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Gap equation for the chiral-symmetry-restoration transition

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The chiral-symmetry-restoration transition in quantum chromodynamics is studied in analogy to the theory of superconductivity. Particular attention is paid to the dynamical quark mass and to the quark condensate as a function of temperature. A critical temperature T_c in the range of 170—250 MeV is obtained. The relevance to heavy-ion collisions is discussed.

I. INTRODUCTION

Now that the zero-temperature order parameter of the spontaneously broken chiral symmetry of quantum chromodynamics (QCD) has been identified and partially understood,[1–5] it is time to investigate the temperature dependence of these parameters. Indeed, this has been done[6] for the gluonic sector of QCD on a lattice and more recently this has also been attempted on a lattice with light quarks.[7] Such a temperature dependence may be probed in the near future in heavy-ion collisions at very high energies. In fact, critical temperatures and chemical potentials may have already been detected in high-energy cosmic-ray events.[8]

In this paper we study two nonperturbative order parameters in QCD which in part play the role of the energy gap Δ in the nonrelativistic BCS theory of superconductivity:[9,10] The dynamically generated quark mass m_{dyn} and the quark condensate $\langle \bar{q}q \rangle_0$. Some time ago the temperature dependence of the chirally symmetric meson sector of the σ-model Lagrangian was investigated,[11] but only recently was the temperature dependence of the fermion mass in gauge theories considered.[12,13]

We shall attempt to work out the critical temperature T_c for the restoration of chiral symmetry at which the nonperturbative quark mass and quark condensate "melt" or vanish. It may be that T_c for m_{dyn} and for $\langle \bar{q}q \rangle_0$ are identical and also coincide with the temperature T_D of color deconfinement. But in the absence of a complete solution of the nonperturbative theory of QCD, we will consider each order parameter separately and study the consequences based on our understanding of the temperature-dependent BCS theory of superconductivity and of the present approaches to zero-temperature QCD. Based on a variety of calculations, we find that T_c is in the range 180 to 250 MeV.

We begin in Sec. II by reviewing the real-time formalism of temperature-dependent field theory proposed by Dolan and Jackiw,[11] in particular, the manner in which they calculate the four-point self-energy loop for bosons. For spinless fermions, the fermion loop recovers the homogeneous gap equation which is the precise relativistic generalization of the BCS gap equation, while the boson loop leads to an inhomogeneous equation whose solution gives $T_c = 2f_\pi \approx 180$ MeV in the σ model.

In Sec. III we show that the "off-diagonal" left-right chiral-symmetry-breaking quark mass acquires a temperature dependence in the BCS-gap-equation fashion. The analog of the Debye momentum cutoff turns out to be $k_D \approx 1260$ MeV and the corresponding temperature T_c where $m_{\text{dyn}}(T_c)=0$ again appears to be at $T_c \approx 180$ MeV for a reasonable range of the QCD coupling α_s. Then, in Sec. IV, we study the melting of the quark condensate $\langle \bar{q}q \rangle_0$, calculating the temperature dependence of the fermion loop. Invoking the accepted energy scale for $\langle \bar{q}q \rangle_0$, we obtain $T_c \approx 250$ MeV at zero chemical potential. Generalizing the quark-condensate equation to account for nonzero chemical potential, we find $T_c \approx 220$ MeV for a typical value of $\mu_c \approx 200$ MeV.

Next, in Sec. V we review the phenomenology associated with heavy-ion collisions and the possible phase transition to a quark-gluon plasma.[8] This suggests typical values for T_c around 180—190 MeV for massless quarks and slightly higher ones for massive quarks ($m=315$ MeV): $T_c \sim 220$ MeV. In either case these estimates are compatible with our findings in Secs. II—IV. However, the naive bag-model estimate appears to underestimate both T_c and μ_c by at least 50 MeV.

Finally, in Sec. VI we survey all of the nonperturbative energy scales of QCD and set up a chain loop of relations such that $\Lambda_{\text{QCD}} \to m_{\text{dyn}} \to f_\pi$, $\langle \bar{q}q \rangle_0 \to T_c \to$ string tension $\to \Lambda_{\text{QCD}}$. In Sec. VII we summarize our findings of how the chiral-symmetry-breaking order parameters of QCD, m_{dyn} and $\langle \bar{q}q \rangle_0$, acquire temperature dependence and why chiral symmetry is restored when T is greater than $T_c \sim 200$ MeV.

