PHYSICAL REVIEW D VOLUME 31, NUMBER 8 15 APRIL 1985

Gravitational models of a Lorentz extended electron

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(Received 18 October 1984)

We show how the Einstein-Maxwell field equations of general relativity can be used to construct a Lorentz model of an electron as an extended body consisting of pure charge and no matter. In contrast with Lorentz's approach using inertial mass, we associate the mass of the electron with its Schwarzschild gravitational mass. The Schwarzschild mass of an extended charged body as seen at infinity arises from the charge as well as the matter that the extended body possesses. The field equations for a Lorentz-type pure-charge extended electron are obtained by setting the matter terms equal to zero in the field equations for a spherically symmetric charged perfect fluid. Several explicit solutions to the pure-charge field equations are examined.

At the turn of the century, Lorentz proposed a model of an electron as an extended body consisting of only pure charge and no matter.[1] The mass of the electron was to arise from the energy in its electric field. The main difficulty with Lorentz's model was that it had no mechanism to overcome the electrostatic repulsion of the charge, so that the body was unstable and would "explode." To maintain stability, Poincaré postulated stresses that would hold the charge together. Because the Poincaré stresses were introduced in essentially an *ad hoc* fashion, the Lorentz model of an extended electron has not been widely accepted.

In his work Lorentz dealt only with the inertial aspect of mass. Lorentz certainly was aware of the Newtonian gravitational aspect of mass, but he probably disregarded gravitational effects because Newtonian gravitational forces are many orders of magnitude smaller than electrical forces for the values of the charge and mass of an electron. Inasmuch as Lorentz's ideas were developed before 1905, no use could have been made of Einstein's theories of special and general relativity.

In previous works[2,3] we have obtained results indicating that it is possible to construct a model of a Lorentz extended pure-charge electron within the context of the Einstein-Maxwell field equations of general relativity. This problem has also been addressed by others,[4–6] and, with particular relevance to the present paper, recently by Tiwari, Rao, and Kanakamedala[7] (TRK). In the approach taken here and by TRK, the electron's mass is associated with the Schwarzschild gravitational mass given by general relativity, and not with the inertial mass used by Lorentz.

We take the metric around an electron to be of a static form

$$ds^2=g_{ij}dx^i dx^j - V^2 dT^2 \tag{1}$$

that for large distances away from the electron reduces to the Reissner-Nordström metric

$$ds^2=dR^2/V^2+R^2d\Omega^2-V^2dT^2\ , \qquad d\Omega^2=d\theta^2+\sin^2\theta\, d\phi^2\ , \tag{2}$$

where

$$V^2=1-2M/R+Q^2/R^2\ . \tag{3}$$

In terms of the gravitational units we are using, we have for an electron

$$M=6.76\times 10^{-58}\ \text{m},\quad Q=1.38\times 10^{-36}\ \text{m}\ ,$$
$$c=M/Q=4.90\times 10^{-22}\ , \tag{4}$$
$$R_e=Q^2/M=2.82\times 10^{-15}\ \text{m (classical electron radius)}\ .$$

The charge Q of the electron is obtained by integrating the electron's charge density μ over its proper volume v_s:

$$Q=\int_{v_s}\mu\, dv_3\ , \tag{5}$$

where the invariant volume element dv_3 is given by

$$dv_3=(\bar{g})^{1/2}dx^1dx^2dx^3\ , \tag{6}$$

where $\bar{g}$ is the determinant of the spatial part of the metric.

The Schwarzschild mass M in (3) can be expressed in a similar manner in terms of the integral of a "Schwarzschild mass density" ρ_M,

$$M=\int_{v_s}\rho_M dv_3\ , \tag{7}$$

where ρ_M is given by[2]

$$\rho_M=(M_i^i-M_4^4)V+\mu\Psi\ . \tag{8}$$

Here the total stress-energy tensor T_ρ^σ is the sum of the material and electrical parts,

$$T_\rho^\sigma=M_\rho^\sigma+E_\rho^\sigma \tag{9}$$

and Ψ is the electrical potential inside the source.

The density expression in (8) shows that in addition to the material components M_ρ^σ, the charge density μ of (5), with its associated electric potential Ψ, also contributes to the Schwarzschild gravitational mass M seen at infinity. It is here where the possibility of a Lorentz pure-charge extended electron is seen to exist within the Einstein-Maxwell equations of general relativity, for even with no actual "matter" present, charge density and stresses will result in a source that has a Schwarzschild gravitational mass.

