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Charged spin fluid in the Einstein-Cartan theory

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We propose a variational principle describing a charged spin fluid in the Einstein-Cartan theory. We show that this fluid can be described by the current vector V_i which has a potential decomposition and generalizes the results given by Taub. We also derive Maxwell's equations in the presence of spin and torsion. The Eulerian description of the fluid is given by an action integral whose Lagrangian is the pressure plus the free Lagrangians of the gravitational and electromagnetic fields. Finally, we analyze the circulation and Bernoulli theorems using the current vector V_i.

I. INTRODUCTION

In previous works[1,2] we dealt with spin fluids in the Einstein-Cartan-Kibble-Sciama theory (ECKS theory) and gave a variational principle and an Eulerian description by means of the potential decomposition of the current vector V_i. Moreover, by considering some particular situations, we described some features of this fluid.

In the present paper we extend this discussion to a charged spin fluid. The theory of a charged fluid in general relativity (GR) is completely established (see Ref. 3 and references therein). Moreover, a Lagrangian formulation for charged polarized media, which are similar to charged spin fluids, may be found in Ref. 4. However, our aim is to give an Eulerian description of a charged spin fluid in the presence of torsion.

The concept of torsion was introduced in 1922 by Cartan,[5] but it was only in the 1960's that Kibble[6] and Sciama[7] utilized it to account for the effects of spin in a general-relativistic theory (a complete discussion of the subject can be found in Ref. 8).

We assume that the Lagrangian for the present case is similar to the Lagrangian given in Ref. 1. We have to modify the total energy density term by adding the interaction between the electromagnetic field and the magnetoelectric moment (proportional to the spin tensor by a factor $e/2m_0$); moreover we have added two terms. One is $(1/16\pi)F_{ij}F^{ij}$, the Lagrangian of the electromagnetic field; the other one is $-J_iA^i$, the electromagnetic potential energy.

We demonstrate that this new material Lagrangian density is given by

$$\mathscr{L}_n=\sqrt{-g}\left[-p+\frac{1}{16\pi}F_{ij}F^{ij}\right],$$

where we now have also to consider the electromagnetic interactions in the pressure p [see (2.13)]. In this formulation it is still possible to construct the current vector V_i and decompose it into potentials, as in Refs. 1, 10, and 11, among which appears now the four-vector electromagnetic potential A_i. We propose, finally, to extend the circulation theorems (see Refs. 2, 3, 11, and 12) to the charged case.

In Sec. II we give the variational principle and the evolution equations and then we discuss the physical properties of the Lagrangian density. In Sec. III we derive Maxwell's equations in U_4 spacetime from a variational principle and extend them to account for the presence of spin. The equations for the tetrad vectors are derived in Sec. IV, where we define the current vector V_i and we also give its decomposition into potentials and finally we construct the canonical energy-momentum tensor. In Sec. V we derive the Einstein-Cartan equations and the dynamical energy-momentum and spin tensors. In Sec. VI we extend the circulation theorems to the present case.

II. VARIATIONAL PRINCIPLE

In Ref. 1 we introduced for a spin fluid the Lagrangian density

$$\mathscr{L}=\sqrt{-g}\,[\,\mu(\rho,S,h_0)+\rho U^i\partial_i\phi+\rho\theta U^i\partial_i S+\rho C U^i\partial_i B+\rho h_0 a^k U^i\nabla_i b_k+\lambda_{ij}(a^ia^j+b^ib^j+\hat{\sigma}^i\hat{\sigma}^j-U^iU^j-g^{ij})]$$

extending to the ECKS theory the special-relativistic approach given by Halbwachs[13] and generalizing also the general-relativistic formulation of the dynamics of a perfect fluid given by Schutz in Ref. 11 using the velocity potentials. In $\mathscr{L}$, $\mu(\rho,S,h_0)$ is the total energy density; ρ is the matter density (the number of particles per unit volume); S is entropy; ϕ, θ, and C are Lagrange multipliers that impose, respectively, the conservation of the number of particles, of the entropy, and of the identity of the particles, B being one of the Lagrangian coordinates; U^i is the four-velocity, normalized such that $U^iU_i=-1$; g_{ij} is the metric tensor of signature $(-,+,+,+)$; σ_i is the spin vector, and $\hat{\sigma}_i=\sigma_i/\rho h_0$ such that $\hat{\sigma}^i\hat{\sigma}_i=+1$ and it is normal to U^i $(U^i\hat{\sigma}_i=0)$; h_0 is a standard spin module function;[13] a^i and b^i are spacelike vectors constrained by the Lagrangian multipliers λ_{ij} to form with $\hat{\sigma}^i$ and U^i a vierbein at each point of spacetime.

