PHYSICAL REVIEW D VOLUME 31, NUMBER 8 15 APRIL 1985

Bargmann structures and Newton-Cartan theory

C. Duval,* G. Burdet, H. P. Künzle,† and M. Perrin
Centre de Physique Théorique,‡ Centre National de la Recherche Scientifique, Luminy Case 907, 13288 Marseille, France
(Received 21 June 1984)

It is shown that Newton-Cartan theory of gravitation can best be formulated on a five-dimensional extended space-time carrying a Lorentz metric together with a null parallel vector field. The corresponding geometry associated with the Bargmann group (nontrivially extended Galilei group) viewed as a subgroup of the affine de Sitter group AO(4,1) is thoroughly investigated. This new global formalism allows one to recast classical particle dynamics and the Schrödinger equation into a purely covariant form. The Newton-Cartan field equations are readily derived from Einstein's Lagrangian on the space-time extension.

INTRODUCTION

It has long been recognized that Newtonian gravitation theory admits a geometric formulation like general relativity. The quest for a geometric approach to Newton theory actually goes back to the eve of Einstein's 1916 theory. We refer to Refs. 1—3 for a survey of the subject and a complete bibliography. Because Cartan first proposed a strictly geometrical definition of the classical gravitational field in terms of linear connections with values in the Lie algebra of the homogeneous Galilei group,[4] this theory is often referred to as the Newton-Cartan theory.

From a more contemporary viewpoint, Newton's theory may be considered as a testing ground to several open problems in general relativity such as the description of extended bodies and the two-body problem. It also helps to better understand some specific features in field theory (e.g., the question of the gauge nature of gravitation). In a systematic classification of all available space-time structures one thus would like to incorporate the eldest viable theory of gravitation.[5]

Apart from its intrinsic interest, the Newton-Cartan theory yields a coherent framework to mathematically investigate various aspects of classical physics [e.g., the principle of general covariance and its plausible extensions,[6,7] the geometric prescriptions of minimal coupling,[2,7] the group-theoretical derivation of the classical gravitational field equations,[8] and the appearance of the Bargmann group (extended Galilei group)[9—11] the covariant Newtonian limit of some solutions of the Einstein equations,[12,33] new trends in analytical mechanics,[13] classical solar system physics,[14] and Newtonian cosmology,[15] etc.].

Recently, the sharp experimental verification of the principle of equivalence at the nuclear level [the so-called Collela-Overhauser-Werner (COW) experiments on neutron interferometry in a weak gravitational field][16,17] naturally raised the question of the reformulation of the Schrödinger wave equation in a covariant guise. However, difficulties stem from the fact that the Bargmann group (the symmetry group of the free Schrödinger equation)[18—20] does not effectively act on flat Galilei space-time. This might explain why the Schrödinger equation retains such an intricate, although gauge-invariant, form on a curved Newtonian space-time.[21,7] The same remark applies to the four-component spinor Levy-Leblond equation[18,19,22] (the nonrelativistic analog of the Dirac equation).

Dealing with Newtonian structures in a way that resembles the geometric approach to general relativity allows one to give the Bargmann group a purely classical (nonquantum) status. For example, the classification of all physically relevant elementary Bargmann systems has been carried out[6,23] and replaces advantageously Souriau's classification of elementary Galilei systems (free-classical-particle symplectic models) without appealing to such elaborate concepts as the symplectic cohomology generated by the mass.[24] Here the mass is simply introduced as a Casimir invariant of the global Bargmann symmetry group of the theory (see Sec. IV).

Moreover, the Schrödinger group (the "conformal" symmetry group of the free Schrödinger equation[25,26] gains a well-defined status in terms of Newtonian structures.[23] The quantum group-theoretical approach hid for some time the geometrical origin of that group of local space-time transformations. See Refs. 27 and 28 for an introduction to the second-order Cartan structures associated with the corresponding "chronoprojective" geometry of classical space-time. This matter will be more appropriately recast in terms of conformal Bargmann automorphisms (in preparation).

Let us recall that a Newton-Cartan space-time is endowed with a degenerate "metric" structure, i.e., a Galilei structure defined by a rank-3 contravariant symmetric tensor (the Euclidean metric of instantaneous three-space slices) and an orthogonal time covector that defines a canonical fibration over the absolute time axis. The degeneracy of a Galilei structure unfortunately results in the nonuniqueness of a "metric" torsion-free connection that would describe the Newtonian gravitational field just as in the relativistic framework: we already know that singling out a class of Galilean observers (e.g., the geodesics of the flat connection of Galilei space-time) is somewhat independent of choosing rulers and clocks in nonrelativistic physics. As a consequence, the theory admits a larger gauge group that extends space-time diffeomorphisms.[7] The eleven-dimensional Bargmann group B (2.1) turns out to be nothing but the stabilizer of the flat Newtonian structure, and plays about the same role as the Poincaré group in special relativity.

It is worth noticing that the Bargmann group can be viewed as a group of transformations of a trivial five-dimensional space-time extension. This point has in fact already been foreseen in earlier work on Galilei spinors connected with the de Sitter group,[29–31] also in Ref. 32 where the 5×5 matrix realization (2.2) of the homogeneous Galilei group H emerges from a careful analysis of point-particle scattering. The same remark appears in filigree in Ref. 11 where Newton's homogeneous field equations (1.15) are derived from a particular property of the 5th component of the Bargmannian torsion. See also Refs. 33 and 34 for earlier arguments in the "light-cone" formalism.

The purpose of this paper is to reconcile to some extent the Newton-Cartan theory based on a four-dimensional space-time manifold and the Bargmann covariance of classical particle and field theory. We wish to show that this program can be achieved by considering a five-dimensional setting from the very beginning. Our investigations will rely on a space-time extension $\widetilde{M}$ viewed as a trivial principal $(\mathbb{R},+)$ bundle endowed with a Lorentz metric. Hint: the homogeneous Galilei group can be realized as a subgroup of the de Sitter group SO(4,1). That principal bundle structure follows, at least locally, from the imposition of an H structure on $\widetilde{M}$. The formalism we will be dealing with is akin to the Kaluza-Klein formalism with two-major differences, however: the structural group is noncompact (mass is not quantized) and the principal fibration is assumed to be null with respect to the Lorentz metric. We find that almost all shortcomings due to the lack of true metric structure on a Newton-Cartan space-time can thus be circumvented.

The paper is organized as follows.

We briefly recall in Sec. I the basic definitions and properties of Galilei and Newton manifolds in terms of H structures.

Section II is devoted to the introduction of the so-called Bargmann structures. Considering the characteristic matrix representation (2.2) of the homogeneous Galilei group H naturally leads us to investigate the case of a Lorentzian $(\mathbb{R},+)$ principal fiber bundle $(\widetilde{M},g,\xi)$ over space-time M. The group generator ξ turns out to be null and parallel with respect to the canonical Lévi-Cività connection. That fibration is designed in such a way that the space-time manifold M is automatically endowed with a canonical Newtonian structure. Several previous results[11] are thus recovered and substantially simplified. The point of view espoused in this section is appropriate for a global definition of the Bargmann automorphisms that extend Galilei automorphisms. We have thus at our disposal a new geometry, namely, the Bargmann geometry associated with the pair (B,H). Local expressions in an adapted coordinate system are explicitly worked out in Sec. III.

Classical free (elementary) Bargmann systems are introduced in Sec. IV on the same footing as Poincaré $\times\,\mathbb{R}$ elementary dynamical systems. For consistency, the relativistic "internal energy" is defined as the rest mass. It actually corresponds to a zero classical internal energy. The simplest case of spinless massive particles is investigated together with the geometric prescription of minimal gravitational coupling. Conservation laws are worked out by means of the symplectic Noether theorem, the mass being associated with the Killing vector field ξ. We do recover from a different point of view earlier results[6] concerned with the symplectic geometry of the space of motions of classical nonrelativistic dynamical systems, in particular the "principle" of geodesic motion.

In Sec. V we outline the method of geometric quantization in the case of free spinless Bargmann dynamical systems. By working on a five-dimensional Lorentzian space-time extension, we find a set of partial differential equations (5.14) and (5.15) which in fact replaces the Schrödinger equation in its most familiar (although not covariant) form. Schrödinger wave functions are essentially harmonic complex functions of $\widetilde{M}$ that transform under the $(\mathbb{R},+)$ structural group according to a multiplicative character that defines the mass of the particle. These equations lead exactly to the Schrödinger equation on a curved Newtonian space-time derived in Refs. 21 and 7 and are globally invariant under Bargmann automorphisms.

The last section is devoted to the derivation of Newton's field equations from a specific Lagrangian density which happens to be nothing but the scalar curvature—just as in a pure Kaluza-Klein setting. The source term (mass density) appears as a Lagrange multiplier to comply with the isotropy of the group generator ξ. Miscellaneous solutions of Newton field equations are finally listed in their Bargmannian form.

I. GALILEI AND NEWTON STRUCTURES: A COMPENDIUM

The (restricted) Galilei group G is usually introduced as the multiplicative group of those 5×5 matrices of the form

$$\begin{bmatrix} R & b & c \\ 0 & 1 & e \\ 0 & 0 & 1 \end{bmatrix}, \tag{1.1}$$

where $R\in SO(3)$, $b,c\in\mathbb{R}^3$, and $e\in\mathbb{R}$. The boosts are parametrized by b, while c and e represent space and time translations, respectively.