II. TEMPERATURE-DEPENDENT FIELD THEORY: GAP EQUATIONS

Calculations in temperature-dependent field theory are most conveniently done for our purpose in the real-time formalism of Dolan and Jackiw.[11] Fermion propagators in this formalism are given by

$$S_F(p)=\frac{i(\not p+m)}{p^2-m^2+i\epsilon}-\frac{2\pi\delta(p^2-m^2)}{e^{|p_0/T|}+1}\ , \tag{1}$$

with T being the temperature, while boson propagators are given by

$$D_F(p)=\frac{i}{p^2-m^2+i\epsilon}+\frac{2\pi\delta(p^2-m^2)}{e^{|p_0/T|}-1}\ . \tag{2}$$

The advantages of this formalism are that the temperature-dependent part is readily isolated and calculations of Feynman diagrams are straightforward.

The showcase for the calculation of a phase diagram in temperature-dependent field theory is the BCS theory of superconductivity. For our purpose we need a relativistic generalization of the BCS gap equation. Such generalizations have been considered by Bailin and Love.[14] We focus, first, on a simplified example to illustrate the workings of the method.

A. BCS-type gap equation

The example we consider is the four-fermion coupling

$$H=\lambda_F(\bar\psi\psi)^2$$

of spinless fermions. The Dyson equation for the proper self-energy is displayed in Fig. 1, and it leads to

$$m=2\lambda_F\int\frac{d^4p}{(2\pi)^4}m\left[\frac{i}{p^2-m^2+i\epsilon}-\frac{2\pi\delta(p^2-m^2)}{e^{|p_0/T|}+1}\right]\ . \tag{3}$$

Performing the contour integration over p_0 and canceling a factor m on both sides of the equation gives

$$1=2\lambda_F\int\frac{d^3p}{(2\pi)^3 2E}\left[1-\frac{2}{e^{E/T}+1}\right]\ . \tag{4}$$

The factor of 2 in the numerator of the second term on the right-hand side of Eq. (4) is due to both $p_0=+E$ and $p_0=-E$ $[E\equiv(\vec{p}^{\,2}+m^2)^{1/2}]$ contributing. This equation can further be written as

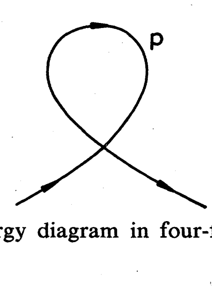

FIG. 1. Self-energy diagram in four-fermion or four-boson coupling theories.

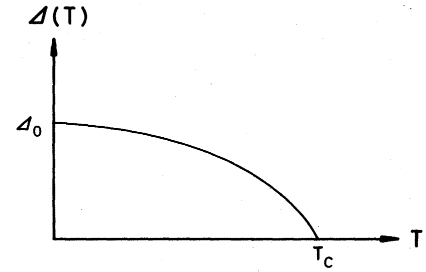

FIG. 2. Dependence of the gap energy Δ on temperature.

$$\begin{aligned}1&=2\lambda_F\int\frac{d^3p}{(2\pi)^3 2E}\tanh\frac{E}{2T}\\ &=\lambda_F\int\frac{d^3p}{(2\pi)^3(p^2+m^2)^{1/2}}\tanh\frac{(p^2+m^2)^{1/2}}{2T}\ .\end{aligned} \tag{5}$$

This result is strikingly similar to the BCS gap equation and can be used to find the temperature dependence of the mass. One solves (5) by cutting off the ultraviolet-divergent integral at the Debye energy ω_D fixed by the zero-temperature mass $m(T=0)=m_0$. For the nonrelativistic theory of superconductivity with $\omega_D \gg T_c$, one integrates an equation similar to (5) by parts to make an asymptotic expansion in the gap energy (Δ_0) relative to T_c, where $\Delta(T_c)=0$, to obtain the dimensionless ratio[9,10]

$$\frac{\Delta_0}{T_c}\simeq\pi e^{-\gamma_E}\simeq 1.76\ . \tag{6}$$

Furthermore, the tanh factor in (5) forces $m(T)$ [or correspondingly $\Delta(T)$] to decrease as T increases toward T_c in the typical quarter-moon pattern of Fig. 2.

