\phi_0{}^2)^2-\frac{\Lambda}{4} -\frac{1}{4}(\rho_m+P_m)+\frac{P'_m}{2}\right]\ , \tag{16b}$$

$$\frac{K_2}{R_2{}^2}+\frac{\ddot R_2}{R_2}+\frac{\dot R_2{}^2}{R_2{}^2}+\frac{3\dot R_3}{R_3}\frac{\dot R_2}{R_2} =8\pi G\left[\frac{3}{4e^2R_2{}^4}+\frac{\lambda}{16}(p^2-\phi_0{}^2)^2 +\frac{\Lambda}{4}+\frac{1}{4}(\rho_m-3P_m)+\frac{P'_m}{2}\right]\ . \tag{16c}$$

(ii) *Higgs equation:*

$$\frac{\ddot p}{p}+\left[\frac{3\dot R_3}{R_3}+\frac{2\dot R_2}{R_2}\right]\frac{\dot p}{p}=\frac{\lambda}{2}(p^2-\phi_0{}^2)\ . \tag{17}$$

(iii) *Yang-Mills equation:*

$$\frac{1}{R_2(t)^2}=e^2p(t)^2\ . \tag{18}$$

III. SOLUTIONS AND ASYMPTOTIC LIMIT

In order to solve Eqs. (16)–(18), we need equations of state relating ρ_m, P_m, and P'_m. Here, we have to face the fact that we do not know how to write an equation of state for the internal pressure. Nevertheless, we may impose some conditions on ρ_m, P_m, and P'_m that provide satisfactory behavior from the physical point of view. As a first attempt, we may set

$$\tfrac{3}{4}(\rho_m+P_m)+\frac{P'_m}{2}=-\tfrac{1}{4}(\rho_m+P_m)+\frac{P'_m}{2}\ , \tag{19a}$$

$$\tfrac{1}{4}(\rho_m-3P_m)+\frac{P'_m}{2}=0\ , \tag{19b}$$

which gives $\rho_m=-P_m=-P'_m/2$ as a solution. This choice is useful if we want to define a four-dimensional effective cosmological constant, Λ_4, from Eqs. (16a) and (16b) at the asymptotic limit *provided that* $\lim_{t\to\infty}R_2(t)=R_{2\infty}=$ constant, $\lim_{t\to\infty}\dot R_2(t)=\lim_{t\to\infty}\ddot R_2(t)=0$. However, we will see that as a consequence of the time behavior of $\ddot R_2(t)$, this choice is ruled out. In order to have Λ_4 at $t\to\infty$, the model naturally offers another possibility which we will adopt. Namely, we can take $P'_m=0$ and $P_m=\rho_m/3$, i.e., we neglect the internal pressure and assume that the matter contribution is in the form of radiation. This choice for the equations of state will prove to be adequate to obtain the desired solutions.

From Eqs. (16) we obtain

$$\frac{\dot R_2}{R_2}=-\frac{\dot p}{p}\ , \tag{20a}$$

$$\frac{\ddot R_2}{R_2}=-\frac{\ddot p}{p}+2\frac{\dot p^2}{p^2}\ . \tag{20b}$$

Using Eqs. (17), (18), and (20), we can write Eq. (16c) as an equation for $R_2(t)$, independent of the equation of state,

$$\dot R_2{}^2=\frac{A}{R_2{}^2}+B+CR_2{}^2\ , \tag{21}$$

where

$$A=\frac{\pi G}{2e^4}(12e^2+\lambda)\ , \tag{22a}$$

$$B=\frac{\lambda}{2e^2}-K_2-\frac{\lambda\pi G\phi_0{}^2}{e^2}\ , \tag{22b}$$

$$C=2\pi G\left[\frac{\lambda(\phi_0{}^2)^2}{4}+\Lambda\right]-\frac{\lambda}{2}\phi_0{}^2\ . \tag{22c}$$

The solution for $R_2(t)$ is then given by

$$R_2(t)^2=\frac{\sqrt{\Delta}-B+(B+\sqrt{\Delta})\alpha(t)}{2C[1-\alpha(t)]}\ , \tag{23}$$

where

$$\Delta\equiv B^2-4AC,\ \ \Delta>0 \tag{24}$$

$$\alpha(t)\equiv f(R_{20}{}^2)\exp[2\Delta(t-t_0)]\ , \tag{25a}$$

$$f(R_{20}{}^2)=\frac{CR_{20}{}^2+(B-\sqrt{\Delta})/2}{CR_{20}{}^2)+(B+\sqrt{\Delta})/2}\ . \tag{25b}$$