Let us recall that a proper *Galilei structure*[2] is defined as a reduction $H(M)\to Gl(M)$ of the frame bundle of a four-dimensional connected smooth manifold M to the homogeneous Galilei group $H\equiv SO(3)\ltimes\mathbb{R}^3\subset G$ which is faithfully represented as the multiplicative group of the 4×4 matrices

$$\begin{bmatrix} R & b \\ 0 & 1 \end{bmatrix}. \tag{1.2}$$

Such a reduction $p:H(M)\to M$ defines (and is characterized by) a pair of tensor fields (γ,ψ) on M,

$$\gamma\equiv\delta^{AB}e_A\otimes e_B \quad (A,B=1,2,3)\,, \tag{1.3}$$

$$\psi \equiv \theta^4 \ . \tag{1.4}$$

Here we denoted by $(\theta^a)=(\theta^a_\alpha dx^\alpha)$ $(a,\alpha=1,\ldots,4)$ the soldering form of $H(M)$ whose typical element is a Galilei frame (e_a) at $x\in M$. We have

$$\ker\gamma=\mathrm{span}\psi \ . \tag{1.5}$$

The signature of γ is given by

$$\mathrm{sign}(\gamma)=(+++0) \ . \tag{1.6}$$

A Galilei space-time (M,γ,ψ) is orientable since there exists a canonical volume element "vol" defined by

$$\theta^1\wedge\theta^2\wedge\theta^3\wedge\theta^4=p^*\mathrm{vol} \ . \tag{1.7}$$

Galilei connections are then introduced as torsion-free connections ω on $H(M)$ [taking their values in the Lie algebra h of H (Ref. 4)—see (1.2)]

$$\omega_{(AB)}=0 \quad (A,B=1,2,3) \ , \tag{1.8}$$

with $\omega_{AB}\equiv\delta_{CA}\omega^C_B$,

$$\omega^4_a=0 \quad (a=1,\ldots,4) \ , \tag{1.9}$$

$$d\theta^A=-\omega^A_B\wedge\theta^B-\omega^A_4\wedge\theta^4 \ , \tag{1.10}$$

$$d\theta^4=0 \ . \tag{1.11}$$

The corresponding covariant derivative ∇ thus satisfies

$$\nabla\gamma=0 \ , \tag{1.12}$$

$$\nabla\psi=0 \ , \tag{1.13}$$

$$\text{torsion }\nabla=0 \ . \tag{1.14}$$

Since ψ is closed, there exists (locally) a fibration $M\to T\equiv M/\ker\psi$ over the absolute Galilean time axis T (which is assumed to have topology $\mathbb{R}$; cf. Refs. 23 and 27 for a solution of Newton's field equations with $T\cong S^1$). A Galilei space-time is thus time orientable and space orientable as well.

Unlike the (pseudo)Riemannian case, ∇ is not fully determined by the Galilei structure (M,γ,ψ). This entails that the Galilei connection must be specified in addition to the Galilei structure to yield the full geometrical setting of Galilean physics.

Newtonian gravitation theory can easily be geometrized in terms of special Galilei connections. *Newtonian connections*[4,2] are merely Galilei connections whose curvature tensor satisfies the nontrivial constraint

$$R_\alpha{}^\beta{}_\gamma{}^\delta=R_\gamma{}^\delta{}_\alpha{}^\beta \ , \tag{1.15}$$

where

$$R_\alpha{}^\beta{}_\gamma{}^\delta\equiv\gamma^{\beta\sigma}R^{\;\delta}_{\alpha\sigma\gamma} \tag{1.16}$$

and

$$R^{\;\delta}_{\alpha\beta\gamma}\equiv R_{\alpha\beta\gamma}{}^\delta\equiv 2(\partial_{[\alpha}\Gamma^\delta_{\beta]\gamma}+\Gamma^\delta_{\sigma[\alpha}\Gamma^\sigma_{\beta]\gamma}) \ .$$

Note that the extra condition (1.15) (which, roughly speaking, means that the Newtonian gravitational field is curlfree) must be introduced heuristically at this stage and cannot definitely be interpreted in terms of Galilei structures. We will presently see how naturally it shows up in terms of Bargmann structures (see Ref. 6 for preliminary considerations related to the Bargmann covariance of classical particle and field theory). That remark constitutes in fact the main motivation and the physical justification of the geometrical structures we will be dealing with in the sequel.

Let us recall that translations $(\mathbb{R}^4,+)$ constitute a reducible subgroup of G, and, hence, that the h restriction of the Maurer-Cartan one-form of G provides us with the distinguished Galilei (or Newton) *flat connection* on space-time G/H.

As for the inhomogeneous field equations, they can be written intrinsically as

$$\mathrm{Ric}=4\pi G\rho\psi\otimes\psi \ , \tag{1.17}$$

where Ric denotes the Ricci tensor of the connection ∇ and G is Newton's constant. Note that mass density ρ is the only source of the classical gravitational field. Equations (1.15) and (1.17) constitute the complete set of Newton(-Cartan) field equations (analogous expressions can be found in Ref. 4).

Let us finally mention an additional constraint on the Newtonian curvature that might be considered as a supplementary field equation,[8,3] namely,

$$R_{\alpha\beta}{}^{\gamma\delta}=0 \tag{1.18}$$

or equivalently (in terms of the curvature two-form)[15]

$$\Omega^A_B=\tfrac{1}{2}\theta^A_\alpha R^{\;\alpha}_{\lambda\mu\beta}dx^\lambda\wedge dx^\mu e^\beta_B=0 \ . \tag{1.19}$$

It implies that the SO(3) bundle $H(M)/\mathbb{R}^3$ of direct orthonormal spacelike frames is flat and, hence, trivial (no rotational holonomy).

II. BARGMANN STRUCTURES AND SPACE-TIME EXTENSIONS

A. Definition

The Bargmann group B (nontrivially extended Galilei group) is isomorphic to the multiplicative group of 6×6 matrices of the form

$$\begin{bmatrix} R & b & 0 & c \\ 0 & 1 & 0 & e \\ -{}^t bR & -b^2/2 & 1 & f \\ 0 & 0 & 0 & 1 \end{bmatrix} . \tag{2.1}$$

The original Levy-Leblond matrix representation[20] has been slightly modified here in order to emphasize the specific semidirect product structure $B\cong H\ltimes\mathbb{R}^5$ which will be extensively used throughout this paper. We have a surjective homomorphism

$$B\to G:(R,b,c,e,f)\mapsto(R,b,c,e) \ ,$$

whose kernel is the center of B generated by $f\in\mathbb{R}$. The homogeneous Galilei subgroup H $(c=0,e=f=0)$ clearly inherits from (2.1) the following 5×5 faithful representation:

$$\begin{bmatrix} R & b & 0 \\ 0 & 1 & 0 \\ -{}^t bR & -b^2/2 & 1 \end{bmatrix} . \tag{2.2}$$

From now on we will be assuming that nonrelativistic physics is governed by H structures. We are therefore led to look for *Bargmann structures,* i.e., reductions $H(\widetilde{M})\to \mathrm{Gl}(\widetilde{M})$ of the frame bundle of a five-dimensional smooth manifold $\widetilde{M}$ to the homogeneous Galilei group H.

The free right action of H (2.2) on Gl $(\widetilde{M})$ is

$$\begin{aligned} &\widetilde{e}_A \mapsto \widetilde{e}_B R^B_A - b_B R^B_A \widetilde{e}_5 \ , \\ &\widetilde{e}_4 \mapsto \widetilde{e}_B b^B + \widetilde{e}_4 - (b^2/2)\widetilde{e}_5 \ , \\ &\widetilde{e}_5 \mapsto \widetilde{e}_5 \ . \end{aligned} \tag{2.3}$$

Computing the tensorial invariants of (2.3), we find that $H(\widetilde{M})$ defines (and is characterized by) the pair (g,ξ) of tensors on $\widetilde{M}$,

$$g \equiv \widetilde{\theta}^A \otimes \widetilde{\theta}^B \delta_{AB} + \widetilde{\theta}^4 \otimes \widetilde{\theta}^5 + \widetilde{\theta}^5 \otimes \widetilde{\theta}^4 \ , \tag{2.4}$$

$$\xi \equiv \widetilde{e}_5 \tag{2.5}$$

[or, equivalently, a pair $(g,\widetilde{\psi})$ with $\widetilde{\psi}\equiv\widetilde{\theta}^4$].

The symmetric tensor g is invertible,

$$g^{-1} = \widetilde{e}_A \otimes \widetilde{e}_B \delta^{AB} + \widetilde{e}_4 \otimes \widetilde{e}_5 + \widetilde{e}_5 \otimes \widetilde{e}_4 \tag{2.6}$$

and has the signature

$$\mathrm{sign}(g) = (++++-) \ . \tag{2.7}$$

The base manifold $\widetilde{M}$ is thus endowed with a Lorentzian structure (2.4) together with a nowhere vanishing null vector field ξ,

$$g(\xi,\xi) = 0 \ . \tag{2.8}$$

B. The Bargmann-Newton morphism

Let us now investigate the relationship between Bargmann structures and the previous Newtonian structures on space-time. Start with the observation that $B/H \to G/H$ is a trivial principal $(\mathbb{R},+)$ bundle over flat space-time:

$$\begin{array}{ccc} B & \to & B/H \cong \mathbb{R}^5 \\ \downarrow & & \downarrow \\ G & \to & G/H \cong \mathbb{R}^4 \ . \end{array} \tag{2.9}$$

We would like to keep that principal fibration in the curved case as much as we consider space-time a fundamental physical concept. Since ξ never vanishes, we may assume that there exists a global one-parameter group of transformations of $\widetilde{M}$ which induce ξ. From now on we will thus confine considerations to the situation of a principal $(\mathbb{R},+)$ bundle $\pi: \widetilde{M} \to M$ with group generator ξ.

Let us show that M is then canonically endowed with a Newtonian structure inherited from the original Bargmann structure.