B. Gap equation for a theory with spontaneous symmetry breaking

A different type of gap equation can arise in theories with spontaneous symmetry breaking. For this purpose we consider the σ model with a $\lambda_B\phi^4$ four-boson coupling. The boson mass $m(T)$ can be calculated in lowest-order perturbation theory according to the rule[11]

$$m^2(T)=12\lambda_B\int\frac{d^4k}{(2\pi)^4}\left[\frac{i}{k^2-m^2+i\epsilon}+\frac{2\pi\delta(k^2-m^2)}{e^{|k_0/T|}-1}\right] \tag{7a}$$

$$=m_0^{\,2}+\frac{6\lambda_B}{\pi^2}T^2J_-(a)\ , \tag{7b}$$

where $a=m/T$, m_0 is the mass at zero temperature, and the function $J_-(a)$ is the dimensionless integral defined by

$$J_\pm(a)=\int_0^\infty\frac{dx\,x^2}{(x^2+a^2)^{1/2}}\left(e^{(x^2+a^2)^{1/2}}\pm 1\right)^{-1}\ . \tag{8}$$

When higher-order loops are summed to all orders, (7b) becomes an inhomogeneous mass equation:

$$m^2(T)=m_0^{\,2}+\frac{6\lambda_B}{\pi^2}T^2J_-\left[\frac{m(T)}{T}\right]\ . \tag{9}$$

At the critical temperature we have $m(T_c)=0$ and then $J_-(0)=\pi^2/6$. Replacing $m_0{}^2$ by the σ-model mass $m_\sigma{}^2=-2m_0{}^2$, we obtain a real temperature:

$$T_c{}^2=\frac{m_\sigma{}^2}{2\lambda_B}>0\ . \tag{10a}$$

If we also invoke the σ-model value[15] $\lambda_B=m_\sigma{}^2/8f_\pi{}^2$ for $f_\pi\simeq 90$ MeV in the chiral limit, then (10a) has the scale

$$T_c=2f_\pi\simeq 180\text{ MeV}\ . \tag{10b}$$

If we further introduce the zero-temperature dynamically generated quark mass[3,5]

$$m_{\text{dyn}}\approx\frac{m_N}{3}\approx 315\text{ MeV} \tag{11a}$$

as the analog of the BCS energy-gap scale Δ_0, then the ratio of (10a) to (10b) is

$$\frac{m_{\text{dyn}}}{T_c}\approx\frac{315\text{ MeV}}{180\text{ MeV}}\approx 1.75\ , \tag{11b}$$

almost identical in magnitude to the BCS ratio (6). Nevertheless, it is important to note that the BCS fermion-loop result (6) follows from a "*homogeneous*" dimensionless equation similar to (5) while the boson-loop result (11b) follows from the *inhomogeneous* equation (7) combined with the nonperturbative mass scales of f_π in (10b) and m_{dyn} in (11a). Furthermore, the inhomogeneous equation does not admit the solution $m(T)=0$ which (presumably) holds for all $T>T_c$.

III. MELTING OF THE CONSTITUENT QUARK MASS

In order to derive a gap equation for the (chiral-symmetry breaking) quark mass we first express the Lagrangian in terms of the left- and right-handed chiral components of the quark field[16] q:

$$L=\bar q_L i\partial\!\!\!/ q_L+\bar q_R i\partial\!\!\!/ q_R-m_{\text{dyn}}(\bar q_L q_R+\bar q_R q_L)\ . \tag{12a}$$

This gives an inverse propagator $S^{-1}(p)$ for the quark which is a 2×2 matrix in the space of (q_L,q_R). Then inverting leads to the propagator

$$S(p)=\frac{i}{p^2-m^2+i\epsilon}\begin{bmatrix} p\!\!\!/ & m_{\text{dyn}}\\ m_{\text{dyn}} & p\!\!\!/\end{bmatrix}\ . \tag{12b}$$

Since the strongest $q\bar q$ scattering amplitude is in the $J=0$ color-singlet channel, the chiral-symmetry-breaking order parameter m_{dyn} is assumed to have these transformation properties. Thus, color would be confined at the chiral phase transition if it were not already confined at a higher temperature. The mass m_{dyn} is determined self-consistently from the (off-diagonal) component of the Dyson equation (Fig. 3)

