Hadron structure in a simple model of quark/nuclear matter

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We study a simple model for one-dimensional hadron matter with many of the essential features needed for examining the transition from nuclear to quark matter and the limitations of models based upon hadron rather than quark degrees of freedom. The dynamics are generated entirely by the quark confining force and exchange symmetry. Using Monte Carlo techniques, the ground-state energy, single-quark momentum distribution, and quark correlation function are calculated for uniform matter as a function of density. The quark confinement scale in the medium increases substantially with increasing density. This change is evident in the correlation function and momentum distribution, in qualitative agreement with the changes observed in deep-inelastic lepton scattering. Nevertheless, the ground-state energy is smooth throughout the transition to quark matter and is described remarkably well by an effective hadron theory based on a phenomenological hadron-hadron potential.

I. INTRODUCTION

The properties of nuclear matter are described remarkably well by theoretical calculations based upon the traditional framework of static nucleons interacting through potentials constrained by nucleon-nucleon scattering phase shifts and deuteron properties. This success is somewhat surprising in view of the fact that nucleons are composite structures with a size approximately equal to the internucleon separation in nuclear matter. Clearly, a major challenge in nuclear physics is that of understanding how models based upon hadron degrees of freedom and effective hadron-hadron interactions emerge from the underlying theory of quarks and gluons. In doing so, one hopes both to understand quantitatively the limits of the potential theory and to identify specific signatures of the quark substructure. There have long been clear indications (e.g., thermal neutron capture by protons) that hadronic degrees of freedom additional to the nucleon are needed for quantitative description of nuclear structure and low-energy hadron dynamics. However, apart from the A dependence of quark structure functions measured in deep-inelastic electron scattering [the European Muon Collaboration (EMC) effect], explicit signatures of the quark substructure in nuclei have proved elusive.

At sufficiently high density or temperature, quarks are believed to become deconfined, rendering descriptions of nuclear matter in terms of hadron degrees of freedom inappropriate. However, the characteristics of the transition from nuclear to quark matter and the effect on observables are not understood. A minimum condition on theoretical models used for investigations of the transition region is a reasonable limit both in the low-density, confined-quark regime and in the high-density, free-quark-gas regime.

We shall investigate the questions outlined above in a simplified model. We use a nonrelativistic quark model similar to the flux-tube model and study the dynamics of one-dimensional matter. The model is a simple extension of that introduced by Lenz *et al.*[1] in a study of two-hadron dynamics, wherein it was found that low-energy properties of the system could generally be reproduced by effective hadron theory while still providing characteristic signatures of quark substructure in selected observables. In contrast, our results apply to the ground state of a many-hadron system (uniform matter) and we study the density dependence of the system from the low-density "nuclear" gas through the transition to a quark Fermi gas. Qualitative insights into the transition and characterization of modified hadron structure in the medium are provided by Hartree-Fock and variational approximations. Exact results are obtained by a stochastic evaluation of the imaginary-time evolution operator. We stress that our model for the many-body system is based entirely upon the quark confining potential, which is characterized by the confinement scale, and quark-exchange symmetry. No additional interactions or scales are introduced.

Correct treatment of the quark-exchange symmetry is important. The variety of "package" models put forward to discuss quark effects in nuclei deal with probabilities rather than amplitudes and thus ignore interference terms. To our knowledge the present work is the first calculation of an explicitly antisymmetric quark wave function for uniform hadron matter.

The nonrelativistic quantum-mechanical approach which we follow obviously will not provide a fundamental description of nuclear matter based upon quantum chromodynamics. Nevertheless, we recall that nonrelativistic quark models have been remarkably effective phenomenologies for correlating data on hadron spectroscopy and hadron properties.[2] They further provide a basis for understanding the phenomena observed in low-energy hadron scattering, such as strong repulsion between nucleons and scattering resonances associated with internal excitation.[1] This is the spirit of our work, namely, to explore the phenomena which arise in the ground state of hadronic matter from hadron substructure and to shed light on the limits of a phenomenology based on hadron degrees of freedom. In this regard, it is important to stress that our model contains only essential features which must be present in any more sophisticated theory, including confinement, exchange symmetry, and cluster separability. Whereas the one-dimensional nature of our calculations may affect the details of the transition from nuclear to quark matter, the model may be solved exactly in one-dimension and the simplicity of the one-dimensional system lends itself to qualitative insights.

In Sec. II, we describe the Hamiltonian for the system of N quarks. Our model hadrons are composed of two quarks bound by a confining potential. However, a satisfactory many-body Hamiltonian cannot be obtained by summing the two-body confining potential over all pairs because it leads to long-range, strong van der Waals forces. Such long-range forces, falling off only with a power of the separation, have no experimental or theoretical basis and would likely have significant effects at high density. Hence, we have employed the model of Lenz *et al.*[1] and write the interaction as a sum of $N/2$ confining "links" between quark pairs such as to minimize the potential energy. This can be thought of as an adiabatic approximation to the energy of a system of heavy quarks in field theory. For a fixed configuration of quarks, the energy of the gluon degrees of freedom is essentially minimized by optimally connecting pairs of quarks by flux tubes. Consequently, our interaction corresponds to an N-body potential. We also present in Sec. II Hartree-Fock estimates of the ground-state energy for one-dimensional quark matter. These provide insight into the density which characterizes the transition from nuclear to quark matter.

Ground-states properties of the model are discussed in Sec. III. While the Brueckner and Fermi-hypernetted-chain approaches appear to be intractable for an N-body potential, no complication is introduced in the stochastic evaluation of the evolution operator by the many-body nature of the interaction and observables are therefore evaluated using a path-integral Monte Carlo technique. We also introduce a variational trial wave function which is valuable for characterizing the modification of hadron structure in the medium and for calculating the quark correlation function and momentum distribution. The numerical results are presented and the relation to the modification of the quark momentum distribution observed in the EMC effect is discussed.

In Sec. IV, we consider the system in terms of an approximation including only "nucleon" degrees of freedom. The low-energy hadron-hadron scattering amplitude is evaluated, both by the resonating-group method and by a stochastic procedure. As in low-energy nuclear physics, a phenomenological hadron-hadron local potential is then parametrized so as to reproduce the scattering amplitude and used to evaluate the ground-state energy of one-dimensional hadron matter. These potential-theory results are compared with the exact quark-matter results and are remarkably similar even at rather large densities.

Our main conclusions and future extensions of the work are discussed in Sec. V.