By the vierbein properties we may define the spin density tensor

$$S_{ij}\equiv\epsilon_{ijkm}U^m\sigma^k=\rho h_0(a_ib_j-a_jb_i) \tag{2.1}$$

and the spin kinetic energy

$$W=\tfrac{1}{2}\omega^{ij}S_{ij}=\rho h_0 a^k\dot{b}_k\ ,$$

where ω^{ij} is the angular velocity associated with the spin of the particle,

$$\omega^{ij}=\tfrac{1}{2}(\dot{a}^ia^j-a^i\dot{a}^j+\dot{b}^ib^j-b^i\dot{b}^j+\dot{\hat{\sigma}}^i\hat{\sigma}^j-\hat{\sigma}^i\dot{\hat{\sigma}}^j-\dot{U}^iU^j+U^i\dot{U}^j)$$

and the dot means $\dot{f}=U^i\nabla_i f$ for a tensorial quantity f and $\dot{\mathscr{F}}=\nabla_i(\rho F U^i)$ for a density tensor $\mathscr{F}=\rho f$.

We shall consider a fluid constituted by only one family of particles characterized by a charge e, a rest mass m_0, and a spin function h_0.

We recall that[8,14] minimal coupling does not apply to the electromagnetic field F_{ij} in a U_4 spacetime, so it is still defined by

$$F_{ij}=\partial_iA_j-\partial_jA_i\ . \tag{2.2}$$

From this the first group of Maxwell's equations may be derived:

$$\partial_{[k}F_{ij]}=0$$

or[14]

$$\nabla_{[i}F_{kj]}=2S_{[ik}{}^mF_{j]m}\ , \tag{2.3}$$

where $S_{ij}{}^k$ is the torsion tensor and square brackets indicate antisymmetrization. The second group of Maxwell's equation will be derived in Sec. III.

As in Ref. 3 the electric current J_i is defined by

$$J_i=\rho eU_i+\sigma_cU_jF_i^j\ , \tag{2.4}$$

where σ_c is the conductivity of the fluid medium.

We shall consider the case with $\sigma_c=0$; i.e., the electric current is purely convective.

In Ref. 1 we have found that the spin alone influences the fluid thermodynamical behavior; but Ray and Smalley[15] showed a similar result in a more general way.

In their approach we have also to consider the interaction between spin and the electromagnetic field; i.e., we have to consider the relativistic extension of that coupling density energy $(e/2m_0)S_{ij}F^{ij}$. Taking into account this new energy contribution, following Lichnerowicz[3] we assume that the first principle of thermodynamics in this case becomes

$$d\bar{\epsilon}=\rho^{-1}dp+T\,dS+\tfrac{1}{2}\omega_{ij}dS^{ij}+\frac{e}{2m_0}S_{ij}dF^{ij}\ , \tag{2.5}$$

where the density of enthalpy ϵ is given by $\epsilon=(\bar{\mu}+p)/\rho_0$, p is the pressure, $\bar{\epsilon}=m_0\epsilon$ and $\rho_0=m_0\rho$;

$$\bar{\mu}=\mu(\rho,h_0,S)+\frac{e}{2m_0}S_{ij}F^{ij}$$

is the total internal energy density. We are assuming that the new coupling energy contribution appears linearly in the expression of the total energy density; the reason for this assumption is that from the expression of the first principle and the definition of ϵ, we have

$$\frac{\partial\bar{\mu}}{\partial F_{ij}}=\frac{e}{2m_0}S^{ij}\ ;$$

furthermore, in the limit of the neutral spin fluid, using this assumption we immediately get back the expression already found in Refs. 1 and 15.