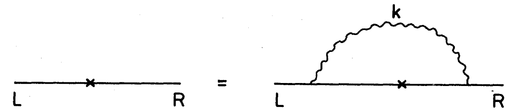

FIG. 3. Self-consistency condition on the quark mass.

$$-im_{\text{dyn}}=(-i)^2g_s{}^2C\int\frac{d^4k}{(2\pi)^4}\frac{\gamma^\mu D_{\mu\nu}(k)\gamma^\nu i m_{\text{dyn}}}{(p-k)^2-m_{\text{dyn}}{}^2+i\epsilon}\ . \tag{13}$$

Working in the Landau gauge, we have $\gamma^\mu D_{\mu\nu}\gamma^\nu=-3i/k^2$. Then with $C_2(R)=\frac{4}{3}$ for the color group SU(3), (13) generalizes to the temperature-dependent form

$$-m(T)=4g_s{}^2m(T)\int\frac{d^4k}{(2\pi)^4}\left[\frac{i}{k^2[(p-k)^2-m^2(T)]}-\frac{2\pi\delta((p-k)^2-m^2(T))}{k^2(e^{|(p_0-k_0)/T|}+1)}+\frac{2\pi\delta(k^2)}{[(p-k)^2-m^2(T)](e^{|k_0/T|}-1)}\right]\ . \tag{14}$$

Since this is a *homogeneous* gap equation, $m(T)=0$ is always a solution, and is the solution actually occurring when $T>T_c$. But $m(T)\neq 0$ divides out of (14) for $T<T_c$, resulting in a dimensionless gap equation similar to the right-hand side of (4).

To demonstrate that (14) in fact has the structure of the right-hand side of (4), we isolate the boson and fermion poles of (14), respectively, at $k_0=\pm|\vec k|$ and $k_0=m\pm E$ $[E=(|\vec k|^2+m^2)^{1/2}]$ at $\vec p=\vec 0$. Then closing the contours in the lower half k_0 plane, the residue of the double propagator integral in (14) is

$$\begin{aligned}\int\frac{d^3k}{(2\pi)^3}&\left[\frac{1}{2|\vec k|(-2)|\vec k|m}+\frac{1}{2E[(m-E)^2-|\vec k|^2]}\right]\\ &=-\int\frac{d^3k}{(2\pi)^3 4m|\vec k|^2}\left[1-\frac{|\vec k|^2}{E(m+E)}\right]\\ &=-\frac{1}{8\pi^2}\int\frac{dk}{E}\ ,\end{aligned} \tag{15}$$

so that the fermion integral upon change of variables $p-k\to k$ becomes at $\vec p=0$, $p_0=m$,

$$\int\frac{d^4k}{(2\pi)^3}\frac{\delta(k^2-m^2)}{(p-k)^2(e^{|k_0/T|}+1)}=-\frac{1}{8\pi^2}\int\frac{dk}{E}\frac{2}{e^{E/T}+1}\ , \tag{16}$$

while the boson integral vanishes. Subtracting (16) from (15) and substituting back into (14), we finally arrive at the homogeneous gap equation for QCD:

$$1=\frac{2\alpha_s}{\pi}\int_0^{k_D}\frac{dk}{[k^2+m^2(T)]^{1/2}}\tanh\frac{[k^2+m^2(T)]^{1/2}}{2T}\ . \tag{17}$$

As in the BCS case we determine k_D at $T=0$, where $m(0)=m_{\text{dyn}}\approx 315$ MeV:

$$1=\frac{2\alpha_s}{\pi}\int_0^{k_D}\frac{dk}{E}=\frac{2\alpha_s}{\pi}\text{arcsinh}\frac{k_D}{m_{\text{dyn}}}\ . \tag{18a}$$

If we take $\alpha_s(m_\sigma)\approx 0.75$, i.e., $\Lambda\approx 250$ MeV for three quark flavors, (18a) implies

$$k_D = m_{\rm dyn}\sinh\frac{\pi}{2\alpha_s}\approx 1260\ {\rm MeV}\ . \tag{18b}$$

Then, returning to (17) at $T=T_c$ with $m(T_c)=0$ or

$$1=\frac{2\alpha_s}{\pi}\int_0^{k_D/2T_c}\frac{dx}{x}\tanh x\ , \tag{18c}$$

we find the upper limit self-consistently to be

$$\frac{k_D}{2T_c}\approx 3.58\ \text{ or }\ T_c\approx 176\ {\rm MeV}\ . \tag{18d}$$