II. THE MODEL AND HARTREE-FOCK ESTIMATES

As noted above, nonrelativistic quark models have enjoyed considerable success in hadron spectroscopy and can form the basis for understanding low-energy hadron dynamics. Such models clearly have advantages over other phenomenological models based upon QCD (such as bag models) in allowing a consistent treatment of confinement dynamics in the many-hadron system. Certain conceptual questions concerning the role and behavior of hadron substructure may thereby be studied precisely and unambiguously.

We study a model system of hadrons composed of two quarks. This affords a certain technical simplicity while retaining the essential features of composite hadrons. We shall work in one space dimension, although extension of the model to three dimensions will be straightforward. The Hamiltonian for one hadron is

$$H_{12}=-\tfrac{1}{2}(\partial_{x_1}{}^2+\partial_{x_2}{}^2)+v(x_{12})\ , \tag{1}$$

where we take the quark mass $m=1$ and v is a confining potential, $v(x_{12})\to\infty$ as $|x_{12}|=|x_1-x_2|\to\infty$. For our calculations, we take a harmonic-oscillator quark-quark potential

$$v=\frac{1}{2}x_{12}{}^2\ . \tag{2}$$

Since we shall assume that the quarks are fermions and, for most of the calculations, without internal degrees of freedom, the ground state will be the lowest odd-parity state

$$\psi_0(x)=\mathcal{N}x\exp[-\tfrac{1}{2}\sqrt{1/2}x^2]\ ,$$
$$\mathcal{N}^{-2}=\pi^{1/2}2^{-1/4}\ . \tag{3}$$

The density $\rho_0(x)=\psi_0(x)^2$ has been normalized to unity. The relevant scale in the problem is given by the ground-state energy

$$\epsilon_0=\frac{3}{\sqrt{2}}=2.12 \tag{4}$$

or by the average quark separation

$$\bar{x}=(\langle x^2\rangle)^{1/2}=3^{1/2}2^{-1/4}=1.46\ . \tag{5}$$

The average quark density in the ground state is

$$\bar{\rho}=2\int_{-\infty}^{+\infty}dx\,\rho_0(x)^2=3\pi^{1/2}2^{-7/4}=0.50\ . \tag{6}$$

The average quark density is useful for giving a rough idea of the quark density in one-dimensional matter analogous to that of nuclear matter. The average quark density in a physical nucleon can be estimated in a uniform density model. Given the nucleon radius $R\equiv(\frac{5}{3})^{1/2}\bar{R}$ with the root-mean-square proton charge radius of $\bar{R}=0.81$ fm, we have

$$\bar{\rho}_N=3\left[\frac{4\pi}{3}R^3\right]^{-1}=0.63\ \text{fm}^{-3}\ . \tag{7}$$

The nuclear-matter density

$$\bar{\rho}_{\rm NM}\approx=3\times0.16\ \text{fm}^{-3}\approx0.5\ \text{fm}^{-3}$$

yields the ratio

$$\frac{\bar{\rho}_N}{\bar{\rho}_{\rm NM}} \approx (1.1)^3 . \tag{8}$$

A smoother nucleon density distribution would not change the ratio appreciably. We express this ratio as a cube in the spirit of a crude transformation between one and three dimensions. Consequently, we take

$$\bar{\rho}_{\rm HM} \approx \frac{\bar{\rho}}{1.1} \approx 0.45 \tag{9}$$

as the average quark density in our one-dimensional hadron matter which corresponds to nucleon overlap in nuclear matter.

In generalizing the two-quark Hamiltonian to a many-body system, we cannot follow the standard approach used for nonconfining potentials. Clearly, a simple two-body interaction would confine the whole system, not preserving cluster separability. In quark models, v is conventionally taken as having a color-dependent factor $\lambda_1 \cdot \lambda_2$, where the λ_i are the SU(3) color matrices for the ith particle. The lowest-order force then cancels between color singlets, but strong long-range van der Waals forces remain. These are not observed in nature and would lead to undesirably strong effects in our treatment of hadron matter. Instead, we use a generalization of the potential used by Lenz *et al.*[1] in their study of two-hadron scattering, giving the Hamiltonian

$$V = \min_{\{P\}} \sum_{n=1}^{N/2} v(\,|x_{P(2n-1)} - x_{P(2n)}|\,) , \tag{10}$$

$$H = T + V , \qquad T = -\tfrac{1}{2} \sum_{i=1}^{A} \partial_{x_i}{}^2 . \tag{11}$$

The minimum in Eq. (10) is taken over all permutations P of the quark labels which pair the N quarks into $N/2$ pairs, with the confining potential or link acting only between the two quarks in each of the $N/2$ pairs. For the four-quark system, the potential is

$$\begin{aligned} V(x_1,x_2,x_3,x_4) &= \min\{V_1,V_2,V_3\} , \\ V_1 &= v(x_{12}) + v(x_{34}) , \\ V_2 &= v(x_{13}) + v(x_{24}) , \\ V_3 &= v(x_{14}) + v(x_{23}) . \end{aligned} \tag{12}$$

Thus, while the potential is an N-body interaction (i.e., it depends upon the configuration of all N quarks), it is basically quite simple. The confining forces operate only between pairs of quarks. At any instant, the $N/2$ links arrange themselves in a way that minimizes the potential energy; this idea is familiar in adiabatic approximations to field theories. The only hadron interactions come from the exchange of quarks between clusters. The model explicitly has no van der Waals forces; the forces between well-separated clusters fall off with the overlap of wave functions. Exchange symmetry is respected. All quarks are confined into hadrons, while allowing the system to separate into free hadron scattering states.

This simple model also incorporates the expected low- and high-density limits. At low density, the system will obviously reduce to a free hadron gas. At sufficiently high density, each quark will be separated from its partner by a distance small compared to the confinement scale. Therefore, the potentials will become irrelevant and the system should behave as a free quark gas. A major aim of our work is examination of the hadron matter observables through the transition region between the two limits.

The matching problem posed by Eq. (10) is trivial in one dimension. (An efficient method for higher dimensions is presented in Ref. 3.) For periodic boundary conditions in one-dimension, corresponding to particles constrained to a ring, there are only two possible matchings to evaluate for any configuration since it is clear that any given quark must be paired with its left or right neighbor in order to minimize the energy. This simplification is important not only for the numerical evaluations described in the next section, but also for estimates which provide qualitative insight. We conclude this section with these estimates.