To describe the charged spin fluid we propose the Lagrangian density

$$\mathscr{L}_n=\Big[\mu(\rho,h_0,S)+\frac{e}{2m_0}S_{ij}F^{ij}+\rho U^i\partial_i\phi+\rho\theta U^i\partial_iS+\rho CU^i\partial_iB+\rho h_0a^iU^k\nabla_kb_i+\lambda_{ij}(a^ia^j+b^ib^j+\hat{\sigma}^i\hat{\sigma}^j-U^iU^j-g^{ij})+\frac{1}{16\pi}F_{ij}F^{ij}-J^iA_i\Big]\sqrt{-g}\ . \tag{2.6}$$

Varying (2.6) with respect to ϕ, θ, S, C, B, and h_0, we have, respectively,

$$\delta\phi:\ \nabla_i(\rho U^i)=0\ , \tag{2.7}$$

$$\delta\theta:\ \dot{S}=0\ , \tag{2.8}$$

$$\delta S:\ \dot{\theta}=\frac{\partial\mu}{\partial S}\Big|_{\rho,h_0,S_{ij}F^{ij}}\equiv T\ , \tag{2.9}$$

$$\delta C:\ \dot{B}=0\ , \tag{2.10}$$

$$\delta B:\ \dot{C}=0\ , \tag{2.11}$$

$$\delta h_0:\ \rho a^k \dot{b}_k + \frac{\partial \mu}{\partial h_0} + \frac{e}{2m_0 h_0} S_{ij} F^{ij} = 0 \,, \tag{2.12}$$

and, by Eqs. (2.8) and (2.10), we get

$$\delta\rho:\ \frac{\partial \mu}{\partial \rho} + \dot{\phi} + h_0 a^k \dot{b}_k - eU^i A_i + \frac{e}{2m_0} \hat{S}_{ij} F^{ij} = 0 \tag{2.13}$$

from which since pressure is given[13] by

$$p = \rho \frac{\partial \mu}{\partial \rho} - \mu \tag{2.14}$$

we have that the Lagrangian is physically interpreted as

$$\mathcal{L}_M = -p + \frac{1}{16\pi} F_{ij} F^{ij} \,. \tag{2.15}$$

As in Newtonian mechanics,[10] special relativity,[13] general relativity,[11] and ECKS theory,[1] on the field equations, the material Lagrangian is the pressure (with a negative sign) plus, in this case, the free electromagnetic Lagrangian.

III. MAXWELL'S EQUATIONS IN A U_4 SPACETIME

Let us derive the second group of Maxwell's equations. From (2.6) the electromagnetic action is

$$I = \int \sqrt{-g} \left[-J^i A_i + \frac{1}{16\pi} F_{ij} F^{ij} + \mu(\rho, h_0, S) + \frac{e}{2m_0} S_{ij} F^{ij} \right] d^4x \,.$$

The variation of I with respect to the electromagnetic four-potential A is

$$\delta I = -\int \sqrt{-g} \left[J^i \delta A_i - \frac{1}{8\pi} F^{ij} \delta F_{ij} - \frac{e}{2m_0} S^{ij} \delta F_{ij} \right] d^4x \,. \tag{3.1}$$

Using (2.5), and the expression for $\bar{\epsilon}$, from the (3.1) we get the Maxwell's equations

$$\overset{*}{\nabla}_j (\sqrt{-g} F^{kj}) + \sqrt{-g} F^{ij} S_{ij}^k = \sqrt{-g}\, 4\pi J^k - 4\pi \overset{*}{\nabla}_j \left[\frac{e}{m_0} \sqrt{-g} S^{kj} \right] - \frac{e}{m_0} (\sqrt{-g} S^{ij}) S_{ij}^k \,. \tag{3.2}$$

Equations (3.2) generalize the second group of Maxwell's equations derived by Prasanna[14] in a U_4 spacetime, the presence of the last two terms being due to the interaction between the magnetoelectric moment density and the electromagnetic field.

IV. THE CURRENT VECTOR V_i AND THE CANONICAL ENERGY-MOMENTUM TENSOR

If we vary (2.6) with respect to the vierbein vectors, we have

$$\delta u^i:\ \rho(\partial_i \phi + \theta \partial_i S + C \partial i B + h_0 a^k \nabla_i b_k - eA_i) - 2\lambda_{ij} U^j = 0 \,, \tag{4.1}$$

$$\delta a^i:\ \rho h_0 \dot{b}_i + 2\lambda_{ij} a^j + \frac{e}{m_0} \rho h_0 b^m F_{im} = 0 \,, \tag{4.2}$$