If instead we take α_s to be α_s (1 GeV)$\simeq 0.50$ with again $\Lambda\simeq 250$ MeV, then although k_D increases substantially, T_c remains almost unchanged. Even if we let $\alpha_s(k^2)$ run under the integral or work in the Feynman gauge, we still retain

$$T_c\approx 180\ {\rm MeV}\ ,$$

so that the dimensionless order-parameter ratio is still $m_{\rm dyn}/T_c\simeq 1.75$ according to (11). With hindsight, since $\omega_D\gg T_c$, the BCS asymptotic expansion (6) is now also valid for QCD with

$$m_{\rm dyn}/T_c\approx \pi e^{-\gamma_E}\approx 1.76$$

independent of the scale used.

Since we have determined the end points of $m(T)$ in the ordered phase, $m_{\rm dyn}$ at $T=0$ and T_c at $m_{\rm dyn}(T_c)=0$, we anticipate a BCS-type quarter-moon-shape curve similar to Fig. 2. The physical significance of the third energy scale, the (large) Debye cutoff $k_D=\omega_D\approx 1.26$ GeV, is that the confining QCD gluon force suggests a "linear string" binding the quarks with Bloch-wave minimum normal-mode wavelength given by $\lambda_D=2\pi/k_D\approx 1$ fm. This (confinement) scale is then related to the interquark distance $d=\lambda_D/2\approx 0.5$ fm, which is approximately what is expected for three quarks confined in a nucleon or quark-antiquark confined in a ρ meson.

IV. MELTING OF THE QUARK CONDENSATE

Besides the quark mass, the second fermion order parameter which measures the breakdown of chiral symmetry in QCD is the quark condensate $\langle\bar{q}q\rangle_0$. Unlike $m_{\rm dyn}$ however, $\langle\bar{q}q\rangle_0$ does not lead to a homogeneous BCS-type gap equation. Instead it parallels the *inhomogeneous* dimensional calculation of Sec. II for the boson self-energy loop, Eq. (9).

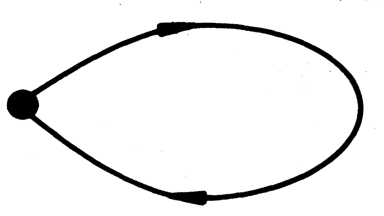

FIG. 4. Quark-condensate diagram.

More specifically, the Feynman rules for the temperature-dependent quark condensate $Q(T)=\langle\bar{q}q(T)\rangle_0$ of Fig. 4 give

$$Q(T)=-N_c\int\frac{d^4p}{(2\pi)^4}{\rm Tr}\left[\frac{i}{\not p-m_{\rm dyn}(p)}-\frac{(\not p+m_{\rm dyn})2\pi\delta(p^2-m^2)}{e^{|p_0/T|}+1}\right] \tag{19a}$$

$$=\langle\bar{q}q\rangle_{0,M}+\frac{6m_{\rm dyn}}{\pi^2}T^2J_+\left[\frac{m}{T}\right]. \tag{19b}$$

Note that the driving term $\langle\bar{q}q\rangle_{0,M}$ in (19) is evaluated at $T=0$, as in the other inhomogeneous integral equation (7), and also that the constant $m_{\rm dyn}\simeq 315$ MeV multiplying the second term in (19b) is analogous to the constant λ_B in (7). By definition $m(T_c)=Q(T_c)=0$, and since $J_+(0)=\pi^2/12$ from (8), the analog of the boson result (10a) is

$$T_c^{\,2}=-\frac{2\langle\bar{q}q\rangle_{0,M}}{m_{\rm dyn}}>0 \tag{20}$$

because $-\langle\bar{q}q\rangle_{0,M}>0$. The subscript M on $\langle\bar{q}q\rangle_{0,M}$ indicates the ultraviolet cutoff of the divergent fermion loop in Fig. 4. When the running quark mass of QCD,

$$m_{\rm dyn}(p^2)\sim p^{-2}(\ln p^2/\Lambda^2)^{d-1}$$

for $d=12(33-2N_F)^{-1}$, is employed in the zero-temperature term one obtains[1,3,5]