We start with a Hartree evaluation of the potential energy for a uniform density in a box of length L. Ignoring the periodic boundary conditions in the potential for the Hartree calculation (this has no effect in the $N \to \infty$ limit), we have

$$V = \tfrac{1}{2} \sum_{i=1}^{N/2} (x_{2i-1} - x_{2i})^2 , \tag{13}$$

where the quark coordinates x_i have been ordered from smallest to largest. We take N quarks confined within a distance L, i.e., density $\rho = N/L$. The Hartree potential energy is then

$$\begin{aligned} \langle V \rangle_H &= \frac{1}{L^N} \int_0^L dx_1 \cdots \int_0^L dx_N V(x_1 \cdots x_N) \\ &= L^2 N! \int_0^1 dx_N \int_0^{x_N} dx_{N-1} \cdots \int_0^{x_2} dx_1 \sum_{i=1}^{N/2} [\tfrac{1}{2}(x_{2i-1}{}^2 + x_{2i}{}^2) - x_{2i-1} x_{2i}] \\ &= \frac{L^2 N}{6} - L^2 N! \sum_{i=1}^{N/2} \int_0^1 dx_A \cdots \int_0^{x_2} dx_1 x_{2i-1} x_{2i} \\ &= \frac{L^2 N}{2} \frac{1}{(N+1)(N+2)} . \end{aligned} \tag{14}$$

Therefore the energy per quark in the $N \to \infty$, constant-ρ limit is

$$\frac{\langle V \rangle_H}{N} = \frac{1}{2\rho^2} . \tag{15}$$

Note that the potential *decreases* with increasing density. Since the Hartree potential per particle for a two-body potential is proportional to ρ, this is a signature of the many-body nature of our interaction.

Fermi anticorrelations will reduce the potential energy. The Hartree-Fock result is obtained by evaluating the expectation value of V with a Slater determinant for the N-quark system. A Monte Carlo evaluation of the expectation value gives, to within a few percent,

$$\frac{\langle V \rangle_{\rm HF}}{N} \approx \frac{1}{4\rho^2} . \tag{16}$$

The value $1/4\rho^2$ is the classical energy for fixed equally spaced quarks; since the average distance between quarks is ρ^{-1}, giving a potential energy per link of $\frac{1}{2}\rho^{-2}$ and thus a potential energy per quark of $\frac{1}{4}\rho^{-2}$. Thus, the Pauli correlations reduce the energy by substantially reducing the variance in the quark separations at high density.

The kinetic energy of the Slater determinant is the Fermi-gas result

$$\frac{\langle T \rangle_{\rm FG}}{N} = \tfrac{1}{6} k_F{}^2 = \frac{\pi^2}{6}\rho^2 , \tag{17}$$

giving a total Hartree-Fock energy

$$\frac{\langle E \rangle_{\rm HF}}{N} \approx \frac{\pi^2}{6}\rho^2 + \frac{1}{4\rho^2} . \tag{18}$$

The vanishing of the potential at high density has already been discussed. Its divergence at low density reflects the lack of confinement (or clustering) in the Slater determinant. The minimum in $\langle E \rangle_{\rm HF}$ occurs at

$$\rho_{\rm min} = \left[\frac{3}{2\pi^2} \right]^{1/4} \approx 0.62 \approx 1.37\bar{\rho}_{\rm HM} \approx 1.24\bar{\rho} . \tag{19}$$

The potential and kinetic terms dominate below and above $\rho_{\rm min}$, respectively, and we expect $\rho_{\rm min}$ to provide a reasonable first estimate of the density at which the wave function loses the clustering property associated with low-density hadronic matter. As will be confirmed by the exact results presented in the next section, a Fermi gas is a good approximation to the wave function above $\rho_{\rm min}$ and thus $\rho_{\rm min}$ characterizes the transition from hadron to quark matter. At this density, the increase in energy per quark above that of free hadrons is

$$\left. \frac{\langle E \rangle_{\rm HF}}{N} \right|_{\rho_{\rm min}} - \left. \frac{\langle E \rangle}{N} \right|_{\rho=0} = \frac{\pi}{\sqrt{6}} - \frac{3}{2\sqrt{2}} = 0.22 = 0.21 \left[\frac{\epsilon_0}{2} \right] . \tag{20}$$

Thus, the Hartree-Fock approximation gives a "critical" quark density about 24% greater than that in a free hadron (37% greater than $\bar{\rho}_{\rm HM}$) with an increase in energy about 20% above that supplied by zero-point motion in isolated clusters. These are modest increases above the isolated-cluster values.

III. THE STOCHASTIC METHOD AND QUARK-MATTER RESULTS

We use the path-integral Monte Carlo method[4] to calculate the ground-state energy of our many-fermion system. The imaginary-time evolution operator $e^{-\beta(H-e_N)}$, where e_N is any constant, is used to project the ground state from an initial variational wave function,

$$|\Psi_0\rangle \equiv \lim_{\beta\to\infty} e^{-\beta(H-e_N)} |\Phi\rangle . \tag{21}$$

For large β, the excited-state components of the variational wave function $|\Phi\rangle$ decay exponentially, leaving only the unnormalized ground-state wave function $|\Psi_0\rangle$ in the $\beta \to \infty$ limit. Consequently, the ground-state energy is given by

$$E_0 = \lim_{\beta\to\infty} E(\beta) , \qquad E(\beta) = \frac{\langle \Phi | H e^{-\beta(H-e_N)} | \Phi \rangle}{\langle \Phi | e^{-\beta(H-e_N)} | \Phi \rangle} . \tag{22}$$

This method is closely related to the Green's function Monte Carlo method[5] in which the Green's function $(E-H)^{-1}$ is used instead of $e^{-\beta(H-E)}$ as a filter to project the exact ground-state wave function.

The Monte Carlo method for evaluating Eq. (22) involves three steps. First, an ensemble of sets of coordinates distributed according to the trial function $|\Phi\rangle$ are generated using the Metropolis algorithm.[6] Next the distribution of coordinates is refined stochastically by evaluating the effect of the operator $e^{-\beta(H-e_N)}$. This is done by breaking the interval β into N small steps of size $\Delta\tau = \beta/N$ and evaluating

$$\langle x_1 \cdots x_N | e^{-\Delta\tau[V-\epsilon_N]} e^{-\Delta\tau T} | y_1 \cdots y_N \rangle$$

for each step. The action of the kinetic energy is given by Gaussian diffusion of the coordinates from their initial values $(y_1 \cdots y_N)$ to new values $(x_1 \cdots x_N)$ and the action of the potential term is given by replicating or deleting coordinate sets according to the value of $\overline{e}^{\Delta\tau(V(x_1 \cdots x_N)-e_N)}$. The normalization energy e_N can be adjusted to keep the population approximately constant, thereby providing an independent estimate of the energy. The outcome of this stochastic evolution is sets of coordinates distributed according to the exact ground-state wave function $|\Psi_0\rangle$ rather than the trial function $|\Phi\rangle$. Finally, the energy is then evaluated by letting H act to the left in Eq. (22):

$$E_0 \approx \frac{\sum_i \langle x_i | H | \Phi \rangle}{\sum_i \langle x_i | \Phi \rangle} , \tag{23}$$

where the x_i are distributed according to $|\Psi_0\rangle$. Full details of this procedure are in Ref. 4.