$$\delta b^i:\ -\rho a_i \dot{h}_0 - \rho h_0 \dot{a}_i + 2\lambda_{ij} b^j + \frac{e}{m_0} \rho h_0 a^m F_{mi} = 0 \,, \tag{4.3}$$

$$\delta \hat{\sigma}^i:\ 2\lambda_{ij} \hat{\sigma}^j = 0 \,. \tag{4.4}$$

Multiplying (4.1) by U_k, (4.2) by a_k, (4.3) by b_k, (4.4) by $\hat{\sigma}_k$, summing and antisymmetrizing, we have

$$\dot{S}_{ki} + \frac{e}{m_0} S_k^m F_{mi} - \frac{e}{m_0} S_i^m F_{mk} + \rho U_k (\partial_i \phi + \theta \partial_i S + C \partial_i B + h_0 a^m \nabla_i b_m - eA_i) - \rho U_i (\partial_k \phi + \theta \partial_k S + C \partial_k B + h_0 a^m \nabla_k b_m - eA_k) = 0 \,. \tag{4.5}$$

Equation (4.5) represents the motion equation for the spin density tensor. Contracting (4.5) with U^k and using (2.13) we get

$$\dot{S}_{ki} U^k + \rho \bar{\epsilon} U_i - \frac{e}{2m_0} S_{km} F^{km} U_i - \frac{e}{m_0} S_i^m F_{mk} U^k = \rho(\partial i \phi + \theta \partial_i S + C \partial_i B + h_0 a^m \nabla_i b_m - eA_i) \,. \tag{4.6}$$

From (4.6) we note that vector V_i,

$$V_i = \bar{\epsilon} U_i + \hat{S}_{ij} \left[\dot{U}^k + \frac{e}{m_0} F^{km} U_m \right] , \tag{4.7}$$

may be decomposed into potentials.

This vector, quoted as current vector according to Ref. 11, generalizes both the current vector defined by Taub[12] and the current vectors defined in Refs. 11 and 1 for the presence of spin and the electromagnetic interactions.

From (4.5), contracting by a^k, b^i, we get

$$\dot{h}_0 = 0 \,, \tag{4.8}$$

i.e., the spin module function is constant along the flow lines. The canonical energy-momentum tensor is defined by

$$\sqrt{-g}\, \Sigma_i{}^j = \frac{\partial \mathcal{L}}{\partial_j \psi} \nabla_i \psi + \frac{\partial \mathcal{L}}{\partial \partial_j A_k} \partial_i A_k - \delta_i^j \mathcal{L} \,,$$

where we do not apply the minimal coupling to the definition of the electromagnetic energy-momentum tensor.

Using (4.6), we get

$$\Sigma_i^j=\rho(\partial_i\phi+\theta\partial_i S+C\partial_i B+h_0 a^k\nabla_i b_k)U^j+\frac{1}{4\pi}F^{jm}\partial_i A_m+\frac{e}{m_0}S^{jm}\partial_i A_m+\delta_i^j\left[p-\frac{1}{16\pi}F_{km}F^{km}\right] . \quad (4.9)$$

In order to symmetrize the electromagnetic terms we add the divergence

$$-\partial_k\left[\frac{1}{4\pi}A_i F^{jk}+\frac{e}{m_0}A_i S^{jk}\right]$$

and by using the second group of Maxwell's equations, written with ordinary derivatives, we get

$$\Sigma_i^j=\rho\bar{\epsilon}U_i U^j+S_{ik}U^j\left[\dot{U}^k-\frac{e}{m_0}F^{km}U_m\right]+\frac{1}{4\pi}F_{im}F^{jm}+\frac{e}{m_0}F_{im}S^{jm}+\delta_i^j\left[p-\frac{1}{16\pi}F_{km}F^{km}\right] \quad (4.10)$$

or by (4.7)

$$\Sigma_i^j=\rho V_i U^j+\frac{1}{4\pi}(F_{im}F^{jm}-\tfrac{1}{4}\delta_i^j F_{km}F^{km})+\frac{e}{m_0}S^{jm}F_{im}+\delta_i^j p . \quad (4.11)$$

In the energy-momentum tensor (4.11) the following three terms appear:

$$\rho V_i U^j+\delta_i^j p ,$$

the energy-momentum tensor of the charged spin fluid;

$$\frac{1}{4\pi}(F_{im}F^{jm}-\tfrac{1}{4}\delta_i^j F_{km}F^{km}) ,$$

the electromagnetic energy-momentum tensor;

$$\frac{e}{m_0}F_{im}S^{jm} ,$$

the energy-momentum tensor due to the interaction between the spin density and the electromagnetic field.