For static, spherically symmetric fields the metric (1) can be written in the form

$$ds^2=dR^2/A^2+R^2d\Omega^2-V^2dT^2 . \quad (10)$$

The Einstein field equations $G^\sigma_\rho=-8\pi T^\sigma_\rho$ yield the following relations:

$$-V^2\frac{d}{dR}(A^2/V^2)=8\pi R(T^1_1-T^4_4) , \quad (11)$$

$$\frac{d}{dR}[R(A^2-1)]=8\pi R^2T^4_4 , \quad (12)$$

$$T^2_2=T^3_3=\frac{Rd}{2dR}(T^1_1)+T^1_1+(R/2V)(T^1_1-T^4_4)\frac{d}{dR}(V) . \quad (13)$$

The electromagnetic stress-energy tensor and Maxwell's equations

$$E_{\rho\sigma}=F^\alpha_\rho F_{\alpha\sigma}-\tfrac{1}{4}g_{\rho\sigma}F_{\alpha\beta}F^{\alpha\beta} , \quad (14)$$

$$F_{\rho\sigma}=\Psi_{\sigma,\rho}-\Psi_{\rho,\sigma} , \quad (15)$$

$$F^{\rho\alpha}{}_{;\alpha}=(-g)^{-1/2}\frac{\partial}{\partial x^\alpha}[(-g)^{1/2}F^{\rho\alpha}]=J^\rho \quad (16)$$

become with the metric form (10)

$$-E^1_1=E^2_2=E^3_3=-E^4_4=(1/8\pi)(A^2/V^2)\left[\frac{d}{dR}\Psi\right]^2 , \quad (17)$$

$$(A/R^2)\frac{d}{dR}\left[(A/V)R^2\frac{d}{dR}\Psi\right]=-4\pi\mu \quad (18)$$

with

$$\Psi_\rho=(0,0,0,\Psi/\sqrt{4\pi}) . \quad (19)$$

If we take the material part of the stress-energy tensor to be a perfect fluid,

$$M_{\rho\sigma}=(\rho+p)V_\rho V_\sigma+pg_{\rho\sigma} , \quad (20)$$

we have

$$M^1_1=M^2_2=M^3_3=p,\ \ M^4_4=-\rho . \quad (21)$$

With $T^\sigma_\rho=M^\sigma_\rho+E^\sigma_\rho$, the following set of equations for a charged perfect fluid are then obtained:

$$-V^2\frac{d}{dR}(A^2/V^2)=8\pi R(\rho+p) , \quad (22)$$

$$(A^2/V^2)\left[\frac{d}{dR}(RV^2)+\left[R\frac{d}{dR}\Psi\right]^2\right]=8\pi R^2p+1 , \quad (23)$$

$$\frac{d}{dR}p=(A/V)\left[\frac{d}{dR}\Psi\right](1/4\pi R^2)\times\frac{d}{dR}\left[(A/V)R^2\frac{d}{dR}\Psi\right]-[(\rho+p)/2V^2]\frac{d}{dR}(V^2) \quad (24)$$

In addition (8) becomes

$$\rho_M=(3p+\rho)V+\mu\Psi . \quad (25)$$

The field of a source composed of only pure charge and no matter will be obtained by setting the matter density terms equal to zero in Eqs. (22)–(25). Before this can be accomplished, though, we must first identify those terms that correspond to actual "matter."

This identification is not at all straightforward. The quantity ρ in (21) measures not only the density of matter, but also the density of energy, including binding energy, that resides in a given infinitesimal volume.[8] If the source contained only matter and no charge, setting the matter density to zero would automatically produce zero binding energy in a given volume, for there would be nothing for the binding energy to bind. For the situation we are here considering, however, even though there is no matter in a given infinitesimal volume, there will be charge on which forces can be exerted. Therefore, setting $\rho=0$ is not necessarily the way to achieve a pure-charge source.

TRK (Ref. 7) and independently the present author[3] have suggested considering as a pure-charge equation of state

$$\rho+p=0 \text{ (pure-charge condition)} . \quad (26)$$

Assuming $\rho>0$, (26) necessarily requires a negative pressure, i.e., the source will be under tension.