As a subgroup of $\mathrm{Diff}(\widetilde{M})$, the structural group $(\mathbb{R},+)$ can be lifted to $H(\widetilde{M})$. Now H and $(\mathbb{R},+)$ actually commute on $H(\widetilde{M})$ and the quotient $H(M)\equiv H(\widetilde{M})/\mathbb{R}$ is indeed a principal H bundle, the bundle of Galilei frames over M. We thus have the commutative diagram

$$\begin{array}{ccc} H(\widetilde{M}) & \xrightarrow{\widetilde{p}} & \widetilde{M} \\ \downarrow \pi_* & & \downarrow \pi \\ H(M) & \xrightarrow[p]{} & M \ , \end{array} \tag{2.10}$$

where π_* denotes the push-forward operation on frame vectors of $\widetilde{M}$. The canonical lift to $H(\widetilde{M})$, Ξ say, of the generator ξ is defined by

$$\pounds_\Xi \widetilde{\theta} = 0, \quad \widetilde{p}_* \Xi = \xi \tag{2.11}$$

so that

$$\widetilde{\theta}^4 = g(\Xi) \tag{2.12}$$

descends to $H(M)$ as

$$\psi \equiv \theta^4 \ . \tag{2.13}$$

Moreover, since

$$\gamma \equiv \pi_* g^{-1} = e_A \otimes e_B \delta^{AB} \quad (e_A \equiv \pi_* \widetilde{e}_A; A = 1,2,3) \tag{2.14}$$

clearly satisfies (1.5) and (1.6), the couple (γ,ψ) is the desired Galilei structure on $H(M)$.

The notion of a Bargmann connection is quite naturally introduced at this stage. Let us first note that H is in fact a subgroup of SO(4,1) where the Lorentz metric (2.4) is given by

$$(g_{a'b'}) = \begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 0 & 1 \\ & & & 1 & 0 \end{bmatrix} , \tag{2.15}$$

$$(a',b',\ldots = 1,\ldots,5) \ .$$

We call any torsion-free connection $\widetilde{\omega}$ on $H(\widetilde{M})$ a *Bargmann connection,*

$$\widetilde{\omega}_{(a'b')} = 0 \ , \tag{2.16}$$

$$\widetilde{\omega}^{a'}_5 = 0 \ (= -\widetilde{\omega}^4_{a'}) \ , \tag{2.16$'$}$$

where

$$\begin{aligned} &\widetilde{\omega}_{a'b'} \equiv \widetilde{\omega}^{c'}_{a'} g_{c'b'} \ , \\ &d\widetilde{\theta}^{a'} = -\widetilde{\omega}^{a'}_{b'} \wedge \widetilde{\theta}^{b'} \ . \end{aligned} \tag{2.17}$$

Equations (2.16) express the fact that $\widetilde{\omega}$ takes its values in $h \subset \mathrm{o}(4,1)$,

$$\widetilde{\omega} = \begin{bmatrix} (\widetilde{\omega}^A_B) & (\widetilde{\omega}^A_4) & 0 \\ 0 & 0 & 0 \\ (-\widetilde{\omega}^B_4 \delta_{AB}) & 0 & 0 \end{bmatrix} \tag{2.18}$$

with $\widetilde{\omega}_{(AB)} = 0$.

In view of (2.18), (2.17) becomes

$$d\widetilde{\theta}^A = -\widetilde{\omega}^A_B \wedge \widetilde{\theta}^B - \widetilde{\omega}^A_4 \wedge \widetilde{\theta}^4 \ , \tag{2.19}$$

$$d\widetilde{\theta}^4 = 0 \ , \tag{2.20}$$

$$d\widetilde{\theta}^5 = -\widetilde{\omega}^5_A \wedge \widetilde{\theta}^A = \widetilde{\omega}^A_4 \wedge \widetilde{\theta}^B \delta_{AB} \ . \tag{2.21}$$

As a Lorentzian connection, $\widetilde{\omega}$ is uniquely determined by

g. To achieve the reduction of the Lorentz bundle of $(\tilde{M},g)$ to $H(\tilde{M})$, the connection $\tilde{\omega}$ must satisfy the additional constraint (2.16′).

Introducing the associated covariant derivative $\tilde{\nabla}$, we get

$$\tilde{\nabla} g = 0 \,, \tag{2.22}$$

$$\tilde{\nabla} \xi = 0 \,, \tag{2.23}$$

$$\text{torsion } \tilde{\nabla} = 0 \,. \tag{2.24}$$

If Ξ denotes the canonical lift (2.11) of ξ to $H(\tilde{M})$, we automatically have

$$\tilde{\theta}^a(\Xi) = 0 \,, \tag{2.25}$$

$$\tilde{\omega}^a_b(\Xi) = 0 \quad (a,b = 1, \ldots, 4) \,. \tag{2.26}$$

Now ξ is a Killing vector field of $(\tilde{M},g)$, whence the affinity relationship

$$\mathfrak{L}_\Xi \tilde{\omega} = 0 \,. \tag{2.27}$$

This remark shows that $(\tilde{\theta}^a)$ and $(\tilde{\omega}^a_b)$ $(a,b=1,\ldots,4)$ are integral invariants of the flow generated by Ξ. They descend to $H(M)$ as (θ^a) and (ω^a_b), respectively. Moreover (2.18)—(2.20) imply that (ω^a_b) is actually a Galilei connection [compare (1.8)—(1.11)].

We have thus proved that there exists a unique Galilei structure associated with a given Bargmann structure. In other words, given a Bargmann bundle $(\tilde{M},g,\xi)$, the base manifold $M \equiv \tilde{M}/\mathbb{R}$ inherits a canonical Galilei structure (γ,ψ) [(2.13) and (2.14)] together with a specific Galilei connection, the projected Levi-Cività connection. Let us emphasize that the Galilei connection is now completely determined by the Bargmann structure.

Of greater physical interest is the fact that the Bargmann connection induces a Newtonian connection on space-time. To prove this, let us rewrite the Newtonian constraint (1.15) in a Galilei frame as

$$R_a{}^c{}_b{}^d = R_b{}^d{}_a{}^c \,, \tag{2.28}$$

where

$$R_a{}^c{}_b{}^d \equiv \gamma^{ce} R^d_{aeb} \quad [(\gamma^{ab} = \gamma^{\alpha\beta}\theta^a_\alpha \theta^b_\beta) = \mathrm{diag}(1110)] \,.$$

Since $(\tilde{\omega}^a_b)$ induces a Galilei connection (ω^a_b) on $H(M)$, the curvature two-form $\tilde{\Omega}$ satisfies

$$\tilde{\Omega}^a_b = \tfrac{1}{2} \tilde{R}^{\,a}_{\;a'b'b} \tilde{\theta}^{a'} \wedge \tilde{\theta}^{b'}$$

$$= \tfrac{1}{2} R^a_{cdb} \theta^c \wedge \theta^d \,, \tag{2.29}$$

where $\tilde{R}$ denotes the Riemann-Christoffel tensor of $\tilde{\nabla}$ while R denotes the curvature of the Galilei connection ∇.

From (2.29) we easily get

$$\tilde{R}^{\,a}_{\;bcd} = R^a_{bcd} \tag{2.30}$$

and

$$\tilde{R}^{\,a}_{\;b5d} = 0 \,. \tag{2.31}$$

Note that in view of (2.23) (recall that $\xi^{a'} = \delta^{a'}_5$),

$$\tilde{R}^{\,a'}_{\;b'c'5} = 0 \,. \tag{2.32}$$

Now the fundamental identity satisfied by the Riemann-Christoffel tensor

$$\tilde{R}_{a'}{}^{b'}{}_{c'}{}^{d'} = \tilde{R}_{c'}{}^{d'}{}_{a'}{}^{b'} \tag{2.33}$$

specialized to indices running from 1 to 4 yields

$$\tilde{R}^{\,d}_{\;aa'c} g^{a'b} = \tilde{R}^{\,b}_{\;ca'a} g^{a'd} \tag{2.34}$$

and (2.15), (2.30), and (2.31) finally lead to

$$R^d_{aec}\gamma^{eb} = R^b_{cea}\gamma^{ed} \tag{2.35}$$

which is identical to condition (2.28) or (1.15).

A Bargmann bundle $\tilde{M}$ is orientable:

$$\tilde{\theta}^1 \wedge \tilde{\theta}^2 \wedge \tilde{\theta}^3 \wedge \tilde{\theta}^4 \wedge \tilde{\theta}^5 = \tilde{p}^* \widetilde{\mathrm{vol}} \,.$$

The same is true for the associated Galilei manifold: $\widetilde{\mathrm{vol}}(\xi) \equiv \pi^* \mathrm{vol}$.

Returning to the original group-theoretical considerations, we find that the *flat Bargmann structure* is the principal bundle $B \to B/H$ endowed with the h restriction of the canonical Maurer-Cartan one-form of B as flat connection form [the translations B/H form an Abelian reducible subgroup of B (Ref. 35)]. The canonical flat Bargmann structure then clearly induces the canonical flat Newtonian structure on space-time [cf. (2.9)]. Note that the additional constraint (1.18) or (1.19) does not seem to fit naturally into our formalism.

C. Structural automorphisms

We recall that the group $\mathrm{Aut}(H(M))$ of automorphisms of a principal bundle $H(M)$ is defined as the subgroup of $\mathrm{Diff}(H(M))$ consisting in those diffeomorphisms which commute with the structural group H. There exists a surjective homomorphism $\mathrm{Aut}(H(M)) \to \mathrm{Diff}(M)$ whose kernel is the group of "vertical" automorphisms of $H(M)$.

Let us now look for the group of automorphisms of a Bargmann structure, i.e., the group of those automorphisms of the principal bundle $H(\tilde{M})$ that preserve the soldering form $\tilde{\theta}$,

$$\mathrm{Barg}(H(\tilde{M})) \equiv \{\tilde{a} \in \mathrm{Aut}(H(\tilde{M})); \tilde{a}^*\tilde{\theta} = \tilde{\theta}\} \,. \tag{2.36}$$

Using standard results known in fiber-bundle theory,[35] we find that $\mathrm{Barg}(H(\tilde{M}))$ is actually isomorphic with the Lie subgroup of $\mathrm{Diff}(\tilde{M})$,

$$\mathrm{Barg}(\tilde{M},g,\xi) \equiv \mathrm{Isom}_0(\tilde{M},g) \cap \mathrm{Aut}(\tilde{M} \to M) \,, \tag{2.37}$$

i.e., the identity component of the group of those isometries of $(\tilde{M},g)$ which also leave invariant the principal fibration $\tilde{M} \to M$ (and hence project onto M as space-time transformations).