The Monte Carlo results contain errors from three principal sources: statistics, use of a finite time $\Delta\tau$, and finite-L effects. Statistical errors are shown with the Monte Carlo results. Typically, these can be reduced to 1% for our problem by using 10^4–10^5 independent samples of Ψ_0 which require less than an hour of VAX 11-780 time. Reasonably high accuracy is required since the full energy E_0 is being calculated directly, while the quantity of physical interest is the energy difference relative to a system of isolated clusters. The errors due to the finite size of $\Delta\tau$ are also small for the value $\Delta\tau=0.1$ used for our calculations. This accuracy is confirmed both by the analytic result for a single harmonic oscillator that the error in the size parameter and ground-state energy are proportional to $(1+\frac{1}{8}\epsilon^2\omega^2)\sim 1+0.0006$ and $(1-\frac{1}{24}\epsilon^2\omega^2)\sim 1-0.0002$, respectively, and by the fact that many-body calculations for $\Delta\tau=0.1$, 0.05, and 0.025 agreed within statistics for a test case. The total interval $\beta=N\Delta\tau$ for N of the order of several hundred was sufficiently large to cause negligible error. The errors due to the finite length for a box length L large compared to the quark correlation length $\lambda^{-1/2}$ [see Eq. (25)] may be qualitatively understood in terms of the N dependence $(1-1/N^2)$ of a noninteracting Fermi gas, and Monte Carlo calculations with the full interaction displayed roughly the same dependence. In the limit of high density, in which $\lambda^{-1/2}$ approaches infinity and necessarily exceeds L, although the evaluation of the potential energy becomes inaccurate, its contribution to the total energy becomes negligible.

We have used two different trial functions $|\Phi\rangle$ in the Monte Carlo procedure. The first is a free-Fermi-gas wave function. For convenience, we choose antiperiodic boundary conditions for an even number N of quarks in a box of length L. This allows the Slater determinant to be rewritten as

$$|\Phi\rangle_{\rm FG}=\prod_{i<j}\sin\frac{\pi}{L}(x_i-x_j)\ . \tag{24}$$

This wave function becomes exact at high density but does not have the clustering needed to provide a good starting point at low density. The second trial function is written as

$$|\Phi\rangle_\lambda \equiv e^{-\lambda V}|\Phi\rangle_{\rm FG}\ , \tag{25}$$

where V is the full many-body potential and λ is a variational parameter. At each value of the density we minimize the expectation value of the Hamiltonian in the trial function $|\Phi\rangle_\lambda$ by using the Metropolis method to evaluate E and $dE/d\lambda$ for a range of values of λ. At high density, $\lambda_{\min}\to 0$ so that the Fermi-gas result is recovered. At very low density, $\lambda\to 1/\sqrt{2}$ reproduces the exact Gaussian wave function of isolated clusters, Eq. (3). Thus, Eq. (25) will become the exact wave function in both the high- and low-density limits, with the variational parameter $\lambda^{-1/2}$ playing the role of a clustering length for quarks in a hadron.

A first orientation is provided by Fig. 1(a) which shows the total energy per quark of hadronic matter compared with the Fermi-gas, Hartree-Fock, and isolated-hadron zero-point energies. In Fig. 1(b), we show the excitation energy per quark for $N=16$ quarks corresponding to $A=8$ hadrons as a function of density. The energy rises smoothly with density from the energy of isolated clusters, $\epsilon_0/2$, to the free Fermi-gas energy. Hadron matter is not bound in our model. As will be seen in the next section, the phase shift for hadron-hadron scattering is characteristic of a repulsive hadron-hadron interaction, as

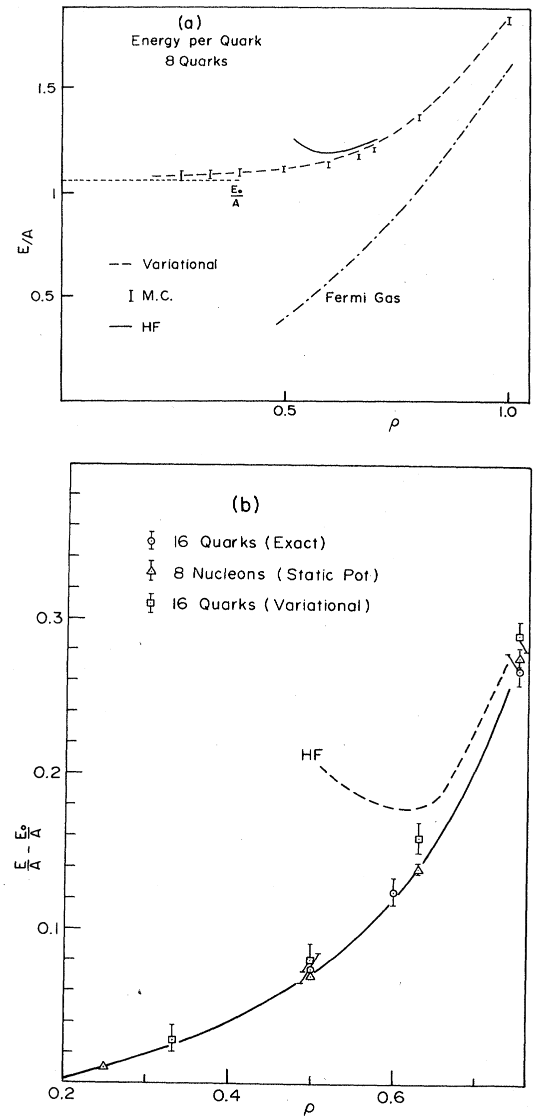

FIG. 1. (a) Energy per quark vs density for eight quarks in a periodic box. The low-density limit $E_0/A=3/2\sqrt{2}$ is Eq. (4) divided by $A=2$. The error bars are the exact path-integral Monte Carlo result, Eq. (23). Also shown is the energy for the variational wave function Eq. (25) (dashed), Hartree-Fock, Eq. (24) (solid), and free Fermi gas, $E/A=\pi^2/6\rho^2(1-1/A^2)$ (dot-dashed). (b) Excitation energy per quark ($\equiv$total energy minus the zero-point energy, $E_0/A=3/2\sqrt{2}$) vs density for either 16 quarks or eight clusters. The solid curve guides the eye through the cluster results calculated using the hadron-hadron potential in Eq. (45). The circles are path-integral Monte Carlo calculations using Eq. (23) while the squares use the variational wave function Eq. (25).

might be associated with the strong central repulsion in the NN potential. An intermediate-range (nonconfining) attractive force could be added to our model. Such long-range behavior may or may not be important at densities relevant for the nuclear to quark-matter transition. In any case, the main features of the two-nucleon correlation function in nuclear matter arise from the repulsive core and from exchange symmetry. We choose therefore to restrict our simple model such that only the confining force and quark exchange dictate the dynamics.