We cannot say, *a priori,* if the condition $\Delta>0$ is always satisfied. However, as we will see, for "realistic" values of the coupling constants this condition is always fulfilled. The other possibility, $\Delta<0$, would give a divergent solution for large t.[13]

An important feature of the solution (23) is that the asymptotic limit ($t\rightarrow\infty$) gives a *constant value* for $R_{2\infty}$:

$$\lim_{t\rightarrow\infty}R_2(t)^2=-\frac{B+\sqrt{\Delta}}{2C}\equiv R_{2\infty}{}^2\ . \tag{26}$$

This result seems to rule out time-varying constants for large values of t. But this is an asymptotic result; whenever the initial value of $R_2(t)$, R_{20}, is different from $R_{2\infty}$ there will be an expanding or shrinking $R_2(t)$ that will produce a well-defined time variation for the coupling constants. To analyze this further, it is convenient to write the expression of $R_2(t)$ in terms of four-dimensional constants. In order to obtain the correct dimensionality we must have[10]

$$\begin{aligned} e^2&=e_4{}^2 4\pi R_2{}^2\ ,\\ G&=G_4 4\pi R_2{}^2\ ,\\ \lambda&=\lambda_4 4\pi R_2{}^2\ ,\\ \phi_0{}^2&=\phi_{04}{}^2(4\pi R_2{}^2)^{-1}\ . \end{aligned} \tag{27}$$

Thus,

$$\frac{\dot{e}_4}{e_4}=\frac{\dot{\lambda}_4}{\lambda_4}=\frac{\dot{G}_4}{G_4}=-\frac{2\dot{R}_2}{R_2}\ . \tag{28}$$

We note that Eq. (9) gives a constant mass scale:

$$\frac{2\dot{\mu}_4}{\mu_4}=\frac{(\dot{\phi}_{04}{}^2)}{\phi_{04}{}^2}+\frac{\dot{\lambda}_4}{\lambda_4}=0\ . \tag{29}$$

So, our model falls in the second category studied by Marciano.[7]

We can obtain an explicit picture of $R_2(t)$ provided that we write A, B, and C in terms of the four-dimensional couplings at their asymptotic limit, i.e., if we substitute $R_2(t)$ by $R_{2\infty}$ in Eq. (27. In order to do this we must analyze the asymptotic limit of the field equations as well, using that $\dot{R}_2\rightarrow 0$ as $t\rightarrow\infty$. However, we still cannot conclude that $\lim_{t\rightarrow\infty}\ddot{R}_2\rightarrow 0$ so we set

$$\lim_{t\rightarrow\infty}\frac{\ddot{R}_2}{R_2}\equiv-\sigma,\ \ \sigma=\text{constant} \tag{30}$$

(the minus sign being chosen for convenience). We can also write the energy-momentum conservation equation as

$$\dot{\rho}_m+\left[\frac{3\dot{R}_3}{R_3}+\frac{2\dot{R}_2}{R_2}\right]\left[\frac{4\rho_m}{3}+T^{00}_{\text{gauge}}\right]+T^{00}_{\text{gauge},0}=0\ . \tag{31}$$

Using Higgs and Yang-Mills equations and the fact that $\lim_{t\rightarrow\infty}\dot{R}_2=0$, we may write that $\lim_{t\rightarrow\infty}\rho_m=\rho_{m\infty}=$ constant.