An elementary calculation shows that we recover the original Bargmann group B (2.1) in the flat case,

$$B = \left\{ \begin{bmatrix} A & C \\ 0 & 1 \end{bmatrix}; A \in \mathrm{SO}(4,1)\uparrow; A\tilde{e}_5 = \tilde{e}_5; C \in \mathbb{R}^5 \right\} \,. \tag{2.38}$$

Since the automorphisms (2.36) also preserve the fibration $H(\tilde{M}) \to H(M)$ (2.10), they induce the group

$$\mathrm{Gal}(H(M),\omega)\equiv\{a\in\mathrm{Aut}(H(M));a^*\theta=\theta;a^*\omega=\omega\} \tag{2.39}$$

isomorphic with the group of Galilei transformations[36]

$$\mathrm{Gal}(M,\gamma,\psi,\nabla)\equiv\{a\in\mathrm{Aff}_0(M,\nabla);a_*\gamma=\gamma;a^*\psi=\psi\}\ , \tag{2.40}$$

where $\mathrm{Aff}(M,\nabla)$ denotes the group of affine transformations of (M,∇). Again, the subscript 0 stands for "identity component."

We already know that $\mathrm{Aut}(\tilde{M}\to M)$ is an extension of $\mathrm{Diff}(M)$; now since the isometric vertical automorphisms actually reduce to a subgroup isomorphic to $(\mathbb{R},+)$, we finally have the exact sequence

$$1\to\mathbb{R}\to\mathrm{Barg}(\tilde{M},g,\xi)\to\mathrm{Gal}(M,\gamma,\psi,\nabla)\to 1 \tag{2.41}$$

which means that Bargmann automorphisms constitute a nontrivial one-dimensional central extension of Galilei automorphisms.

III. LOCAL EXPRESSIONS OF BARGMANN STRUCTURES

From now on a *Bargmann manifold* is defined as a principal $(\mathbb{R},+)$ bundle $\pi:\tilde{M}\to M$ over a four-dimensional connected smooth manifold M such that

$\tilde{M}$ is endowed with a Lorentzian metric g

$$\text{of signature } (+\,+\,+\,+\,-)\ , \tag{3.1}$$

the group generator ξ satisfies

$$g(\xi,\xi)=0\ , \tag{3.2}$$

$$\tilde{\nabla}\xi=0\ , \tag{3.3}$$

where $\tilde{\nabla}$ denotes the Levi-Cività connection of $(\tilde{M},g)$.

Now all $(\mathbb{R}^n,+)$ principal bundles can be made trivial;[37] this entails that $\tilde{M}$ is isomorphic with the trivial principal bundle $M\times\mathbb{R}$. We are thus dealing with a rather loose structure whose global topology (and cohomology) is in fact governed by that of the base manifold M.

The results of the preceding section can be quite easily recast as follows. The one-form $g(\xi)$ turns out to be a nowhere vanishing closed basic one-form of $\tilde{M}$; hence

$$g(\xi)\equiv\pi^*\psi\ \ (d\psi=0)\ . \tag{3.4}$$

Since ξ is a Killing vector, the pushed-forward twice contravariant tensor

$$\gamma\equiv\pi_* g^{-1} \tag{3.5}$$

is well defined on M and satisfies

$$\gamma(\psi)=0\ . \tag{3.6}$$

Using (3.1) and (3.2) we prove by inspection that the signature of γ is as in (1.6). Clearly (γ,ψ) is a Galilei structure on M.

As for the induced Newtonian connection, it turns out to be defined by

$$\nabla_X Y\equiv\pi_*\tilde{\nabla}_{\tilde{X}}\tilde{Y}\ , \tag{3.7}$$

where $\tilde{X}$ and $\tilde{Y}$ are any infinitesimal automorphisms of $\tilde{M}$ that project onto X and Y, respectively ($\tilde{X}\in\mathrm{aut}(\tilde{M}\to M)$ iff $[\xi,\tilde{X}]=0$): we simply use the fact that ξ is a Killing (hence affine) vector field of $(\tilde{M},g)$ together with the following expression of the Lie derivative of a connection,[35]

$$\pounds_\xi\tilde{\Gamma}(\tilde{X},\tilde{Y})=\pounds_\xi\tilde{\nabla}_{\tilde{X}}\tilde{Y}-\tilde{\nabla}_{\tilde{X}}[\xi,\tilde{Y}]-\tilde{\nabla}_{[\xi,\tilde{X}]}\tilde{Y}\ , \tag{3.8}$$

to prove that

$$[\xi,\tilde{\nabla}_{\tilde{X}}\tilde{Y}]=0\ . \tag{3.9}$$

Thus $\tilde{\nabla}_{\tilde{X}}\tilde{Y}$ lies again in $\mathrm{aut}(\tilde{M}\to M)$, hence projects on M as a well-defined vector field. The proof that (3.7) really defines a connection on M uses the fact that the right-hand side of (3.7) only depends upon X and Y. A direct computation then shows that ∇ is Newtonian.

Since $\tilde{M}$ is isomorphic with the trivial bundle $M\times\mathbb{R}$, we will choose to work in the adapted coordinate system $(x^j)=(x^\alpha,x^5)$ induced by a local chart (x^α) of M $(j=1,\dots,5;\alpha=1,\dots,4)$.

Setting

$$\gamma=\gamma^{\alpha\beta}\partial_\alpha\otimes\partial_\beta\ , \tag{3.10}$$

$$\psi=\psi_\alpha dx^\alpha\ , \tag{3.11}$$

we obtain $g^{\alpha\beta}=\gamma^{\alpha\beta}$ in (3.5); $g_{5\alpha}=\psi_\alpha$ in (3.6) and (3.11) and $g_{55}=0$ in (3.2). Hence,

$$g^{\alpha\beta}=\gamma^{\alpha\beta},\ g_{\alpha\beta}=\overset{V}{\gamma}_{\alpha\beta}-2\phi\psi_\alpha\psi_\beta\ , \tag{3.12}$$

$$g^{\alpha5}=V^\alpha,\ \ g_{\alpha5}=\psi_\alpha\ , \tag{3.13}$$

$$g^{55}=2\phi,\ \ g_{55}=0 \tag{3.14}$$

with

$$\psi_\alpha V^\alpha=1,\ \ \overset{V}{\gamma}_{\alpha\beta}V^\beta=0,\ \ \overset{V}{\gamma}_{\alpha\sigma}\gamma^{\sigma\beta}=\delta^\beta_\alpha-V^\beta\psi_\alpha\ . \tag{3.15}$$

All components [(3.12)–(3.14)] g_{ij} depend on x^α only ($\xi=\partial_5$ is a Killing vector field). The metric g in that coordinate system defines a unit vector field V (a preferred observer) and a function ϕ of M.

Let us recall that every Newtonian connection on M can be decomposed (once a choice of observer U [$\psi(U)=1$] has been performed) according to[2]

$$\Gamma^\gamma_{\alpha\beta}=\overset{U}{\Gamma}{}^\gamma_{\alpha\beta}+\psi_{(\alpha}F_{\beta)\sigma}\gamma^{\sigma\gamma}\ , \tag{3.16}$$

where[13]

$$\overset{U}{\Gamma}{}^\gamma_{\alpha\beta}=\tfrac{1}{2}\gamma^{\sigma\gamma}(2\partial_{(\alpha}\overset{U}{\gamma}_{\beta)\sigma}-\partial_\sigma\overset{U}{\gamma}_{\alpha\beta})+U^\gamma\partial_{(\alpha}\psi_{\beta)} \tag{3.17}$$

and

$$F_{\alpha\beta}=2\partial_{[\alpha}A_{\beta]}\ . \tag{3.18}$$

The one-form A (reminiscent of the local expression of an Abelian connection form) can be physically interpreted as the combination of gravitational and inertial potentials with respect to the observer U. Dealing with a pair (U,A) in place of the given Newtonian connection Γ amounts to choosing a "Bargmann" gauge. The corresponding gauge group has been studied in Ref. 7.

If we calculate the Levi-Cività connection components

$\widetilde{\Gamma}^k_{ij}$ from (3.12)–(3.14) we find that

$$\widetilde{\Gamma}^{\gamma}_{\alpha\beta}=\overset{V}{\Gamma}{}^{\gamma}_{\alpha\beta}+\gamma^{\sigma\gamma}\partial_\sigma\phi\psi_\alpha\psi_\beta\ , \tag{3.19}$$

i.e., the space-time components define a Newtonian connection in a particular gauge. By equating expressions (3.16) and (3.19) we find the relation to a "boost" gauge (U,A), namely,

$$V^\alpha=U^\alpha-\gamma^{\alpha\beta}A_\beta\ , \tag{3.20}$$

$$\phi=A^2/2-A_\alpha U^\alpha\quad(A^2\equiv\gamma^{\alpha\beta}A_\alpha A_\beta) \tag{3.21}$$

[with reference to the general gauge group discussed in Ref. 7 the expressions for V,ϕ in terms of (U,A) are invariant under the boost subgroup: $U^\alpha\mapsto U^\alpha+\gamma^{\alpha\beta}W_\beta$, $A_\alpha\mapsto A_\alpha+W_\alpha-(W_\beta U^\beta+W^2/2)\psi_\alpha$]. The general gauge group of Newton theory thus reduces here to $\mathrm{Aut}(\widetilde{M}\to M)$ (see Sec. II C).