$$-\langle\bar{q}q\rangle_{0,M}=\frac{3\times 4i}{(2\pi)^4}\int^M\frac{d^4p\,m_{\rm dyn}(p^2)}{p^2-m_{\rm dyn}^{\,2}(p^2)}\approx\frac{3m_{\rm dyn}^{\,3}}{4\pi\alpha_s(M^2)} \tag{21a}$$

$$\approx(246\ {\rm MeV})^3 \tag{21b}$$

for $M\approx 1$ GeV, α_s (1 GeV^2)≈ 0.50, and $m_{\rm dyn}\approx 315$ MeV, in close agreement with the empirical estimate.[2] However, T_c is presumably determined by the minimum value of $\langle\bar{q}q\rangle_{0,M}$ which we suggest occurs at[5] $M=m_\sigma\approx 2m_{\rm dyn}$, where $\alpha_s(m_\sigma^{\,2})\approx 0.75$ and the QCD coupling freezes out. Then (20) and (21) imply

$$T_c\approx\left[\frac{3}{2\pi\alpha_s(m_\sigma^{\,2})}\right]^{1/2}m_{\rm dyn}\approx 250\ {\rm MeV}\ , \tag{22}$$

which is reasonably close to our other estimates (18d) and (10b).

Since quark and antiquark densities are different in several interesting physical situations, we now generalize the above to include the effect of a chemical potential $\mu\neq 0$. Then p_0 in the statistical factor in (19) is replaced by $p_0-\mu$, where p_0 is the time component of the quark four-momentum. The temperature-dependent integral in (19a) now becomes

$$Q(T,\mu)=\langle\bar{q}q\rangle_{0,M}+\tfrac{3}{2}\frac{4\pi}{(2\pi)^3}\int_0^\infty \frac{p^2dp}{E}\mathrm{Tr}(\not p-m_{\mathrm{dyn}})\left[\frac{1}{e^{(E-\mu)/T}+1}+\frac{1}{e^{(E+\mu)/T}+1}\right] \tag{23a}$$

$$=\langle\bar{q}q\rangle_{0,M}+\frac{3m_{\mathrm{dyn}}T^2}{\pi^2}\left[K_+\left[\frac{m(T)}{T}\right]+K_-\left[\frac{m(T)}{T}\right]\right]. \tag{23b}$$

At the critical temperature we again take $Q(T_c,\mu_c)=m(T_c)=0$ and then the $K_\pm(0)$ integrals in (23) become

$$K_+(0)+K_-(0)=\int_0^\infty \frac{p\,dp}{T_c^2}\left[\frac{1}{e^{(p-\mu_c)/T_c}+1}+\frac{1}{e^{(p+\mu_c)/T_c}+1}\right]=\frac{\pi^2}{6}+\frac{\mu_c^2}{2T_c^2}. \tag{24}$$

Consequently the critical-temperature equation (22) becomes generalized to

$$T_c^2+\frac{3\mu_c^2}{\pi^2}=-\frac{2\langle\bar{q}q\rangle_{0,M}}{m_{\mathrm{dyn}}}\approx(250\ \mathrm{MeV})^2. \tag{25}$$

This curve in the T_c,μ_c plane is plotted in Fig. 5. In particular, we note that for cold nuclear matter with $T_c=0$, Eq. (25) requires

$$\mu_{c,\max}\cong 450\ \mathrm{MeV}. \tag{26}$$

Furthermore, for a typical value of $\mu_c\sim 200$ MeV, (25) gives $T_c\sim 225$ MeV.

V. HEAVY-ION COLLISIONS

There exist several indications from cosmic rays showing that, for incident energies above the TeV region, substantial deviations from usual hadronic interactions occur[8,17–19] for interactions involving heavy ions. In the c.m. frame this would correspond to energies of about 50 GeV per ion and such energies are accessible to present-day accelerators. Estimates of the energy densities reached in such collisions have been performed.[8,20–24] All these estimates lead to values around 2–3 GeV/fm^3. These values can be understood with the following simplified argument: Look at the energy density in the center-of-mass frame in one unit of rapidity. The number of pions is known to be about 2.4 for p-p collisions with average energy about 0.5 GeV. Multiplying these two numbers with the number of targets per fm^3 in a nucleus, one obtains a first rough estimate of the energy density as