Thus Eqs. (16)—(18) give, for $t\rightarrow\infty$:

(i) *Yang-Mills equation:*

$$\frac{1}{R_{2\infty}^2}=e^2p_\infty{}^2\ . \tag{32}$$

(ii) *Higgs equation:*

$$\sigma=\frac{\lambda}{2}(p_\infty{}^2-\phi_0{}^2)\ . \tag{33}$$

(iii) *Einstein equations:*

$$\frac{3\ddot{R}_3}{R_3}=-8\pi G\left[\frac{1}{4e^2R_{2\infty}{}^4}-\frac{\lambda}{16}\left[\frac{1}{e^2R_{2\infty}{}^2}-\phi_0{}^2\right]^2-\frac{\Lambda}{4}+\rho_{m\infty}-\frac{\sigma}{4\pi G}\right]\ , \tag{34a}$$

$$\frac{2K_3}{R_3{}^2}+\frac{\ddot{R}_3}{R_3}+\frac{2\dot{R}_3{}^2}{R_3{}^2}=-8\pi G\left[\frac{1}{4e^2R_{2\infty}{}^4}-\frac{\lambda}{16}\left[\frac{1}{e^2R_{2\infty}{}^2}-\phi_0{}^2\right]^2-\frac{\Lambda}{4}-\frac{\rho_{m\infty}}{3}\right]\ , \tag{34b}$$

$$\frac{K_2}{R_{2\infty}{}^2}-\sigma=8\pi G\left[\frac{3}{4e^2R_{2\infty}{}^4}+\frac{\lambda}{16}\left[\frac{1}{e^2R_{2\infty}{}^2}-\phi_0{}^2\right]^2+\frac{\Lambda}{4}\right]\ . \tag{34c}$$

The right-hand side of Eqs. (34a) and (34b) suggests that we may define an effective four-dimensional cosmological constant, Λ_4, if we write that

$$\sigma=\frac{16\pi G}{3}\rho_{m\infty}\ . \tag{35}$$

Λ_4 is given by

$$\Lambda_4=\frac{1}{4e^2R_{2\infty}{}^4}-\frac{\lambda}{16}\left[\frac{1}{e^2R_{2\infty}{}^2}-\phi_0{}^2\right]^2$$
$$-\frac{\Lambda}{4}-\frac{\sigma}{16\pi G}\ . \tag{36}$$

As the present value of Λ_4 is believed to be zero (or nearly), we may fine tune it to be so. This allows us to express Λ in terms of other quantities and also gives an asymptotic behavior for $R_3(t)$ of the form $R_3(t)=at$, a = constant. Also, from (34b) we obtain $K_3=-a^2$, which characterizes an expanding Ricci flat four-dimensional space-time. In fact, we have a Minkowski space written in terms of Robertson-Walker coordinates. This result agrees with our previous power-law solutions but with the new feature that now this is an asymptotic behavior, not valid for small values of t.

Using that $\Lambda_4=0$ in (34c) gives

$$K_2=\frac{8\pi G}{e^2R_{2\infty}{}^2}+\frac{\sigma R_{2\infty}{}^2}{2}\ . \tag{37}$$

Without loss of generality, we can set $K_2=1$. From the asymptotic values of the four-dimensional couplings together with Eqs. (32), (33), and (37) it is easy to obtain the expression for A, B, and C in terms of the four-dimensional couplings and $R_{2\infty}$,

$$A_4=\frac{\pi G_4}{2e_4{}^4}(12e_4{}^2+\lambda_4)\ , \tag{38a}$$

$$B_4=\frac{\lambda_4}{2e_4{}^2}-1-\frac{\pi G_4}{e_4{}^2R_{2\infty}{}^2}\left[\frac{32\pi G_4}{e_4{}^2R_{2\infty}{}^2}+\frac{\lambda_4}{e_4{}^2}-4\right]\ , \tag{38b}$$

$$C_4=-\frac{10\pi G_4}{e_4{}^2R_{2\infty}{}^4}+\frac{\pi G_4\lambda_4}{2e_4{}^4R_{2\infty}{}^4}-\frac{\lambda_4}{2e_4{}^2R_{2\infty}{}^2}$$
$$+\frac{32\pi^2G_4{}^2}{e_4{}^4R_{2\infty}{}^6}+\frac{1}{R_{2\infty}{}^2}\ . \tag{38c}$$

It is then easy to check, using Eq. (38), that $\lim_{t\to\infty}\ddot{R}_2/R_2$ is not zero in general while $\lim_{t\to\infty}\dot{R}_2/R_2=0$:

$$\lim_{t\to\infty}\frac{\dot{R}_2{}^2}{R_2{}^2}=\frac{A_4}{R_{2\infty}{}^4}+\frac{B_4}{R_{2\infty}{}^2}+C_4=0\ , \tag{39a}$$

$$\lim_{t\to\infty}\frac{\ddot{R}_2}{R_2}=\frac{-A_4}{R_{2\infty}{}^4}+C_4\neq 0\ . \tag{39b}$$