V. EINSTEIN-CARTAN EQUATIONS

Going back to the results of Sec. II, where the material Lagrangian is given by (2.15), and considering that, by Eqs. (2.14) and (4.7), the pressure p can be written

$$p=-V_i U^i-\mu , \quad (5.1)$$

where V_i is expressed through its decomposition into potentials [see Eq. (4.6)]

$$V_i=\partial_i\phi+\theta\partial_i S+C\partial_i B+h_0 a^k\nabla_i b_k-eA_i \quad (5.2)$$

we can now consider the variational principle according to Refs. 10, 11, and 1, whose action integral is

$$I=\int(L_F-2kp)\sqrt{-g}\,d^4x ,$$

L_F being the free Lagrangian density of the electromagnetic and gravitational fields, i.e.,

$$L_F=R+2k\left[\frac{1}{16\pi}F_{ij}F^{ij}\right] ,$$

R being the scalar curvature of U_4 and k the relativistic gravitational constant. By (5.2) we shall get both the field equations of ECKS theory and the results of the preceding sections. Varying (5.2) with respect to the contortion tensor $K_{ij}{}^k$ defined by

$$K_{ij}{}^k=-S_{ij}{}^k+S_j{}^k{}_i-S^k{}_{ij}$$

we get

$$\delta K_{ij}{}^k:\ T_{ij}{}^k=k\tau_{ij}{}^k=k\tfrac{1}{2}S_{ij}U^k \quad (5.3)$$

where $T_{ij}{}^k\equiv S_{ij}{}^k+2\delta^k_{[i}S_{j]m}{}^m$ is the modified torsion tensor given by

$$\frac{1}{\sqrt{-g}}\frac{\delta R}{\delta K_{ij}{}^k}=-2T_k{}^{ji}$$

and $\tau_{ij}{}^k$ is the dynamical spin tensor defined by

$$\sqrt{-g}\,\tau_k{}^{ji}\equiv\frac{\delta p}{\delta K_{ij}{}^k}=\tfrac{1}{2}S_k{}^j U^i . \quad (5.4)$$

We want to note that the canonical spin tensor connected to the Lagrangian density (2.5) has the expression

$$\tau_{ij}{}^k=\tfrac{1}{2}S_{ij}U^k+\frac{1}{4\pi}A_{[i}F_{j]}{}^k+\frac{e}{m_0}A_{[i}S_{j]}{}^k \quad (5.5)$$

which is different from the one given for the dynamical spin tensor [equation (5.4)], because of the presence of the gauge-dependent terms $(1/4\pi)A_{[i}F_{j]}{}^k$ which represents the spin of the photon, and $(e/m_0)A_{[i}S_{j]}{}^k$ a term due to the interaction between the electromagnetic field and the spin of the particles. This happens because in (5.2) we have not applied the minimal coupling to the electromagnetic field, so F_{ij} cannot be coupled with the contortion tensor in the Lagrangian. Moreover, from a dynamical point of view, the spin tensor $\tau_{ij}{}^k$ is linked through the field equation (5.3) to the torsion, which is a measurable quantity.

Varying (5.1) with respect to the metric tensor g_{ij}, we get

$$\delta g^{ij}:\ G_{ij}-\overset{*}{\nabla}_k(T_{ij}{}^k+T^k{}_{ji}+T^k{}_{ij})=k\left[\rho V_{(i}U_{j)}+g_{ij}\left[p-\frac{1}{16\pi}F_{km}F^{km}\right]+\frac{1}{4\pi}F_{im}F_j{}^m+\frac{e}{m_0}F_{im}S_j{}^m\right] , \quad (5.6)$$

G_{ij} being the Einstein tensor: $G_{ij}=R_{ij}+\frac{1}{2}g_{ij}R$. Applying equation (5.3) in (5.6), we have

$$G_{ij}=k\left[\tfrac{1}{2}\overset{*}{\nabla}_k(S_{ij}U^k)+\rho V_{(i}U_{j)}+g_{ij}\left[p-\frac{1}{16\pi}F_{km}F^{km}\right]+\frac{1}{4\pi}F_{im}F_j{}^m+\frac{e}{m_0}S_{jm}F_i{}^m\right]+\frac{k}{2}\overset{*}{\nabla}_k(S^{ki}U^j+S^{kj}U^i)\ . \quad (5.7)$$