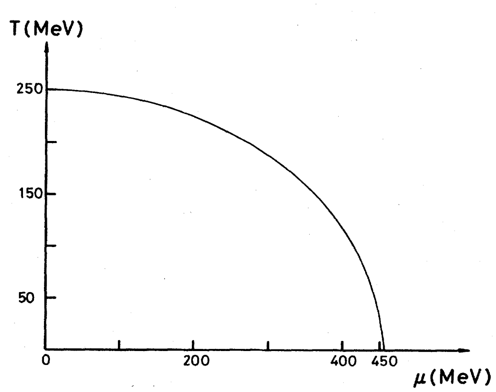

FIG. 5. Critical line in the temperature—chemical-potential plane.

$$\epsilon=(0.5\ \mathrm{GeV})\times 2.4\times(\text{nuclear density})\times 2A^{1/3}$$
$$\approx 0.432\times A^{1/3}\ \mathrm{GeV/fm}^3 \tag{27}$$

per unit of rapidity in the central region. For a uranium-uranium collision ($A=238$) the above estimate gives

$$\epsilon\approx 2.7\ \mathrm{GeV/fm}^3. \tag{28}$$

More involved estimates based on the hydrodynamic equations of motion reduce this value[23] because of dilution effects in the final state; the actual values fluctuate around 2 GeV/fm^3 depending on the model used.

Estimates of the baryon-number density have also been made. An essential difference with the energy density appears here because baryons are mainly produced in the fragmentation regions. Dilution effects will therefore be more severe in this case. The early estimates based on the presence of a hot fireball neglected this final-state dilution.[8,20] This leads to a baryon-number density strongly enhanced by the compression factor:

$$n_B\approx 0.5/\mathrm{fm}^3. \tag{29}$$

Only a few other estimates exist since those based upon the hydrodynamic equations concentrate on the central region of rapidities where contamination from baryons is negligible. The estimates of Kajantie and collaborators[22,23] lead to values which are smaller than the one given above by about 20%. The basic difference, however, is that in their estimate these high baryon densities are only reached in a region where the energy density is less than 1 GeV/fm^3 and thus no phase transition occurs. Because of this basic uncertainty as regards the relevance of the baryon-number density to the phase transition, we consider two cases: (a) hot neutral gas, corresponding to $\epsilon\sim 2$ GeV/fm^3 and $n_B\simeq 0$; and (b) hot baryonic gas, corresponding to $\epsilon\sim 2$ GeV/fm^3 and $n_B\sim 0.5$/fm^3. We now want to translate these numbers into parameters relevant to the physical description of the situation, namely, temperature T and chemical potential μ.

For a free gas of quarks and gluons the energy and baryon-number densities are given by

$$\epsilon=2\times 3\sum_{\text{flavors}}\int\frac{d^3p}{(2\pi)^3}\left[\frac{E}{e^{(E-\mu)/T}+1}+\frac{E}{e^{(E+\mu)/T}+1}\right]+2\times 8\int\frac{d^3p}{(2\pi)^3}\frac{E}{e^{E/T}-1}, \tag{30a}$$

$$n_B=2\sum_{\text{flavors}}\int\frac{d^3p}{(2\pi)^3}\left[\frac{1}{e^{(E-\mu)/T}+1}-\frac{1}{e^{(E+\mu)/T}+1}\right]. \tag{30b}$$

Factors of 2, 3, and 8 refer to spin, colors, and number of gluons, respectively. The first term in the large parentheses corresponds to the contribution of quarks while the second term corresponds to the contribution of antiquarks. We will now focus on two possibilities: in the first one we consider ϵ and n_B for massless quarks; in the second one we fix all quark masses to be 315 MeV.

For the massless-quark case we obtain

$$\epsilon=\frac{3}{\pi^2}\sum_{\text{flavors}}\left[\frac{7\pi^4}{60}T^4+\frac{\pi^2}{2}T^2\mu^2+\frac{1}{4}\mu^4\right]+\frac{8\pi^2}{15}T^4\,, \tag{31a}$$

$$n_B=\tfrac{1}{3}\sum_{\text{flavors}}\left[\mu T^2+\frac{\mu^3}{\pi^2}\right]. \tag{31b}$$

For a hot neutral gas this leads to $T\approx188$ MeV (two flavors) and to ≈177 MeV (three flavors). For a hot baryonic gas it leads to $T\approx178$ MeV and $\mu\approx171$ MeV (two flavors). For a quark mass of 315 MeV one finds for a hot neutral gas $T\approx194$ MeV (two flavors) and for a hot baryonic gas (≈180 MeV and $\mu\approx230$ MeV (two flavors).