Using Eqs. (30) and (33) we obtain a relation for $R_{2\infty}{}^2$ in terms of G_4,λ_4, and $e_4{}^2$,

$$R_{2\infty}{}^2=\frac{8\pi G_4[4\pm(10+\lambda_4/e_4{}^2)^{1/2}]}{6e_4{}^2-\lambda_4}\ . \tag{40}$$

We see that the model provides a prediction for $R_{2\infty}$ which is indeed of order of L_p ($L_p\sim 1.6\times 10^{-33}$ cm), for either root. However, we must remember that the solution for $R_2(t)$ is based on the assumption that $\Delta>0$. Below are the bounds on the value of $e_4{}^2$, for positive Δ

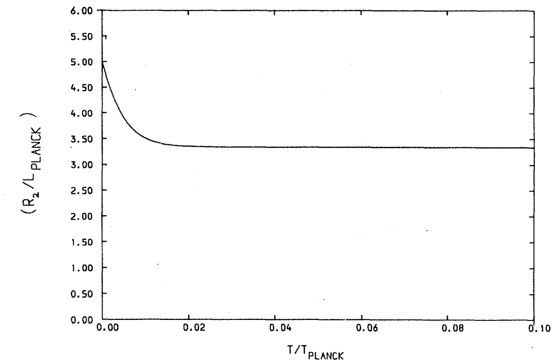

FIG. 1. The time behavior of the internal radius, $[R_2(t)]^2$, is shown. We can see that it reaches the asymptotic limit $R_{2\infty}$ in a very short time of order $10^{-2}t_{\text{Planck}}$. This behavior would not be changed for another choice of R_{20}, the "initial value."

using two sample values for λ_4, 0.1 and 0.01, respectively.

(a) Negative root: $\begin{cases} e_4{}^2\gtrsim 0.064\ ,\\ e_4{}^2\gtrsim 0.064\ .\end{cases}$

(b) Positive root: $\begin{cases} 0.01\lesssim e_4{}^2\lesssim 0.041,\ \ e_4{}^2\lesssim 0.85\ ,\\ e_4{}^2\gtrsim 1.21\ .\end{cases}$

It is easy to check that the positive root gives bigger values for $R_{2\infty}/G_4$.

Figures 1 and 2 show the graphs of $R_2(t)^2$ and $\dot{R}_2/R_2$, respectively. We have defined dimensionless variables

$$r_2{}^2=\frac{R_2{}^2}{\pi G_4},\quad r_3{}^2=\frac{R_3{}^2}{\pi G_4},\quad t=\frac{t}{t_{\text{Planck}}}$$

and used the negative root with $e_4{}^2=\frac{1}{3}$ and $\lambda_4=\frac{1}{10}$. It is not difficult to check that within the bounds, the smaller the values for $e_4{}^2$ and λ_4 the quicker the curve reaches its asymptotic value.

There is, of course, a freedom in the choice of R_{20}

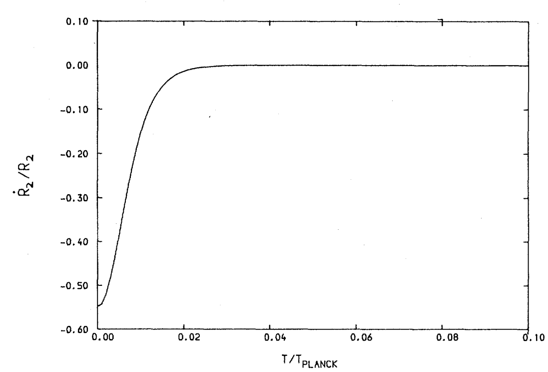

FIG. 2. We display the time variation of $\dot{R}_2/R_2$ since it is intimately connected with the variation of the coupling constants. As $R_2\to R_{2\infty}$, $\dot{R}_2/R_2\to 0$ and no time variation can be detected.