By using the Weyssenhoff condition ($S_{ij}U^j=0$) and the equation (4.5) one can write

$$G_{ij}=k\left[\rho V_iU_j+\frac{1}{4\pi}F_{im}F^{jm}+\frac{e}{m_0}S_{jm}F_i^m+g_{ij}\left[p-\frac{1}{16\pi}F_{km}F^{km}\right]\right]\ , \quad (5.8)$$

where the last term in (5.7) can be eliminated adding, in the integral action, the null term:[1]

$$\frac{k}{2}\int\sqrt{-g}\,g_{ij}\overset{*}{\nabla}_k(S^{ki}U^j+S^{kj}U^i)d^4x\ .$$

The right-hand side (RHS) of Eq. (5.8) is the dynamical energy-momentum tensor which is equivalent[8] to the canonical energy-momentum tensor Σ_{ij} of Eq. (4.11).

The variations of (5.2) with respect to ϕ, θ, S, C, B, h_0, a_k, and b_K give us the equations from (2.7) to (2.12) and the equations (4.2) and (4.3).

The Bianchi identities imply immediately the conservation laws for Σ_i^j and $\tau_{ij}{}^k$:

$$\overset{*}{\nabla}_j\Sigma_i^j=2S_{ij}{}^k\Sigma_k^i+\tau^j_{km}R_{ij}^{km}\ , \quad (5.9)$$

$$\overset{*}{\nabla}_k\tau_{ij}{}^k=\Sigma_{[ij]}\ . \quad (5.10)$$

By Eq. (5.3) and by Weyssenhoff condition, the projection along U^i of (5.9) becomes

$$U^i\nabla_j\Sigma_i^j=0\ , \quad (5.11)$$

i.e.,

$$-\rho\dot{\bar\epsilon}-\frac{e}{m_0}\dot S^{\ m}_{i}F_{mk}U^kU^i+\frac{e}{m_0}S^{jm}U^i\nabla_jF_{im}+\dot p=0\ . \quad (5.12)$$

Equation (5.10) is the motion equation for the spin density tensor, equivalent to (4.5),

$$\dot S_{ij}=\dot U^k(S_{ik}U_j-S_{jk}U_i)+\frac{e}{m_0}F^{km}U_m(S_{ik}U_j-S_{jk}U_i)+\frac{e}{m_0}(F_i^mS_{jm}-F_j^mS_{im})\ . \quad (5.13)$$

By using (2.3), (2.6), and (5.13) in (5.12), we get

$$T\dot S=0$$

according to (2.8). Contracting (5.9) with the projector $P_k^i=(\delta_k^i+U_kU^i)$, if we use (5.3) and the Weyssenhoff condition, we get

$$-P_k^i\partial_ip=\rho\bar\epsilon\dot U_k+S_{km}\ddot U^m-\frac{1}{m_0}\hat f^{(L)}_{\ m}P_k^i\dot S_{\ i}^{m}-\frac{e}{m_0}S_k^m(\dot F_{mj}U^j+F_{mj}\dot U^j)+\frac{e}{m_0}S^{jm}P_k^i\nabla_jF_{im}-f_k^{(L)}-f_k^{(M)}-\frac{2k}{m_0}\rho^2h_0{}^2\hat f_{\ i}^{(L)}\hat\sigma^i\hat\sigma_k\ , \quad (5.14)$$

where $\hat f_k^{(L)}=eU^jF_{kj}$ is the Lorentz force, $\hat f_k^{(M)}=\frac{1}{2}\hat S_{im}U^jR_{kj}^{im}$ is the Mathisson force.

Equation (5.14) is the Euler equation for a charged spin fluid; it generalizes the one given in Ref. 1 in the case of a neutral spin fluid because of the presence of the electromagnetic terms and of a corrective term due to the coupling between the electromagnetic energy-momentum and the torsion.

VI. CIRCULATION THEOREMS

In Sec. IV we introduced the current vector V_i which generalizes the current vector introduced in Refs. 1, 2, 3, 11, and 12. As it can be seen by Eq. (4.6), V_i is gauge invariant since the RHS is a gauge-invariant expression; moreover, if we make the transformation $A_i\to A_i+\partial_i\psi$ the gradient can be balanced by the substitution $\partial_i\phi\to\partial_i(\phi-e\psi)$. This does not affect the Lagrangian (2.5) and the equation (2.13).