Also shown in Fig. 1 are the results with the variational wave functions, Eqs. (24) and (25). The former is the Hartree-Fock energy, displaying the features discussed qualitatively in the last section. The result with the more realistic trial wave function, Eq. (25), is seen to agree extremely well with the full Monte Carlo results at all densities. The clustering parameter λ has been chosen to minimize the energy at each density; the resulting $\lambda_{\min}(\rho)$ is shown in Fig. 2, displaying the expected drop from $1/\sqrt{2}$ to zero. Clearly, $\lambda_{\min}$ drops fairly rapidly implying that a free quark gas provides a very good description of the system for densities $\rho \gtrsim 1.5\bar{\rho}$. It is interesting that the energy shows no dramatic behavior despite the disappearance of clustering. Note also that, while the clustering parameter $\lambda^{-1/2} \approx 1.4\lambda_0^{-1/2}$ at $\rho = \bar{\rho}$ (i.e., a 40% increase in the quark correlation length at a hadron matter density equal to the hadron internal average quark density), the energy per quark is only $\sim 5\%$ greater than the isolated-hadron zero-point energy.

The Monte Carlo results shown in Fig. 1 are for $N = 16$. We verified that the finite-size errors are within the statistical error bars shown in Fig. 1 by explicitly calculating with $N = 32$ and 64.

We have also evaluated the quark-quark correlation function and the single-quark momentum distribution for our model, in both cases using the fact that the variational wave function $|\Phi\rangle_{\lambda\min} \equiv |\Phi_0\rangle$ provides a very good description of the ground state. The quark-quark correlation function or two-body density is defined by

$$\begin{aligned}\tilde{\rho}_2(x_1 - x_2) &= \frac{1}{N(N-1)} \sum_{i \neq j} \delta(x_i - x_1)\delta(x_j - x_2) \\ &= \frac{1}{(N-1)L} \sum_{i \neq j} \delta(|x_1 - x_2| - |x_i - x_j|) \,. \end{aligned} \tag{26}$$

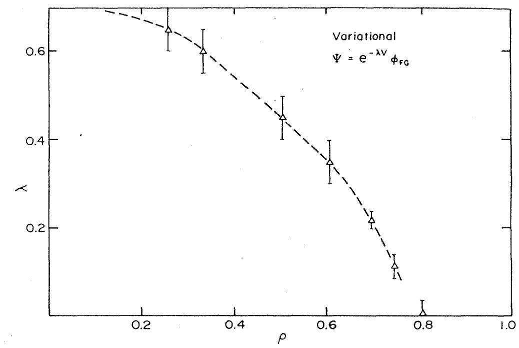

FIG. 2. Variational clustering parameter λ of Eq. (25) which minimizes the energy at each density for a 16-quark system. At low density $\lambda \to 2^{-1/2}$ while at high density λ is expected to go to zero as ρ^{-2}.

The second line in Eq. (26) follows for an infinite system. We evaluated $\rho_2(x)$ only to first order in the difference between $|\Psi_0\rangle$ and $|\Phi\rangle$,

$$\langle \rho_2(x) \rangle \approx 2\frac{\langle \Phi | \tilde{\rho}_2 | \Psi \rangle}{\langle \Phi | \Psi \rangle} - \frac{\langle \Phi | \tilde{\rho}_2 | \Phi \rangle}{\langle \Phi | \Phi \rangle} \,. \tag{27}$$

Since $|\Phi\rangle$ incorporates the correct high- and low-density limits, we expect this approximation should be good and indeed we find that $\langle \rho_2(x) \rangle$ is close to that calculated with the variational wave function itself (i.e., $\lesssim 20\%$ difference at $\rho = 0.27$ and much less at $\rho = 0.5$). Statistical errors (not shown) are less than 5%, and finite-size errors are small for $x \ll L$. The results are shown in Fig. 3, together with those for a free hadron and for the quark Fermi gas, at two values of the density. At a low density, $\rho = 0.27$, the quark-quark correlation is dominated by the peak at small x, which represents the second quark bound in a hadron, with depletion at larger distances arising from the hadron-hadron repulsion. However, at moderate density $\rho = \bar{\rho} = 0.5$, the correlation function displays little of the free hadron clustering and instead resembles closely that for a free quark gas. Note, however, as seen in Fig. 1(a) that the energy per quark at this density is much greater than the Fermi-gas energy, but only slightly greater than the energy for free hadrons.

The quark momentum distribution $n(p)$ was calculated for the variational wave function $|\Phi\rangle$,

$$n(p) = \int_{-L/2}^{L/2} \frac{dy}{2\pi} e^{ipy} \int_{-L/2}^{L/2} dx_1 \cdots dx_N \Phi^*_{\lambda\min}(x_1, x_2, \ldots, x_N) \Phi_{\lambda\min}(x_1 + y, x_2, \ldots, x_N) \,. \tag{28}$$

We expect the variational wave function Φ_λ to yield an adequate approximation for $n(p)$ both because of its correct $\rho \to 0$ and $\rho \to \infty$ limits and because of its accuracy for the correlation function. For a free hadron, the ground-state momentum distribution is

$$\begin{aligned} n(p) &= \frac{2^{7/4}}{\sqrt{\pi}} p^2 e^{-\sqrt{2}p^2} \,, \\ \int_{-\infty}^{+\infty} dp\, n(p) &= 1 \,. \end{aligned} \tag{29}$$

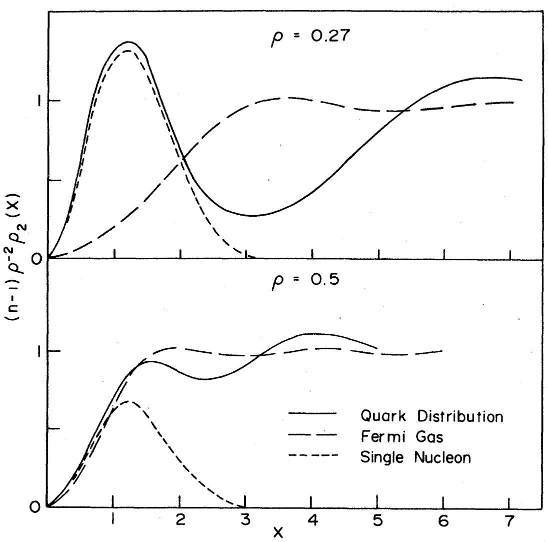

FIG. 3. Quark-quark correlation function ρ_2 at low and moderate densities for an eight-quark system (normalized to one at large distances). Also shown are results for a free Fermi gas and an isolated hadron with the wave function of Eq. (3).