We recognize in (3.21) the most familiar expression of the scalar potential with respect to the observer U [the "gravitational" potential $-A(U)$ plus the "Coriolis" rotational potential $A^2/2$].[21] We can now consider A_α as the local components of a connection form

$$\omega_*=A_\alpha dx^\alpha+dx^5 \tag{3.22}$$

on the $(\mathbb{R},+)$ principal bundle $\widetilde{M}$ over M. This connection form then has the special property of being null, i.e.,

$$g^{-1}(\omega_*,\omega_*)=0\ . \tag{3.23}$$

Now $\widetilde{U}\equiv g^{-1}(\omega_*)$ is an infinitesimal automorphism of $\widetilde{M}\to M$ that projects onto the unit vector field U (3.20); observers can thus be viewed as null $(\mathbb{R},+)$ connection forms on $(\widetilde{M},g,\xi)$. As far as we are considering Bargmann structures intrinsically, no fundamental physical status should be attached to these observers. However, they happen to be often necessary in the four-dimensional picture (e.g., the formulation of the Schrödinger equation on a curved Newtonian space-time.)[21,7]

The expression [(3.12)–(3.14)] of the Lorentz metric g, leads to the only nonvanishing components of the Christoffel symbols, namely,

$$\widetilde{\Gamma}^{\gamma}_{\alpha\beta}=\Gamma^{\gamma}_{\alpha\beta}\ , \tag{3.24}$$

where Γ is given by (3.19) and

$$\widetilde{\Gamma}^{5}_{\alpha\beta}=-\gamma_{\sigma(\alpha}\overset{V}{\nabla}_{\beta)}V^\sigma-\partial_{(\alpha}\phi\psi_{\beta)}\ . \tag{3.25}$$

We thus see that the Bargmann connection $\widetilde{\Gamma}$ projects onto M as a well-defined Newtonian connection Γ (3.24). (3.24).

As for the Riemann-Christoffel tensor, its nonvanishing components read

$$\widetilde{R}^{\ \delta}_{\alpha\beta\gamma}=R^{\ \delta}_{\alpha\beta\gamma}\ , \tag{3.26}$$

$$\widetilde{R}^{\ 5}_{\alpha\beta\gamma}=2\nabla_{[\alpha}\Phi_{\beta]\gamma},\quad \Phi_{\alpha\beta}\equiv\widetilde{\Gamma}^{5}_{\alpha\beta}\ . \tag{3.27}$$

The Ricci tensor then satisfies

$$\widetilde{R}_{\alpha\beta}=R_{\alpha\beta},\quad \widetilde{R}_{\alpha5}=0,\quad \widetilde{R}_{55}=0\ , \tag{3.28}$$

and the scalar curvature is finally given by

$$\widetilde{R}=R\ , \tag{3.29}$$

where $R\equiv R_{\alpha\beta}\gamma^{\alpha\beta}$ denotes the "scalar curvature" of the connection Γ. Again, the Newtonian constraint (1.15) is merely a consequence of the identity (2.33), and (3.26), (3.27).

IV. BARGMANN DYNAMICAL SYSTEMS

Let us now turn to the formulation of classical dynamics in terms of Bargmann structures. We will use in this section the structural relationship between relativistic and nonrelativistic mechanics which appear to be quite easily unified—at least in the free case—in a five-dimensional setting.

Let us start with $\mathbb{R}^{4,1}$ endowed with the Lorentzian metric

$$g=\delta_{AB}d\underset{\sim}{x}^A\otimes d\underset{\sim}{x}^B-c^2d\underset{\sim}{x}^4\otimes d\underset{\sim}{x}^4+c^{-2}d\underset{\sim}{x}^5\otimes d\underset{\sim}{x}^5 \tag{4.1}$$

viewed as a $(\mathbb{R},+)$ fiber bundle $\mathbb{R}^{4,1}\to\mathbb{R}^{3,1}:(\underset{\sim}{x}^A,\underset{\sim}{x}^4,\underset{\sim}{x}^5)\mapsto(\underset{\sim}{x}^A,\underset{\sim}{x}^4)$ over Minkowski space-time.

The group generator $\underset{\sim}{\xi}\equiv\partial/\partial\underset{\sim}{x}^5$ is spacelike $[g(\underset{\sim}{\xi},\underset{\sim}{\xi})=c^{-2}]$ and parallel transported by the flat Levi-Civita connection of $\mathbb{R}^{4,1}$. As for the group of automorphisms of $(\mathbb{R}^{4,1},g,\underset{\sim}{\xi})$, a simple calculation already leads to the group

$$\widetilde{P}\equiv\mathrm{Isom}_0(\mathbb{R}^{4,1},g)\cap\mathrm{Aut}(\mathbb{R}^{4,1}\to\mathbb{R}^{3,1})\cong P\times\mathbb{R}\ , \tag{4.2}$$

where P denotes the restricted Poincaré group, i.e., the identity component of the affine-orthogonal group $AO(3,1)$.

Since $\widetilde{P}$ is merely a trivial extension of P, we should not reasonably expect to learn anything new from it in the framework of special relativity. There is a subtlety, however. If we slightly change the point of view by means of the following linear diffeomorphism

$$\begin{aligned}x^A&=\underset{\sim}{x}^A\ ,\\ x^4&=(\underset{\sim}{x}^4+c^{-2}\underset{\sim}{x}^5)/2\ ,\\ x^5&=\underset{\sim}{x}^5-c^2\underset{\sim}{x}^4\ ,\end{aligned} \tag{4.3}$$

the Lorentz metric g turns out to be c independent in this new coordinate system

$$g=\delta_{AB}dx^A\otimes dx^B+dx^4\otimes dx^5+dx^5\otimes dx^4\ . \tag{4.4}$$

Moreover, the principal fibration $\pi:\mathbb{R}^{4,1}\to\mathbb{R}^{3;1}:(x^A,x^4,x^5)\mapsto(x^A,x^4)$ induced by the null parallel vector field $\xi\equiv\partial/\partial x^5$ is designed in such a way that the pair (γ,ψ) defined by (3.5) and (3.4)

$$\gamma=\delta^{AB}\partial_A\otimes\partial_B\ , \tag{4.5}$$

$$\psi=dx^4\ , \tag{4.6}$$

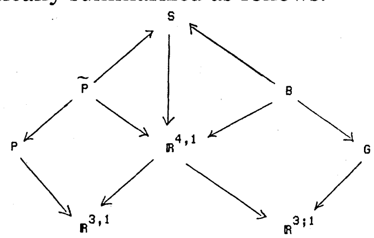
is in fact the canonical flat Galilei structure of $\mathbb{R}^{3;1}$ (see Sec. I). (The semicolon reminds us that the signature is now Galilean rather than Lorentzian.)

Both Minkowski and Galilei structures lift isometrically to their common five-dimensional Lorentzian extension. The interesting feature is that they are actually associated with two different $(\mathbb{R},+)$ principal fibrations, the Galilean one being null as previously noticed [cf. (4.4), also Sec. II].

We find that the group of automorphisms of the new structure $(\mathrm{R}^{4,1},g,\xi)$

$$B \equiv \mathrm{Isom}_0(\mathbb{R}^{4,1},g) \cap \mathrm{Aut}(\mathbb{R}^{4,1} \to \mathbb{R}^{3;1}) \tag{4.7}$$

is isomorphic with the Bargmann group (2.1) and (2.38). We thus have at our disposal two different geometries respectively associated with the pairs $(\widetilde{P},L)$ and (B,H) $[L \equiv \mathrm{SO}(3,1)\uparrow]$. If we denote by S the identity component of the affine de Sitter group $A0(4,1)$, these results can be diagrammatically summarized as follows:



(4.8)

Although the Bargmann group originally occurred in the quantum-mechanical context,[9,18–20] we believe that appealing to the Bargmann covariance of classical particle mechanics helps to bridge the gap between relativistic and nonrelativistic dynamics by embedding the limiting procedure (e.g., group contraction $c \to \infty$) into the common five-dimensional Lorentzian framework.

A. (Poincaré$\times\mathbb{R}$) dynamical systems

The mathematical setup for particle dynamics is concerned with the canonical symplectic structure of the space of motions (locally, the phase space) of a dynamical system.[24] According to the Kirillov-Kostant-Souriau orbit method,[38,39,24] elementary relativistic dynamical systems (free particles) are classified as coadjoint orbits of the (restricted) Poincaré group P' (the universal covering of P is not needed for our purpose). These orbits turn out to be labeled by two Casimir invariants, namely, spin s and mass m. For example, the symplectic structure of a spinless massive Poincaré-orbit $\Omega(s=0,m>0)$ is defined by the exterior derivative of the one-form

$$\alpha \equiv -mc^2\underset{\sim}{\theta}^4 \tag{4.9}$$

of P viewed as an "evolution" space above Minkowski space-time P/L inasmuch as the leaves of the characteristic foliation $\ker(d\alpha)$ do project onto timelike Minkowski geodesics. The space of motions $P/\ker(d\alpha)$ is nothing but the orbit $\Omega(0,m) \cong \mathbb{R}^6$ itself endowed with the symplectic two-form σ,

$$d\alpha \equiv [P \to \Omega(0,m)]^*\sigma\ . \tag{4.10}$$

In the case of a curved Lorentzian space-time (M,g), the evolution space to consider is rather the bundle of Lorentz frames $L(M)$ endowed with the same dynamical one-form (4.9) (which amounts to minimal gravitational coupling). See Refs. 40, 41, and 23 for a more detailed account on these questions, especially those related to classical spin which we will skip here for the sake of simplicity.

What is the situation in the case of $\widetilde{P}$ elementary dynamical systems? Since $\widetilde{P}$ is isomorphic with the direct product $P \times \mathbb{R}$, all $\widetilde{P}$-coadjoint orbits are clearly symplectomorphic with the direct products $\Omega(s,m) \times \{E_0\}$, where E_0 is the $(\mathbb{R},+)$ Casimir readily interpreted as the *relativistic internal energy.*

It is not hard to find the one-form of $\widetilde{P}$ (pulled-back from S) that gives rise to the symplectic structure of the coadjoint orbit $\Omega(0,m) \times \{E_0\}$,

$$\varpi \equiv -mc^2\underset{\sim}{\theta}^4 + E_0c^{-2}\underset{\sim}{\theta}^5\ . \tag{4.11}$$

The velocity of light "c" enters formula (4.11) in order to comply with the physical dimension of the soldering form $\underset{\sim}{\theta}$ [the $(\mathbb{R}^5,+)$ components of the Maurer-Cartan one-form of $\widetilde{P}$]: $[\underset{\sim}{\theta}^B]=L$; $[\underset{\sim}{\theta}^4]=T$; $[\underset{\sim}{\theta}^5]=A\cdot M^{-1}$ where $A \equiv M \cdot L^2 \cdot T^{-1}$ $(=[\varpi])$ stands for "action."