We propose, now, to extend the definition of circulation given in Ref. 2 by using the current vector defined here; the circulation along a closed line is

$$\Gamma(W)=\oint V_i\lambda^id\tau\ , \quad (6.1)$$

where λ^i is the vector tangent to the closed line, and W parametrizes the integral curves of velocity or the integral curves of the vorticity vector ω^i defined by

$$\omega^i=\epsilon^{ijkm}U_m\nabla_kU_j\ .$$

Let us define the V-vorticity tensor

$$\Omega_{ij}=2\nabla_{[j}V_{i]}\ , \quad (6.2)$$

that is,

$$\Omega_{ij}=2\partial_{[i}V_{j]}+2S_{ij}{}^kV_k\ . \quad (6.3)$$

If in (6.2) we consider V_k decomposed into potentials, according to (4.6) we get

$$\Omega_{ij}=2S_{ij}{}^kV_k+h_0a^mb_kR_{ijm}{}^k+2\partial_{[j}\theta\partial_{i]}S+2\partial_{[j}C\partial_{j]}B+2\nabla_{[j}h_0a^m\nabla_{i]}b_m+eF_{ij}\ . \quad (6.4)$$

By comparing (6.4) with (6.3), we find that

$$2\partial_{[j}V_{i]}=2\partial_{[j}\theta\partial_{i]}S+2\partial_{[j}C\partial_{i]}B+h_0a^m b_k R_{ijm}{}^k +2\nabla_{[j}h_0 a^m\nabla_{i]}b_m+eF_{ij} , \quad (6.5)$$

Eq. (6.5) generalizing the expressions given in Refs. 2 and 11.

Since it holds that $\Gamma(W)=0$ if and only if $\partial_{[i}V_{j]}=0$, then, from (6.5), this is equivalent to

$$\Omega_{ij}=2S_{ij}{}^k V_k \quad (6.6)$$

or

$$\Omega_{ij}=2S_{ij}{}^k\bar{\epsilon}U_k+2S_{ij}{}^k\hat{S}_{km}\left[\dot{U}^m-\frac{e}{m_0}F^{mj}U_j\right] .$$

This expression may be reduced if one considers the Weyssenhoff condition and the field equation (5.3),

$$\Omega_{ij}=-kS_{ij}\bar{\epsilon} ,$$

i.e., in the irrotational case, the torsion is linked to the macroscopical quantity Ω_{ij} by

$$-\tfrac{1}{2}\Omega_{ij}U^k=\bar{\epsilon}S_{ij}{}^k . \quad (6.7)$$

Moreover, in the irrotational case, if we contract (6.5) by U^j, we obtain

$$T\partial_i S-a^m\dot{b}_m\partial_i h_0+h_0(\dot{a}^m\nabla_i b_m-\dot{b}^m\nabla_i a_m) =\hat{f}_i^{(m)}-\hat{f}_i^{(L)} , \quad (6.8)$$

Eq. (6.8) being reduced to Eq. (13) of Ref. 2 in the limit $e\to 0$. Finally, we may observe that also the Bernoulli theorem holds, in the form given in Ref. 2, for the charged spin fluid

$$\bar{\epsilon}+\tfrac{1}{2}q^2-\varphi+T_0=\text{const} , \quad (6.9)$$

where

$$T_0=\hat{S}_{0k}\left[\dot{U}^k-\frac{e}{m_0}F^{km}U_m\right] . \quad (6.10)$$

VII. CONCLUSIONS

In this work we have dealt with a charged spin fluid, giving a Lagrangian description of it in a Riemann-Cartan spacetime. We have considered a one-family fluid with null conductivity ($\sigma_c=0$), i.e., the case in which the electric current is purely convective. By this variational principle we have derived Maxwell's equations for a spacetime with torsion in the presence of spin.

We have demonstrated that the material Lagrangian reduces to pressure plus the free electromagnetic Lagrangian. This suggests that a more realistic model may be constructed by considering the sum of different pressure terms for a fluid with many families of particles.

In our opinion, the next step is to study the more general cases of charged spin fluids with $\sigma_c\neq 0$ and the magnetohydrodynamic case in which conductivity is infinite.

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