Note that, because of the antiperiodic boundary conditions, $n(p)$ is calculated only at odd multiples of π/L. The results are shown in Fig. 4. Statistical errors in the evaluation of Eq. (28) are about 2%. Evaluation for both $N=8$ and $N=16$ shows that finite-size effects are also very small. As with the correlation function, the quark momentum distribution displays a rapid transition as a function of density from the free cluster result ($\rho=0$) to the Fermi-gas result (our $\rho=0.8$ result cannot be distinguished from the Fermi-gas momentum distribution).

Our results have an interesting connection with the EMC effect. In nonrelativistic quantum mechanics, the response function measures the momentum distribution, $n(p)$. In relativistic field theory, the structure function measures the momentum distribution in the infinite momentum frame, which corresponds to the distribution of $p^+=p_0+p_{\|}$ in the rest frame (where $p_{\|}$ denotes the component of $\mathbf{p}$ parallel to $\mathbf{q}$). In our nonrelativistic model, we find that the momentum distribution of quarks within a hadron is modified when that hadron is immersed in hadronic matter in the same way as the distribution of p^+ in a physical nucleon is observed to be modified when that nucleon is immersed in a nucleus.

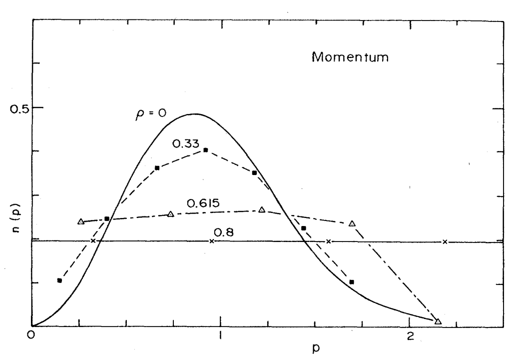

FIG. 4. Single-quark momentum distribution for an eight-quark system. This is a variational calculation using Eqs. (25) and (28). The zero-density curve is Eq. (29).

Qualitatively, the distribution of p in a nonrelativistic system may be compared with the distribution of p^+ in a relativistic system. For spinless particles, the structure function measured in deep-inelastic scattering may be written

$$f(x)=\int dp\, n(p)\delta(p^+-xM)\ , \tag{30}$$

where $x=Q^2/2M\nu$ is the Bjorken variable and M is the hadron mass. The scaling variable x thus specifies the fraction of p^+ borne by the constituent with which the electron interacts. If one assumes that $p \ll m$, where m is the quark mass, $p_0 \sim m$ so that $p^+-xM \approx p+m-xM$ and

$$f(x)\approx n\left[M\left|x-\frac{m}{M}\right|\right]\ . \tag{31}$$

Hence, the same variable x which describes the fraction of p^+ borne by a single constituent in a relativistic theory is related to the distribution of the ordinary momentum p in our nonrelativistic theory. The same result may also be derived straightforwardly using the impulse approximation.

The quark momentum distribution is enhanced at low and high momenta and depleted (as required by normalization) at "average" momenta when compared to the free cluster $n(p)$. Consequently, the structure function in hadron matter is enhanced around $x_0 \equiv M_q/M_n = 1/n_q$ where n_q is the number of quarks in a hadron. The structure function is enhanced at very large x because of "Fermi motion." In between, the structure function must be suppressed in hadron matter because of the normalization constraint. The symmetry of $f(x)$ about x_0 is a result of the nonrelativistic quark model; clearly the physics at small x in the real world is much more complicated than could be discussed within a valence quark picture. Nevertheless, the qualitative features of the EMC effect for $x \gtrsim 0.3$ emerge very simply from the general features of confinement and exchange symmetry. The same behavior was found by Lenz *et al.*[1] in their study of the $q^2\bar{q}^2$ system.

We have generated an explicitly antisymmetric many-quark wave function. One would like to relate this to the many-hadron wave functions commonly discussed. There have been a number of package models put forth to describe deep-inelastic lepton-scattering data. Here one assumes quarks are bound into individual hadrons (nucleons, pions, Δ's, six-quark bags, etc.) and then totally neglects the antisymmetry among quarks assigned to different packages. We emphasize that there is no unique way to project from our exact quark wave function an excited hadron (Δ) or multibaryon ("six-quark bag") probability. Such probabilities must necessarily be model dependent.

Instead we have calculated the quark correlation function. Here one sees the peak at small separation, which represents the structure of a single hadron, change with density. It may be useful in the future to ask questions about quarks in nuclei in terms of the correlation function. Finally, Ref. 7 has analyzed deep-inelastic scattering data in terms of the change of a simple parameter specifying the confinement scale in the ground-state wave function. In our model the "length scale for quark confinement" is given by the variational clustering parameter $\lambda^{-1/2}$ of Eq. (25). To our knowledge, this is the first dynamical calculation of how this scale depends on density.

We find a suppression of the peak height of $n(p)$ roughly linear with density for densities $\lesssim\bar{\rho}$. This linear dependence is consistent with the observed A dependence of the EMC effect. The fact that the same dependence arises in simple overlap models suggests that this A dependence is not very discriminating.

Before going on to a discussion of an effective hadron model for our system, we note that results have also been obtained for spin-$\frac{1}{2}$ and spin-$\frac{3}{2}$ quarks with spin-independent forces. The internal degrees of freedom result in the ground states of isolated clusters now being symmetric under space exchange. In the spin-$\frac{3}{2}$ case, four quarks actually bind. However, there are no qualitative differences in either case when compared with the model results already discussed. The many-hadron system relaxes towards a free quark gas as the density increases with no unusual structure produced in the ground-state energy.