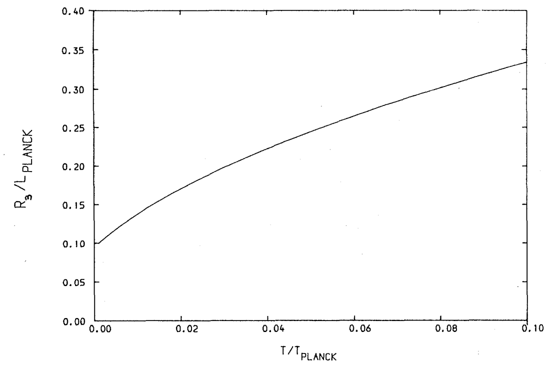

FIG. 3. The time variation of the physical radius, $R_3(t)$. We have chosen $R_{20}/R_{30}=50$ as an initial condition in order to maximize the effects from $R_2(t)$. As $R_2 \to R_{2\infty}$, R_3 evolves linearly in time.

which is unavoidable in these models. Nevertheless, the fact that such a well-behaved solution for $R_2(t)$ comes naturally from the field equations is a reassuring fact, especially when one notes that the time variation for the couplings lies well inside the presently accepted limits.[14]

We can also use the fourth-order Runge-Kutta method to solve for $R_3(t)$ and $\rho_m(t)$. Equations (16a) and (16b) combine to give the following equation for $R_3(t)$:

$$\frac{6K_3}{R_3^2}+\frac{6\ddot{R}_3}{R_3}+\frac{6\dot{R}_3^2}{R_3^2}\left[\frac{A_4}{R_2^4}+\frac{B_4}{R_2^2}+C_4\right]^{1/2}=-\frac{\alpha}{R_2^6}-\frac{\beta}{R_2^4}-\frac{\gamma}{R_2^2}+\delta\,, \quad (41)$$

where

$$\alpha=\frac{8A_4}{e_4^2}\,, \quad (42a)$$

$$\beta=\frac{8}{e_4^2}\left[1+B_4-\frac{\lambda_4}{4e_4^2}-\frac{2A_4e_4^2}{8}\right], \quad (42b)$$

$$\gamma=\frac{8}{e_4^2}\left[C_4+\frac{\phi_{04}\lambda_4}{2}\right], \quad (42c)$$

$$\delta=\frac{4}{e_4^2R_{2\infty}^4}\left[11+\frac{\lambda_4}{4e_4^2}\right]+\frac{64}{e_4^4R_{2\infty}^6}+\frac{1}{R_{2\infty}^2}\left[\frac{\lambda_4}{e_4^2}-6\right]. \quad (42d)$$

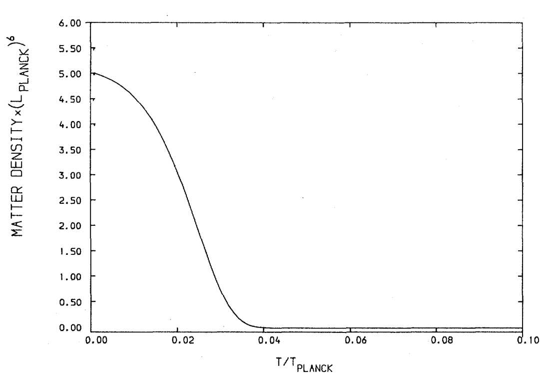

FIG. 4. The solution of the energy-momentum conservation equation for the time behavior of the matter density is shown. The asymptotic limit, $\rho_{m\infty}$, can be calculated analytically from the requirement of a zero four-dimensional cosmological constant. From dimensional analysis, it is easy to check that $[\rho_m]=L_p^{-6}$.

Figure 3 shows the time behavior of $R_3(t)$ with $K_3=-1$, $R_{20}^2=10.0$, and λ_4 and e_4^2 as before. We can see that because of the rapid fall of $R_2(t)$, the linear behavior sets in very rapidly. We have used a "small" initial value of R_{30} in order to maximize the effects of $R_2(t)$.

For $\rho_m(t)$ we use Eq. (31) together with the Higgs equation and, of course, the solution for $R_2(t)$. The result is a very complicated first-order equation for $\rho_m(t)$ which, nevertheless, has the interesting solution shown in Fig. 4. It is not difficult to check that the asymptotic limit agrees precisely with the one assumed in Eq. (35) in the appropriate units. The matter term is responsible for the asymptotic stability of the solutions, since it is balancing the fact that $\lim_{t\to\infty}(\ddot{R}_2/R_2)\neq 0$, in the first Einstein equation.