With the condition (26) we find from (22) that A/V=const, and since outside the source $A=V$, we have from continuity of $g_{\rho\sigma}$ that everywhere

$$A=V . \quad (27)$$

When the pure-charge condition (26) or equivalently (27) is imposed, Eqs. (23)–(25) and Maxwell's equation (18) take the form

$$\frac{d}{dR}(RV^2)+\left[R\frac{d}{dR}\Psi\right]^2=8\pi R^2p+1 , \quad (28)$$

$$dp/dR=(d\Psi/dR)(1/4\pi R^2)\frac{d}{dR}\left[R^2\frac{d}{dR}\Psi\right] , \quad (29)$$

$$(1/4\pi R^2)\frac{d}{dR}\left[R^2\frac{d}{dR}\Psi\right]=-\mu/V , \quad (30)$$

$$\rho_M=2pV+\mu\Psi . \quad (31)$$

A particular solution for the pure-charge condition (26) has been obtained by TRK (Ref. 7) by observing that (30) can be integrated directly if the charge density satisfies

$$\mu/V=\mu_0=\text{const} \quad (32)$$

in which μ_0 is the charge density at $R=0$. The details of the TRK solution can be found in Ref. 7. We note, though, a point not brought out by TRK that at the radius R_s of the source the boundary conditions $\Psi_s=Q/R_s$, $q(R_s)=Q$, and $m(R_s)=M$ require that

$$R_s=\tfrac{4}{5}Q^2/M=\tfrac{4}{5}R_e . \quad (33)$$

Thus the radius of the TRK "constant" charge-density electron equals $\frac{4}{5}$ of the classical electron radius

$R_e = Q^2/M$. It is interesting that this is the same value for R_s that arises from a classical analysis of the momentum of the electromagnetic field of an extended electron with a uniform charge density (Feynman *et al.*, Ref. 1).

Another pure-charge solution different from the TRK solution can be obtained by assuming that the source is structured such that the ratio of the Schwarzschild mass density ρ_M given in (8) to the charge density μ defined in (5) is a constant c over the extent of the source:

$$\rho_M/\mu = c = M/Q \ . \qquad (34)$$

We have previously investigated this type of source in detail,[2] and have shown in general that the metric component V, which represents the gravitational potential, and the electric potential Ψ are related both inside and outside the source by

$$V^2 = 1 - 2c\Psi + \Psi^2 = (\Psi - c)^2 + 1 - c^2 \ . \qquad (35)$$

This relationship for the exterior region $R > R_s$ was first obtained by Weyl[9] in an investigation where V and Ψ are functionally related, so we refer to such fields as "Weyl-type" fields. We thus have here the general-relativistic analog of a Newtonian situation in which a source with a constant ratio of mass and charge density produces gravitational and electrical potentials that are everywhere functionally related.

From (31) we obtain with (34)

$$\mu/V = -2p/(\Psi - c) \ . \qquad (36)$$

Substituting this into (30) and using (29) we get

$$(1/p)dp/dR = [2/(\Psi - c)]\frac{d}{dR}(\Psi - c) \qquad (37)$$

whose solution is

$$p/p_0 = (\Psi - c)^2/(\Psi_0 - c)^2 \ , \qquad (38)$$

where p_0 and Ψ_0 are the values of p and Ψ at $R = 0$. Substituting (38) and (35) into (28) we obtain

$$\left\{\frac{d}{dR}[R(\Psi - c)]\right\}^2 - [8\pi p_0/(\Psi_0 - c)^2][R(\Psi - c)]^2 = c^2 \ . \qquad (39)$$

The solution to (39) will depend on the sign of p_0. We will show below that the boundary conditions at $R = R_s$ will require p_0 to be negative. From (38) it is seen that p will then be negative throughout the source; that is, the charge in the source is under tension. In turn, from (26), a negative value for p corresponds to a positive value for ρ. Setting

$$a^2 = -(\Psi_0 - c)^2/8\pi p_0 \qquad (40)$$

the solution to (39) is

$$\Psi - c = (ac/R)\sin(R/a) \ , \qquad (41)$$

where the integration constant in the argument of the sine function has been set equal to zero in order for Ψ not to be infinite at $R = 0$. From (41) we then find

$$\Psi_0 = 2c \ , \qquad (42)$$

so that

$$a^2 = -c^2/8\pi p_0 \ . \qquad (43)$$

The charge $q(R)$ and the Schwarzschild mass $m(R) = cq(R)$ as a function of R inside the source are obtained by using (36) to obtain

$$q(R) = m(R)/c = \int_0^R \mu\, dv_3 = cR[(a/R)\sin(R/a) - \cos(R/a)] \ . \qquad (44)$$