The one-form ϖ descends as the canonical one-form

$$\varpi \equiv \underset{\sim}{\widetilde{p}}_j d\underset{\sim}{x}^j \quad (j=1,\ldots,5) \tag{4.12}$$

on the sub-bundle of $T^*\mathbb{R}^{4,1}$ defined by the constraints

$$g^{ij}\widetilde{p}_i\widetilde{p}_j = (E_0{}^2 - m^2c^4)/c^2\ , \tag{4.13}$$

$$\widetilde{p}_j\underset{\sim}{\xi}^j = E_0/c^2\ . \tag{4.14}$$

Note that $\widetilde{p}$ is interpreted as the momentum, energy, mass (co)vector of the spinless particle, and since

$$d\underset{\sim}{\theta}^5 = 0 \quad (\underset{\sim}{\theta}^5 = d\underset{\sim}{x}^5)\ , \tag{4.15}$$

introducing the relativistic internal energy E_0 in this five-dimensional setting does not modify space-time dynamics—i.e., the space-time projection of the leaves of the characteristic $\widetilde{P}$-foliation $\ker(d\varpi)$. We simply have the latitude of choosing mass and internal energy quite independently.

B. Bargmann dynamical systems

Let us now consider the situation from the point of view of the Bargmann fibration $\mathbb{R}^{4,1} \to \mathbb{R}^{3;1}$.

If we assume that the one-form ϖ in (4.11) of the affine de Sitter group S should also rule nonrelativistic spinless particle dynamics, we are led to consider this time the one-form

$$\beta \equiv [B \to S]^*\varpi\ . \tag{4.16}$$

The soldering form $\widetilde{\theta}$ of the principal bundle $B \to B/H$ is related to $\underset{\sim}{\theta}$ by

$$\underset{\sim}{\theta}^A = \widetilde{\theta}^A\ ,$$
$$\underset{\sim}{\theta}^4 = (\widetilde{\theta}^4 - c^{-2}/2\widetilde{\theta}^5)\ , \tag{4.17}$$
$$\underset{\sim}{\theta}^5 = (\widetilde{\theta}^5/2 + c^2\widetilde{\theta}^4)$$

[cf. (4.3)] and thus

$$\beta = (E_0 - mc^2)\widetilde{\theta}^4 + \tfrac{1}{2}(E_0c^{-2} + m)\widetilde{\theta}^5\ . \tag{4.18}$$

But c-dependent coefficients should not occur in a nonrelativistic expression such as (4.18). Using the flexibility in the choice of the relativistic internal energy E_0, we would like to have the velocity of light c removed from (4.18) by canceling some unsatisfactory coefficient. If we assume positivity of energy, the only possibility we are left with is indeed

$$E_0 = mc^2 \tag{4.19}$$

and the dynamical one-form β is finally given by the very simple expression

$$\beta = m\tilde{\theta}^5 . \tag{4.20}$$

It appears that the celebrated Einstein formula (4.19) can be considered as providing the structural coherence between relativistic and nonrelativistic dynamical systems built from a common five-dimensional space-time extension. It implies a *zero classical internal energy* [the "time" coefficient in (4.18)].

The one-form (4.20) can be shown to give rise to the canonical symplectic structure of the B-coadjoint orbit with spin zero, mass m, and vanishing (classical) internal energy. Let us stress that (4.19) also implies the identification of relativistic and nonrelativistic mass Casimir invariants. See Refs. 6 and 23 for a classification of Bargmann elementary dynamical systems. It has already been proved there that the leaves of the characteristic foliation of the B-presymplectic two-form $d\beta$ project onto flat Galilei space-time as timelike geodesics (world lines). Hence the straightforward generalization to the curved case: the evolution space of a spinless test particle in the Newtonian gravitational field is the bundle of Galilei frames $H(\tilde{M})$ endowed with the one-form β (4.20) whose exterior derivative (2.21),

$$d\beta = m\tilde{\omega}_4^A \wedge \tilde{\theta}^B \delta_{AB} , \tag{4.21}$$

defines the symplectic structure of the space of motions $U \equiv H(\tilde{M})/\ker(d\beta)$. Compare Ref. 6. [The assumption that the space of motions U be actually a smooth Hausdorff manifold forces us to pay special attention to some particular cases such as the regularization of the Kepler (or Newton) problem.[42]]

Again the one-form β descends as the canonical one-form

$$\beta \equiv p_j dx^j \tag{4.22}$$

on the sub-bundle $V \subset T^*\tilde{M}$ defined by the constraints

$$g^{ij} p_i p_j = 0 , \tag{4.23}$$

$$p_j \xi^j = m , \tag{4.24}$$

i.e., the momentum-energy-mass covector p [see (4.34)] turns out to be null and the mass is related to the canonical Killing vector field ξ. Note that the constraints (4.13) and (4.14) are identical to (4.23) and (4.24) as long as (4.19) holds.

The equations of motion are quite easily derived by computing the kernel of the presymplectic two-form $d\beta = dp_j \wedge dx^j$ of the new evolution space V. We find that a curve $[t \mapsto p_j(t), x^j(t)]$ is tangent to $\ker(d\beta)$ iff

$$\dot{p}^j = 0 , \tag{4.25}$$

$$\dot{x}^j = \lambda p^j + \mu \xi^j \quad (\lambda, \mu \in \mathbb{R}) , \tag{4.26}$$

where $\dot{p}^j \equiv \dot{x}^k \tilde{\nabla}_k p^j$ and $\dot{x}^j \equiv dx^j/dt$.

The particle is not localized on $(\tilde{M}, g, \xi)$ although it appears to be localized on a timelike geodesic of $(M, \gamma, \psi, \nabla)$,

$$\dot{p}^\alpha = 0 , \tag{4.27}$$

$$\dot{x}^\alpha = \lambda p^\alpha \quad (\lambda \in \mathbb{R}; \psi_\alpha p^\alpha = m) . \tag{4.28}$$

C. Conservation laws

Conservation laws are associated with the symplectomorphisms of the space of motions of a dynamical system. We will investigate here those generated by the Bargmann automorphisms (2.37). Let us recall that a vector field Z lies in the Lie algebra of $\mathrm{Barg}(\tilde{M}, g, \xi)$ iff

$$\pounds_Z g = 0, \quad \pounds_Z \xi = 0 . \tag{4.29}$$

Its canonical lift ζ to $H(\tilde{M})$ satisfies $\pounds_\zeta \tilde{\theta} = 0$ from (2.36). It follows from (4.20) that necessarily

$$\pounds_\zeta \beta = 0 , \tag{4.30}$$

hence that the Bargmann automorphisms form a subgroup of all symplectomorphisms of the space of motions U. The function

$$h \equiv \beta(\zeta) \quad (= p_j Z^j) \tag{4.31}$$

on $H(\tilde{M})$ satisfies $X(h) = (\pounds_\zeta \beta)(X) - d\beta(\zeta, X) = 0$ for all $X \in \ker(d\beta)$. It thus turns out to be actually defined on U (the Hamiltonian associated with ζ). In other words, h is a constant of the motion.

Let us compute this Hamiltonian in the flat case. From (2.1) we can find the general expression of an infinitesimal Bargmann automorphism

$$Z = (\omega_B^A x^B + \beta^A x^4 + \gamma^A)\partial_A + \epsilon \partial_4 + (\chi - \beta_A x^4)\partial_5 , \tag{4.32}$$

where $\omega \in \mathrm{so}(3)$, $\beta, \gamma \in \mathbb{R}^3$, $\epsilon, \chi \in \mathbb{R}$. Since h depends linearly upon Z, we can put

$$h \equiv -\tfrac{1}{2} L_{AB}\omega^{AB} - G_A \beta^A + P_A \gamma^A - E\epsilon + M\chi \tag{4.33}$$

and readily find with the help of Eqs. (4.23), (4.24), (4.31), and (4.32) that

$$\begin{aligned} L_{AB} &= 2x_{[A}p_{B]} \quad \text{(angular momentum)} , \\ G^A &= mx^A - p^A x^4 \quad \text{(center of mass)} , \\ P^A &= p^A \quad \text{(linear momentum)} , \\ E &= -p_4 = p_A p^A/(2m) \quad \text{(energy)} , \\ M &= p_5 = m \quad \text{(mass)} . \end{aligned} \tag{4.34}$$

V. THE COVARIANT SCHRÖDINGER EQUATION

The question arises as to whether the wave equations of nonrelativistic quantum mechanics can be recast in a different guise by means of Bargmann structures. The point is that the geometric apparatus previously introduced can be most naturally incorporated into Schrödinger theory. Remember that the Bargmann group was originally discovered[9] in the quantum group-theoretical context to get rid of the unsuccessful attempts to let the Galilei group act on the solutions of the free Schrödinger equation (cohomological obstruction). We will again deal with the spinless case for the sake of brevity. The spin-$\frac{1}{2}$ (Levy-Leblond) equation[19,22] will be revisited along the same lines elsewhere.