IV. CONCLUDING REMARKS

We have shown that the inclusion of the scalar field in generalized Kaluza-Klein models provides a useful way of studying the time behavior of the extra dimensions since it offers the possibility of solving the field equations for the internal radius. Although the model is nonrealistic due to the small SO(3) symmetry, we believe that this treatment can be extended to any SO(N) group provided that the internal space is not a product of spaces. Also, an extension to a $D=6$, $N=2$ supergravity theory coupled with a Maxwell field seems interesting. It is believed that it could describe physics at the preonic level.[15]

We must note that, as we move towards the singularity, there is an ambiguity in the initial conditions typical of any cosmological model. Nevertheless, the time behavior of the internal radius is always predictable from Eq. (23): If initially $R_2(0)=0$, (or very small), the internal space would evolve quickly to its asymptotic value for any choice of e_4^2 and λ_4. If our universe can be described by a higher-dimensional theory, the effects of the time variation of the internal radius are restricted to the very early stages of its cosmological evolution.

Finally, we believe that the present model can be of help in a serious analysis of the effects of the extra dimensions. We note that, in the recent literature, interesting predictions have been obtained for the behavior of the internal radius based in quantum effects and finite-temperature analysis but that a dynamical model was lacking.[16] Although most of this work was done for spheres of odd dimensionality, an extension of the present work is possible and can then be adapted to these previous results. This analysis is now in progress.

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[1]B. DeWitt, in *Relativity, Groups and Topology* (Gordon and Breach, New York, 1964); Y. M. Cho and P.G.O. Freund, Phys. Rev. D **12**, 1711 (1975); E. Witten, Nucl. Phys. **B186**, 412 (1981); E. Cremmer, in *Supergravity '81*, edited by S. Ferrara and J. G. Taylor (Cambridge University Press, London, 1982).

[2]A. Salam and J. Strathdee, Ann. Phys. (N.Y.) **141**, 316 (1982); W. Mecklenburg, Fortschr. Phys. **32**, 207 (1984); J. F. Luciani, Nucl. Phys. **B135**, 141 (1978).

[3]L. Alvarez-Gaumé and E. Witten, Nucl. Phys. **B234**, 269 1984.

[4]E. Witten, Princeton report, 1983 (unpublished), and references therein. For an explicit example, see S. Randjbar-Daemi, A. Salam, and J. Strathdee, Nucl. Phys. **B214**, 491 (1983).

[5]We are not considering here the "pure Kaluza-Klein" monopole that arises from the metric. See, for example, D. Gross and M. Perry, Nucl. Phys. **B226**, 29 (1983); P. Nelson, Harvard report, 1983 (unpublished).

[6]P.G.O. Freund and M. A. Rubin, Phys. Lett. **97B**, 233 (1980).

[7]P.G.O. Freund, Nucl. Phys. **B209**, 146 (1982); S. Randjbar-Daemi, A. Salam, and J. Strathdee, Phys. Lett **135B**, 388 (1984); W. J. Marciano, Phys. Rev. Lett. **52**, 489 (1984); M. Gleiser, S. Rajpoot, and J. G. Taylor, Phys. Rev. D **30**, 756 (1984).

[8]D. Sahdev, Phys. Lett. **137B**, 155 (1984); E. Alvarez and M. Belen Gavela, Phys. Rev. Lett. **51**, 931 (1983).

[9]M. Gleiser, S. Rajpoot, and J. G. Taylor, Ann. Phys. (N.Y.) (to be published); Phys. Lett. **138B**, 377 (1984).

[10]E. Cremmer and J. Scherk, Nucl. Phys. **B108**, 409 (1976).

[11]See, for example, A. Linde, Rep. Prog. Phys. **41**, 389 (1979).

[12]P.G.O. Freund, Phys. Lett. **120B**, 335 (1983).

[13]I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series and Products* (Academic, New York, 1980).

[14]See the paper by W. J. Marciano in Ref. 7.

[15]A. Salam and E. Sezgin, Trieste report (unpublished).

[16]P. Candelas and S. Weinberg, Nucl. Phys. **B237** 397 (1984); T. Appelquist and A. Chodos, Phys. Rev. Lett. **50**, 141 (1983); M. Nakahara, University of Alberta report, 1984 (unpublished).