IV. APPROXIMATION BY HADRONIC DEGREES OF FREEDOM

We have seen that the quark-matter energy is very smooth even as the clustering parameter changes rapidly. It remains to understand the same system in terms of an effective "nuclear physics" Hamiltonian containing only hadronic degrees of freedom. That is, we introduce an effective hadron-hadron potential chosen to reproduce the scattering phase shifts and then evaluate the ground-state energy of hadron matter.

The Hamiltonian for the four-quark system is given by Eqs. (11) and (12). Following Ref. 1, we introduce new coordinates,

$$x=\frac{x_1+x_2}{2}-\frac{x_3+x_4}{2}\,,\quad y=\frac{x_1+x_3}{2}-\frac{x_2+x_4}{2}\,,\quad z=\frac{x_1+x_4}{2}-\frac{x_2+x_3}{2}\,. \tag{32}$$

These coordinates play the role of channel coordinates for the relative motion of two hadrons given any of the three possible quark pairings. Dropping the center-of-mass motion the Hamiltonian becomes

$$H=-\tfrac{1}{2}(\partial_x{}^2+\partial_y{}^2+\partial_z{}^2)+V\,,$$

where

$$V=x^2+y^2+z^2-x_>^2$$

and (33)

$$x_>^2=\max(x^2,y^2,z^2)\,.$$

We see that only one of the coordinates can become large, corresponding to the scattering channel coordinate, while the other two coordinates describe the two confined free-hadron wave functions. We have calculated the low-energy scattering amplitude for ground-state clusters in one dimension in two ways, a resonating-group expansion and a path-integral Monte Carlo method. The resonating-group approach starts from an ansatz for the full wave function,

$$\Psi_{(x,y,z)}\equiv\mathscr{A}\sum_i\phi_i(x,y)f_i(z)\,, \tag{34}$$

where $\mathscr{A}$ is the antisymmetrization operator, $f_i(z)$ is a channel scattering wave function, and $\phi_i(x,y)$ is the product of cluster wave functions for the two hadrons. The index i labels the internal states for both hadrons. An effective coupled-channel theory is defined by projection on the Schrödinger equation with hadron bound-state wave functions,

$$\int dx\,dy\,\phi_j^*(x,y)(H-E)\Psi(x,y,z)=0\,. \tag{35}$$

The effective channel potentials defined by Eq. (35) are nonlocal. We truncate the equations by including only the state in which both clusters are in their ground states, $\phi_0(x,y)=\psi_0(x-y)\psi_0(x+y)$. This was seen to give very accurate results for low-energy scattering in the work of Lenz *et al.*[1] and will be checked below for our specific model. With this truncation, we have the Schrödinger equation,

$$[-\tfrac{1}{2}\partial_x{}^2+V_{\rm L}(z)-\epsilon]f_0(z)=\int_{-\infty}^{+\infty}dy\,V_{\rm NL}(z,y;\epsilon)f_0(y)\,, \tag{36}$$

$$V_{\rm L}(z)=\int_{-\infty}^{\infty}dx\,dy(z^2-x_>^2)\phi_0^2(x,y)\,, \tag{37}$$

$$V_{\rm NL}(z,y:\epsilon)=2\int_{-\infty}^{\infty}dx(-\tfrac{1}{2}\partial_z{}^2+z^2-x_>^2-\epsilon)\phi_0^*(x,y)\phi_0(x,z)\,, \tag{38}$$

where ϵ is the scattering energy,

$$\epsilon=E-6/\sqrt{2} \tag{39}$$

and

$$\phi_0(x,y)=\frac{2^{5/4}}{\sqrt{\pi}}(x^2-y^2)e^{-(x^2+y^2)/\sqrt{2}}\,. \tag{40}$$

The local potential $V_{\rm L}$ results from the direct term in Eq. (35), i.e., projection of $\phi_0^*(x,y)$ onto $\phi_0(x,y)$ in Ψ. It is given by

$$V_{\rm L}(z)=\frac{1}{\sqrt{2}}(z^2-\tfrac{3}{2})[1-{\rm erf}^2(z)]-\frac{\sqrt{2}}{\sqrt{\pi}}(z^3+3z)e^{-z^2}{\rm erf}(z)-\frac{\sqrt{2}}{\pi}(z^2+2)e^{-2z^2} \tag{41}$$

This is strongly attractive at the origin: $V_L(0)=-1.96$ is almost half the zero-point energy of the isolated clusters. It drops to half its value at $z\approx 1$ and falls off as a Gaussian at large separation. The nonlocal potential arises from exchange terms in Eqs. (35) and (34). The nonlocality range must be the range of the ground-state cluster wave function. Since this term accounts for the Pauli exclusion principle, it is clearly repulsive.

The nonlocal Schrödinger equation was solved numerically in a momentum representation. Although our two quark clusters are bosons (with reduced mass one for the two hadrons), note that by virtue of the repulsive "core," the scattering would not be drastically different for fermions. The scattering amplitude satisfies the Lippmann-Schwinger equation,

$$F(q,k)=\mathcal{U}(q,k)-2\mathrm{P}\int_0^\infty \frac{dt}{2\pi}\frac{\mathcal{U}(q,t)}{t^2-k^2}F(t,k)\ , \tag{42}$$

where P denotes the principal value and the phase shift is given by

$$\tan\delta(k)=-F(k,k)/2k\ . \tag{43}$$

The Fourier transformed potentials in Eq. (42) are defined by

$$\begin{aligned}
\mathcal{U}(q,k)&=V(q,k)+V(q,-k)\ ,\\
V(q,k)&=V_L(q-k)+V_{NL}(q,k)\ ,\\
V_L(q-k)&=\int_{-\infty}^{\infty}dx\cos[(q-k)x]V_L(x)\ ,\\
V_{NL}(q,k)&=-\int_{-\infty}^{\infty}dx\cos(qx)\int_{-\infty}^{\infty}dy\cos(ky)V_{NL}(x,y;\epsilon)\ .
\end{aligned} \tag{44}$$

The singularity in Eq. (42) is removed by the usual Kowalski-Noyes method[8] and the resulting finite equation is solved by matrix inversion. The phase shift is shown in Fig. 5. We note that for the bosons in one dimension, the phase shift goes to $\pi/2$ at $k=0$ for any repulsive potential because by symmetry the unscattered wave function is $\cos(kr)$, whereas the exact zero-energy wave function joins onto an asymptotic wave function having a node near the origin. The phase shift is characteristic of a repulsive potential and yields a scattering length of 1.7. This is close to the size of a free cluster.