A. Geometric quantization and the free Schrödinger equation

We rephrase in a Bargmann covariant manner the method of geometric quantization[39,24,43] to systematically derive the quantum nonrelativistic wave equation for a free (elementary) spinless dynamical system. Using the expression (4.22) of the dynamical one-form β on the evolution space $V \equiv \{(p,x) \in T^*\mathbb{R}^{4,1},\ g^{-1}(p,p)=0,\ p(\xi)=m\}$, we get

$$\beta = d(p_j x^j) - q^A dp_A \tag{5.1}$$

with

$$q^A \equiv x^A - p^A x^4/m \quad (=G^A/m) . \tag{5.2}$$

Setting

$$z \equiv e^{(i/\hbar)p_j x^j} \tag{5.3}$$

[the Planck constant $\hbar$ is introduced in such a way that the phase $z \in U(1)$ is dimensionless] we find that β is actually the pull-back to V of the canonical U(1)-connection one-form

$$\omega \equiv (\hbar/i)dz/z - q^A dp_A \tag{5.4}$$

of the (pre)quantum bundle $(Y \equiv \mathbb{R}^6 \times U(1), \omega)$ over the symplectic space of motions $(\mathbb{R}^6, d\omega = dp_A \wedge dq^A)$ already introduced in Sec. IV.

The geometric quantization of the model starts with the choice of a Planck polarization F (in the real case, a maximal involutive distribution of horizontal hence isotropic subspaces $F_y \subset T_y Y$, i.e., $[F,F] \subset F$ and $\omega(F) = d\omega(F,F) = 0$ (Ref. 24)). This amounts to the choice of a "representation" (e.g., the p or q representation). For convenience we will choose here the polarization [see (5.4)]

$$F \equiv \text{span}\{\partial/\partial q^A\} \quad (A=1,2,3) . \tag{5.5}$$

The next step consists in looking for (polarized) wave functions, i.e., complex-valued functions $f: Y \to \mathbf{C}$ constant along F,

$$X(f) = 0 \quad (\text{all } X \in F) , \tag{5.6}$$

and equivariant with respect to the U(1) action on Y,

$$f(yz) = z f(y) \quad (\text{all } z \in \text{U}(1)) . \tag{5.7}$$

We easily find that in our case (5.5) the F-polarized functions are of the form

$$f(p,q,z) = z\widetilde{\Phi}(p) , \tag{5.8}$$

where $\widetilde{\Phi}$ denotes a complex function of (p_A) only. Pulling them back to the evolution space V we get

$$f(p,x) = e^{(i/\hbar)p_j x^j}\widetilde{\Phi}(p) . \tag{5.9}$$

We can then integrate (5.9) along the fibers (x=const) and assuming for mere technical reasons that $\widetilde{\Phi}$ be compactly supported, we readily obtain the function $\Phi: \mathbb{R}^{4,1} \to \mathbb{C}$,

$$\Phi(x) = \int e^{(i/\hbar)p_j x^j}\widetilde{\Phi}(p)d\mu(p) , \tag{5.10}$$

where $d\mu(p)$ is the Euclidean density $|dp_1 \wedge dp_2 \wedge dp_3|$ of the fibers.

Our claim is that (5.10) is the general solution of the free Schrödinger equation

$$[\hbar^2/(2m)]\Delta\Phi + i\hbar\partial_4\Phi = 0 . \tag{5.11}$$

With the help of (4.34) we find that (5.10) can be rewritten as

$$\Phi_Z(x^A, x^4) = Z \int \exp\{(i/\hbar)[p_A x^A - p^2/(2m)x^4]\} \times \widetilde{\Phi}(p)d\mu(p) \tag{5.12}$$

which we recognize as the general solution of Schrödinger equation (5.11) modulo an overall phase factor

$$Z \equiv e^{imx^5/\hbar} \tag{5.13}$$

which quite naturally shows up in our formalism.

The crucial remark is that the wave function Φ (5.10) really satisfies the following set of partial differential equations—compare (4.23) and (4.24):

$$\Delta_g \Phi = 0 , \tag{5.14}$$

$$\xi(\Phi) = (im/\hbar)\Phi , \tag{5.15}$$

where Δ_g denotes the Laplace-Beltrami operator of the flat Bargmann manifold $(\mathbb{R}^{4,1}, g, \xi)$ [see (4.4)]. The pair of equations of (5.14) and (5.15) thus turns out to be strictly equivalent to the Schrödinger equation (5.11).

B. The Schrödinger equation on a curved Newtonian space-time

It is known that minimal gravitational coupling cannot easily be formulated on the Schrödinger equation (5.11) in a Galilei-covariant manner. The difficulties attached to that problem were studied in Refs. 21 and 7. On the other hand, geometric quantization does not provide us with a general scheme to tackle the problem in full generality, there being (up to now) no quite satisfactory formulation of the so-called "pairing" method.[43]

Just as in the framework of quantum mechanics on a general relativistic space-time, we heuristically introduce the prescription of minimal Newtonian gravitational coupling by assuming that the Schrödinger equation retains the same form as in (5.14) and (5.15) on any Bargmann manifold $(\widetilde{M}, g, \xi)$.

A *Schrödinger wave function* describing a spinless particle of mass m (5.15) is thus a mapping $\Phi: \widetilde{M} \to \mathbb{C}$ such that

$$\Phi(xa) = e^{ima/\hbar}\Phi(x) \quad (\text{all } a \in \mathbb{R}) , \tag{5.16}$$

where $(x \mapsto xa)$ denotes the free $(\mathbb{R},+)$ action on the Bargmann manifold [the wave function is associated with a character of $(\mathbb{R},+)$ labeled by the mass of the particle].

The *Schrödinger equation* would then be nothing but the harmonicity condition

$$\Delta_g \Phi = 0 . \tag{5.17}$$

Let us explicitly work out the local expression of the Laplace-Beltrami operator in an adapted coordinate system (Sec. III). We have

$$\begin{aligned}\Delta_g \Phi &= g^{ij}\tilde{\nabla}_i \partial_j \Phi \\ &= g^{\alpha\beta}\tilde{\nabla}_\alpha \partial_\beta \Phi + g^{5\alpha}(\tilde{\nabla}_\alpha \partial_5 \Phi + \tilde{\nabla}_5 \partial_\alpha \Phi) + g^{55}\tilde{\nabla}_5 \partial_5 \Phi \\ &= \gamma^{\alpha\beta}[\nabla_\alpha \partial_\beta \Phi - \tilde{\Gamma}^{\,5}_{\alpha\beta}(im/\hbar)\Phi] + (2im/\hbar)V^\alpha \partial_\alpha \Phi \\ &\quad -(2m^2/\hbar^2)\phi\Phi\end{aligned}$$

in view of (3.12)—(3.14), and (3.24) and $\partial_5\Phi=(im/\hbar)\Phi$ [(5.15) and (5.16)].

Using (3.6) and (3.25) together with (3.20) and (3.21) we find that Eq. (5.17) takes the form

$$\begin{aligned}&[\hbar^2/(2m)]\nabla^\alpha \partial_\alpha \Phi + i\hbar(U^\alpha - \gamma^{\alpha\beta}A_\beta)\partial_\alpha \Phi \\ &\quad + [(i\hbar/2)\nabla_\alpha(U^\alpha - \gamma^{\alpha\beta}A_\beta) + m(A_\alpha U^\alpha - A^2/2)]\Phi = 0\end{aligned} \tag{5.18}$$

which turns out to exactly correspond to the Schrödinger equation on a curved Newtonian space-time derived from quite different arguments in Refs. 21 and 7.

Let us stress that the group of Bargmann automorphisms (2.37) maps the space of solutions (5.16) of the Schrödinger equation (5.17) into itself,

$$\Phi \mapsto (\tilde{a}^{-1})^* \Phi \quad [\tilde{a} \in \mathrm{Barg}(\tilde{M},g,\xi)] , \tag{5.19}$$

since the Laplace operator is invariant under isometries and the mass $(\mathbb{R},+)$ character (5.16) preserved by bundle automorphisms. In the flat case, the action (5.19) turns out to reduce to the well-known Bargmann representation on the solutions of the free Schrödinger equation.[18]

The larger "chronoprojective" invariance of Eq. (5.18) has been elucidated in Ref. 44 and will be reformulated in terms of conformal Bargmann invariance in a forthcoming paper.

VI. NEWTON'S FIELD EQUATIONS

Strangely enough, Newton's field equations (1.15) and (1.17) cannot be easily derived from a specific space-time Lagrangian density. It just seems that there might exist some puzzling geometric obstruction to the existence of a well-defined variational problem in the four-dimensional picture. Also the fact that only matter density enters the source term of Newton's field equations has not been quite understood so far in a covariant formalism. The role of the mass flow and stress-energy tensor of matter distributions[6] still remains to be clarified. Moreover, since Newton's theory can be given a geometric description that borrows some aspects from general relativity, why does the "natural" Lagrangian $L \equiv R\,\mathrm{vol}$ fail to yield the correct field equations? See Ref. 8 for a group-theoretical approach and Ref. 45 for a survey of the problem.

We will show here that the introduction of Bargmann structures improves this situation over the four-dimensional Newtonian formulation. Replacing the space-time arena by a Lorentzian five-dimensional extension $(\tilde{M},g,\xi)$ already prompts us to try the Einstein Lagrangian,

$$\tilde{L} \equiv \tilde{R}\,\widetilde{\mathrm{vol}} \tag{6.1}$$

and to formulate the following variational problem,

$$\delta \int_{\tilde{M}} \tilde{L} = 0 \tag{6.2}$$

on a given $(\mathbb{R},+)$ principal bundle $(\tilde{M},\xi)$ as though we were actually dealing with vacuum field equations. We are deliberately ignoring matter field Lagrangians in (6.1) and (6.2).