These estimates indicate that the temperature expected to be reached in a heavy-ion collision will be in the 180—200-MeV range while the chemical potential will be either close to zero in the central region or around 170—230 MeV in the fragmentation regions.

Our analysis in the previous sections shows that the above estimates make us optimistic about reaching the phase transition in future high-energy heavy-ion collisions.

VI. QCD ORDER-PARAMETER CHAIN

Finally we review how the QCD energy scales, invoked so far in a somewhat *ad hoc* fashion, are in fact related to one another. To begin, we introduce $\Lambda_{\overline{\rm MS}}$ as the overall energy scale in QCD. Asymptotic freedom requires

$$\alpha_s(p^2)=\frac{\pi d}{\ln p^2/\Lambda_{\overline{\rm MS}}{}^2}$$

in the deep Euclidean region, where $d=12(33-2N_f)^{-1}$. To match the quarkonium decay widths for $\phi\to e^+e^-$ and $\psi\to e^+e^-$, one infers[25] $\alpha_s(1\text{ GeV})\simeq0.5$ and α_s (3 GeV)$\simeq0.28$, both of which imply $\Lambda_{\overline{\rm MS}}(3)\simeq250$ MeV for $N_f=3$ ($\overline{\rm MS}$ denotes the modified minimal-subtraction scheme). The former coupling also fits[26] the nonleptonic baryon $B\to B'\pi$ decays. Furthermore, scaling violations and the R value for larger p^2 suggest[27] $\Lambda_{\overline{\rm MS}}(5)\approx130$ MeV for $N_f=5$ and $p^2\sim(10\text{ GeV})^2$. It has been shown[28] that in fact $\Lambda_{\overline{\rm MS}}(5)\approx130$ MeV corresponds to $\Lambda_{\overline{\rm MS}}(3)\approx250$ MeV.

Given this latter "unique" Λ, we follow Ref. 4 which uses renormalization-group techniques and two-loop order to obtain

$$m_{\rm dyn}=\Lambda_{\overline{\rm MS}}(3)e^{1/6}\approx300\text{ MeV}\,, \tag{32}$$

where $m_{\rm dyn}$ increases to ~317 MeV for $\alpha_s(1\text{ GeV})\approx0.53$. Next we relate $m_{\rm dyn}$ to the pion decay constant by calculating the nonperturbative quark loop for $\langle 0|A_\mu|\pi\rangle=if_\pi q_\mu$, from which one obtains in the chiral limit

$$f_\pi\approx\frac{\sqrt{3}}{2\pi}m_{\rm dyn}\approx87\text{ MeV} \tag{33}$$

for $N_c=3$. Also the quark condensate $\langle\bar{q}q\rangle_{0,M}$ then follows from $m_{\rm dyn}\approx315$ MeV according to (21).

The next link is the string tension $\sqrt{\sigma}$ where it is approximated[3] that

$$\sqrt{\sigma}\approx\left[\frac{\pi}{2}\right]^{1/2}m_{\rm dyn}\simeq400\text{ MeV}\,. \tag{34}$$

Since the heavy-quark $q\bar{q}$ potential is σr for r large, σ can be inferred from the charmonium decay spectrum where Richardson[29] also finds $\sqrt{\sigma}\approx400$ MeV. Moreover, the ratio

$$\frac{\sqrt{\sigma}}{\Lambda_{\overline{\rm MS}}}\approx1.5 \tag{35}$$

has been independently obtained.[30] Lattice QCD (Ref. 6) then gives the next link in the chain by finding

$$T_c/\sqrt{\sigma}\sim0.5\,, \tag{36}$$

albeit for pure glue or massless-quark theories. Given $\sqrt{\sigma}\approx400$ MeV, (36) predicts $T_c\approx200$ MeV, which is verified by the melting of $m_{\rm dyn}$ and $\langle\bar{q}q\rangle_0$ as given in this paper. Lastly, (35) of this paper relates T_c back to $\Lambda_{\overline{\rm MS}}$ and to $m_{\rm dyn}$ and the QCD order-parameter chain is closed by returning to (32).

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