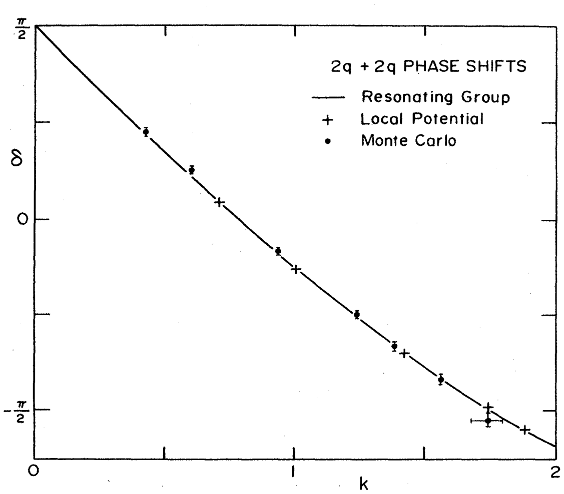

FIG. 5. The phase shift for the scattering of two hadrons. The solid curve is a resonating-group calculation, Eqs. (42)—(44) while the error bars are from a path-integral Monte Carlo calculation. This phase shift can be fit with the local hadron-hadron potential Eq. (45) (crosses).

To check the resonating-group truncation, the phase shift can also be calculated exactly with the Monte Carlo method.[9] The basic idea is to impose the boundary condition that the relative wave function vanish at a specified cluster separation. This defines the energy at which the scattering wave function has the specified node, and as long as the interactions are negligible, the phase shift at that energy is thus defined. These results are shown on Fig. 5 as error bars (representing the statistical accuracy of the Monte Carlo calculation). The two methods are in excellent agreement for energies below inelastic threshold ($\epsilon_{IN}=2\sqrt{2}$).

The phase shift can be fit easily with a repulsive phenomenological local potential $\tilde{V}(x)$. Since the cluster wave functions are Gaussian, we assumed such a form for $\tilde{V}$. The parametrization

$$\tilde{V}(x)=\frac{25}{\sqrt{\pi}}\exp\left[-\frac{1}{\sqrt{2}}x^2\right] \tag{45}$$

produced an excellent fit to the phase shifts, as shown in Fig. 5. We will use this potential as our nuclear interaction.

For comparison with our quark-matter calculations, we performed Monte Carlo calculations for a Bose system of mass-two hadrons interacting through the phenomenological hadron-hadron interaction $\tilde{V}$. Because of finite-volume effects in the energy, small differences between the nucleon and quark-energy calculations are sensitive to finite-volume effects even for large systems. These effects have been controlled in this work by directly comparing A-nucleon systems with $(N=2A)$-quark systems. No many-hadron forces are included, although they must in principle be there for our quark model. The energy per quark (i.e., half the energy per hadron) is compared in Fig. 1(b) with the full quark-matter energy minus the zero-point energy. Surprisingly, the agreement is very good even for densities well beyond $\bar{\rho}$, i.e., at densities where the quark-matter wave function is essentially that of a free quark gas.

V. CONCLUSIONS

We have studied a simple model of the ground state of hadron matter with many of the essential features for examining the nuclear-to-quark-matter transition and the validity of effective models based upon hadron degrees of freedom. Our hadrons are composite, with quarks confined by static potentials. The potential, reminiscent of flux-tube models, guarantees cluster separability and eliminates strong van der Waals forces as would be present in conventional Hamiltonians based upon confining forces. The many-body nature of the interaction leads to a free-quark Fermi gas as the large-density limit of the theory, thereby allowing us to study the "transition" from the clustered low-density hadron matter.

The model has no parameters introduced to enforce any special dynamics. Indeed, there is only one overall scale parameter, the confinement scale, with all dynamics generated solely by compositeness and exchange symmetry. For technical simplicity, we considered in this paper only two-quark hadrons in one dimension. This allowed some insight into the transition dynamics, particularly through the Hartree-Fock results.

The energy of the system showed no remarkable features as the density increased from the clustered to the free-quark configuration. Further, the effective "nuclear theory," defined through a phenomenological local hadron-hadron potential fit to the low-energy scattering amplitude, provided remarkably good agreement over this same range of densities. We repeat that the description in terms of hadron degrees of freedom was accurate even when the quark-quark correlation function showed no evidence of clustering. Moreover, deconfinement of clusters occurs at rather low densities, $\rho \gtrsim \bar{\rho}$ where $\bar{\rho}$ is the average quark density inside a free hadron.

We found that a variational wave function with a variational parameter characterizing hadron size in the medium produced excellent agreement with the full results. With this as a measure, the quark correlation scale or effective hadron size was increased over that of a free hadron by $\sim 40\%$ at $\rho = \bar{\rho}$. Such ideas about increased confinement scales in nuclei have been discussed in many contexts recently. As far as we know, ours is the first *dynamically* consistent model within which the quark correlation function is calculated.

Despite the success of the hadron phenomenology in reproducing the quark-matter energy, we did find significant changes in the quark momentum distribution. Within a nonrelativistic model, this is measurable in the quark structure function measured by deep-inelastic lepton scattering. We found behavior qualitatively similar to that observed in the EMC effect. The quark low-momentum components are enhanced.

The variational parameter, λ, is observed to decrease indicating a "partial deconfinement" of quarks with increasing density. Again, this change is accompanied by a remarkably small increase in the energy per quark.

Work is underway to extend the results presented here to three space dimensions. These calculations are using the variational wave function of Eq. (25) and an efficient algorithm[3] to determine the optimal quark pairing. However, it is expected that the quark antisymmetrization is the most important feature of this model so the changes in the nucleon substructure will persist in three dimensions.

In summary, we have consistently studied the ground-state dynamics of a system of hadrons composed of confined quarks, demonstrating the compatibility of a nuclear description of bulk properties (energy per quark) with signatures of quark substructure in selected observables (quark momentum distribution). This compatibility is particularly surprising in its persistence into density regimes where the hadrons are strongly modified in the medium. This may be especially significant since the transition to the free-quark gas is well advanced at rather modest densities, $\rho \sim \bar{\rho}$. Clearly, more quantitative comparisons with the properties of nuclear matter require elaboration of the model. A more realistic description of the nucleon is needed, including both three quarks per hadron and quark internal degrees of freedom; the calculations must be extended to three dimensions; and nonconfining interactions must also be included to generate the strong channel dependence of the nuclear force. Ultimately, the nonrelativistic approach may be too limiting for quantitatively correlating hadron properties with nuclear observables. Nevertheless, our results indicate the importance of consistently treating the quark-confining dynamics and exchange symmetry and offer some hope for a comparatively simple characterization of quark effects in nuclei.

ACKNOWLEDGMENT

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