Since the bundle structure is given from the outset, we simply must take into account the fact that ξ be null—(2.8), (3.2). Introducing this constraint via a Lagrange multiplier λ, (6.1) and (6.2) now read

$$\delta \int_{\tilde{M}} [\tilde{R} + \lambda g(\xi,\xi)]\widetilde{\mathrm{vol}} = 0 . \tag{6.3}$$

An elementary calculation then yields

$$\int [\tilde{R}^{\,ij} - \tfrac{1}{2}(\tilde{R} + \lambda g_{kl}\xi^k\xi^l)g^{ij} + \lambda \xi^i \xi^j]\delta g_{ij}\widetilde{\mathrm{vol}} = 0 \tag{6.4}$$

for all variation δg with compact support (remember that $\delta\xi=0$), hence

$$\tilde{R}^{\,ij} - \tfrac{1}{2}\tilde{R}g^{ij} + \lambda\xi^i\xi^j = 0 . \tag{6.5}$$

Taking the trace of this expression we end up with [see (3.2) and (3.29)]

$$\tilde{R} = 0 \quad (=R) \tag{6.6}$$

and

$$\tilde{R}^{\,ij} = -\lambda\xi^i\xi^j , \tag{6.7}$$

where λ is *a priori* a (real) function of $\tilde{M}$. Using the identity

$$\tilde{\nabla}_i(\tilde{R}^{\,ij} - \tfrac{1}{2}\tilde{R}g^{ij}) = 0 , \tag{6.8}$$

we get

$$\xi(\lambda) = 0 . \tag{6.9}$$

The Lagrange multiplier λ is actually a function of space-time M. A quick dimensional inspection of formula (6.7) shows that $[\lambda] = G \times$ mass density (G denotes Newton's gravitational constant); we thus would like to set

$$\lambda \equiv -4\pi G\rho \tag{6.10}$$

in order to rewrite (6.7) as

$$\tilde{R}_{ij} = 4\pi G\rho\tilde{\psi}_i\tilde{\psi}_j \quad (i,j=1,\ldots,5) \tag{6.11}$$

with $\tilde{\psi}_i \equiv g_{ij}\xi^j$ (3.4).

Taking advantage of the previous results (3.12)—(3.14), (3.28), and (3.29), we find that Eq. (6.11) is strictly equivalent to

$$R_{\alpha\beta} = 4\pi G\rho\psi_\alpha\psi_\beta \quad (\alpha,\beta=1,\ldots,4) \tag{6.12}$$

which is nothing but Newton's (inhomogeneous) field equation (1.17). Note that (1.15) is automatically satisfied in our formalism [see (2.35)]. The source of the Newtonian gravitational field can thus be viewed as the Lagrange multiplier associated with the null Bargmann fibration over space-time. This somewhat special appearance of mass density is indeed characteristic of our formalism.

Let us finish with some well-known solutions of Newton's field equations expressed in terms of Bargmann structures. The spherically symmetric static Newtonian field is clearly described by the triple $((\mathbb{R}^3-\{0\}\times\mathbb{R}\times\mathbb{R},g,\partial/\partial x^5)$ with [cf. (3.14)]

$$g=\delta_{AB}dx^A\otimes dx^B+dx^4\otimes dx^5+dx^5\otimes dx^4 +2GM/r\,dx^4\otimes dx^4\ , \tag{6.13}$$

where $r\equiv(x_Ax^A)^{1/2}\neq 0$, and M denotes the total mass of the source.

As for the classical model of Newtonian cosmology (e.g., Ref. 46) it turns out to correspond to the Bargmann structure $(\mathbb{R}^{4,1},g,\partial/\partial x^5)$ with

$$g=\delta_{AB}dx^A\otimes dx^B+dx^4\otimes dx^5 +dx^5\otimes dx^4-Br^2/\beta^3dx^4\otimes dx^4\ , \tag{6.14}$$

where β is a function of T (the absolute time axis) that satisfies

$$\beta^2\ddot{\beta}=-B=\text{const}<0\quad\text{(expansion)} \tag{6.15}$$

and can be interpreted as the nonrelativistic version of the Friedmann universe "radius"—or cotemperature (from a thermodynamical point of view).

The Hubble coefficient is given by

$$H\equiv\dot{\beta}/\beta\ . \tag{6.16}$$

If we then perform the following coordinate transformation,

$$\begin{aligned} y^A&\equiv x^A/\beta\ ,\\ y^4&\equiv\int\beta^{-2}dx^4\ ,\\ y^5&\equiv x^5+Hr^2/2\ , \end{aligned} \tag{6.17}$$

the metric (6.14) turns out to be conformally flat

$$g=\beta^2(\delta_{AB}dy^A\otimes dy^B+dy^4\otimes dy^5+dy^5\otimes dy^4)\ . \tag{6.18}$$

This result of our theory helps to relate even more closely Friedmann's and Newton's cosmological models on the grounds of maximal conformal invariance. See Refs. 23, 27, and 28 for an alternative proof of the so-called chronoprojective (or nonrelativistic conformal) flatness of classical cosmological space-time.

ACKNOWLEDGMENTS

We would like to express our gratitude to J. Brooke, P. Iglesias, and J. M. Souriau for enlightening discussions. Special thanks are due to J. Ehlers for several suggestions and ideas which have greatly influenced this work.

*Faculté des Sciences de Luminy et Centre de Physique Théorique, CNRS, 70 route Léon Lachamp, Case 907, F-13288 Marseille Cedex 9.

†Present address: Department of Mathematics, The University of Alberta, Edmonton, Canada T6G 2G1.

‡Laboratoire Propre, Centre National de La Recherche Scientifique.

[1]P. Havas, Rev. Mod. Phys. **36**, 938 (1964).

[2]H. P. Künzle, Ann. Inst. Henri Poincaré **27**, 337 (1972).

[3]J. Ehlers, in *Grundlagenprobleme der Modernen Physik*, edited by J. Nitsch *et al.* (Bibl. Inst. Mannheim, Wien, Zürich, 1981).

[4]E. Cartan, Ann. Scient. Ec. Norm. Sup. **40**, 325 (1923); **41**, 1 (1924).

[5]A. Trautman, Rep. Math. Phys. **10**, 297 (1976).

[6]C. Duval and H. P. Künzle, Rep. Math. Phys. **13**, 351 (1978).

[7]C. Duval and H. P. Künzle, Gen. Relativ. Gravit. **16**, 333 (1984).

[8]W. G. Dixon, Commun. Math. Phys. **45**, 167 (1975).

[9]V. Bargmann, Ann. Math. **59**, 1 (1954).

[10]C. Duval and H. P. Künzle, C. R. Acad. Sci. **285A**, 813 (1977).

[11]F. Müller-Hoissen, Dissertation Göttingen, 1983 (unpublished).

[12]H. P. Künzle, Gen. Relativ. Gravit. **7**, 445 (1976).

[13]M. Trümper, Ann. Phys. (N.Y.) **149**, 203 (1983).

[14]H. Goenner, Phys. Lett. **93A**, 469 (1983).

[15]F. Müller-Hoissen, Gen. Relativ. Gravit. **15**, 1051 (1983).

[16]R. Collela, A. W. Overhauser, and S. A. Werner, Phys. Rev. Lett. **34**, 1472 (1975).

[17]L. Stodolski, Gen. Relativ. Gravit. **11**, 391 (1979).

[18]J. M. Levy-Leblond, in *Group Theory and its Applications*, edited by E. M. Loebl (Academic, New York, 1971), Vol. 2.

[19]J. M. Levy-Leblond, Commun. Math. Phys. **6**, 286 (1967).

[20]J. M. Levy-Leblond, Riv. Nuovo Cimento **4**, 99 (1974).

[21]K. Kuchar, Phys. Rev. D **22**, 1285 (1980).

[22]H. P. Künzle and C. Duval, Ann. Inst. Henri Poincaré (to be published).

[23]C. Duval, Thèse, Université d'Aix-Marseille II, 1982 (unpublished).

[24]J. M. Souriau, *Strucure des systèmes dynamiques* (Dunod, Paris, 1970).

[25]C. R. Hagen, Phys. Rev. D **5**, 337 (1972).

[26]U. Niederer, Helv. Phys. Acta **45**, 802 (1972).

[27]G. Burdet, C. Duval, and M. Perrin, J. Math. Phys. **24**, 1752 (1983).

[28]G. Burdet, C. Duval, and M. Perrin, Publ. RIMS Kyoto Univ. **19**, 813 (1983).

[29]J. A. Brooke, J. Math. Phys. **19**, 952 (1978).

[30]J. A. Brooke, J. Math. Phys. **21**, 617 (1980).

[31]J. A. Brooke, Ph.D. thesis, University of Alberta, 1980 (unpublished).

[32]J. Ehlers, in *Space, Time and Mechanics*, edited by D. Mayr and G. Sussmann (Reidel, Boston, 1983).

[33]G. Burdet, J. Patera, M. Perrin, and P. Winternitz, J. Math. Phys. **19**, 1758 (1978).

[34]G. Burdet, M. Perrin, and P. Sorba, Commun. Math. Phys. **34**, 123 (1973).

[35]S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry* (Wiley, New York, 1963), Vol. 1.

[36]A. Trautman, C. R. Acad. Sci. **257**, 617 (1963).

[37]J. Dieudonné, *Eléments d'Analyse* (Gauthier-Villars, Paris, 1970), Vol. 3.

[38]A. Kirillov, Russ. Math. Surveys **17**, 53 (1962).

[39]B. Kostant, *Quantization and Unitary Representations*, Vol. 170 of *Lecture Notes in Mathematics* (Springer, Berlin, 1970).

[40]H. P. Künzle, J. Math. Phys. **13**, 739 (1972).

[41]J. M. Souriau, Ann. Inst. Henri Poincaré **20**, 315 (1974).

[42]J. M. Souriau, in Proc. IUTAM-ISIMM, Modern Developments in Analytical Mechanics, Att. del. Acad. Sci. Torino

Supp. Vol. 117 (1983).
[43]J. Sniaticki, *Geometric Quantization and Quantum Mechanics* (Springer, New York, 1980).
[44]G. Burdet and M. Perrin, J. Math. Phys. (to be published).
[45]H. Goenner, Gen. Relativ. Gravit. **16**, 513 (1984).
[46]J. M. Souriau, in *Geométrie Symplectique et Physique Mathématique,* Coll. Int. CNRS (edited by J. M. Souriau) no. 237 Editions du CNRS, 59 (1975).