Using this, (35) can be written as

$$A^2 = V^2 = 1 - 2m(R)/R + q^2(R)/R^2 - c^2[1 + \cos(R/a)][1 + \cos(R/a) - (2a/R)\sin(R/a)] \ . \qquad (45)$$

The boundary conditions at $R = R_s$ that $\Psi(R_s) = Q/R_s$ and $q(R_s) = Q$ [or $m(R_s) = M$] yield

$$R_e/R_s = 1 + (a/R_s)\sin(R_s/a) \ , \qquad (46)$$

$$R_e/R_s = (a/R_s)\sin(R_s/a) - \cos(R_s/a) \ . \qquad (47)$$

The two conditions (46) and (47) require

$$\cos(R_s/a) = -1 \ , \qquad (48)$$

which fixes the value of R_s/a to be

$$R_s/a = \pi \ . \qquad (49)$$

[If we had assumed a positive value for p_0 in (39), the relationship corresponding to (48) would be $\cosh(R/a) = -1$, showing that p_0 must be negative.]

The condition (49) means that p will vanish at the surface of the source. In turn, E^1_1 and T^1_1 will be continuous across R_s. In addition, (46) or (47) shows that

$$R_s = R_e \ . \qquad (50)$$

Thus the radius of a Weyl-type pure-charge electron is exactly equal to the classical electron radius $R_e = Q^2/M$.

To summarize the above results, the following expressions hold in the interior region $R \leq R_s = R_e$ of a Weyl-type pure-charge electron:

$$\Psi = c + (cR_e/\pi R)\sin(\pi R/R_e) \ , \qquad (51)$$

$$\rho = -p = (c^2/8\pi R^2)\sin^2(\pi R/R_e) \ , \qquad (52)$$

$$A^2 = V^2 = [(cR_e/\pi R)\sin(\pi R/R_e)]^2 + 1 - c^2 \ . \qquad (53)$$

We thus see that within the formalism of the Einstein-Maxwell equations of general relativity it is possible to construct a model of an electron envisioned by Lorentz as an extended body composed of pure charge without matter. In contrast with inertial mass considered by Lorentz, we have here dealt with the general-relativistic Schwarzschild gravitational mass of an electron arising from a source structured such that it has only pure charge and no matter. The pure-charge condition given by (26) seems reasonable, but other possible conditions relevant to the idea of Lorentz should be explored.

We have pointed out elsewhere[10] that the effective mass M_T of a source with Schwarzschild mass M and charge Q

that attracts uncharged test particles is

$$M_T = M - Q^2/R = M(1 - R_e/R) . \tag{54}$$

It is seen that if the radius R_s of the source is such that $R_s < R_e$, in the region outside the source M_T will become negative for sufficiently small values of R, resulting in a repulsion instead of attraction of uncharged test particles that approach the neighborhood of the source.[10] With the TRK model, $R_s = \frac{4}{5}R_e < R_e$, so that M_T will be negative in the vicinity of an electron, with corresponding repulsive forces being exerted on uncharged test particles in this region. With the Weyl-type model of an electron $R_s = R_e$ resulting in $M_T = 0$ at the boundary of the electron, so that there will be no repulsive forces on uncharged test particles around this type of electron. Effects such as repulsive forces on uncharged test particles produced by the gravitational effective mass M_T around elementary particles like electrons could be subject to experimental test.

I would like to thank Professor Banesh Hoffmann for enlightening discussions on the matters of this paper.

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[1]Lorentz's ideas have been widely discussed in the literature. See, for example, F. Rohrlich, *Classical Charged Particles* (Addison-Wesley, Reading, Mass., 1965); R. P. Feynman, R. B. Leighton, and M. Sands, *The Feynman Lectures on Physics* (Addison-Wesley, Palo Alto, 1964), Vol. II, Chap. 28.

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