Semirelativistic potential model for charmonium

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The charmonium spectrum is investigated with the use of a semirelativistic quantum-chromodynamic potential model. As in our earlier nonrelativistic model, the quark-antiquark potential consists of a perturbative part, which includes the complete one-loop radiative correction to the one-gluon-exchange interaction, and a linear scalar-exchange confining part. We find that the semirelativistic model leads to closer agreement between the theoretical and observed energy levels, yields a larger value of the charmed-quark mass, and significantly affects the leptonic and $E1$ transition widths.

I. INTRODUCTION

The $b\bar{b}$ and $c\bar{c}$ spectra were investigated by us in an earlier paper[1] with the use of a nonrelativistic quantum-chromodynamic potential model. An essential feature of our model was the inclusion of the complete one-loop radiative correction in the perturbative part of the quark-antiquark potential, and our results have been found to be in excellent agreement with experiments.[1,2] It would, however, be desirable to improve upon the nonrelativistic model, especially for the treatment of the lighter $c\bar{c}$ system, and for this purpose we shall explore a semirelativistic model.[3]

Our semirelativistic treatment, which is considerably more complicated than the nonrelativistic approach, will be described in Sec. II. Our investigation shows that the semirelativistic results for the observed energy levels of $b\bar{b}$ are virtually the same as those obtained earlier[1] with the nonrelativistic treatment, while the semirelativistic results for $c\bar{c}$ are more interesting and will be discussed in Sec. III. As we shall see, the semirelativistic model of $c\bar{c}$ leads to closer agreement between the theoretical and observed energy levels, yields a larger value of the charmed-quark mass, and significantly affects the leptonic and $E1$ transition widths.

II. SEMIRELATIVISTIC MODEL

We shall take the Hamiltonian for our semirelativistic model as

$$\mathscr{H}=2(m^2+\vec{p}^{\,2})^{1/2}+\mathscr{V}_p(\vec{r})+\mathscr{V}_c(\vec{r}) , \tag{2.1}$$

where the perturbative and confining potentials $\mathscr{V}_p(\vec{r})$ and $\mathscr{V}_c(\vec{r})$ are the same as in Ref. 1.

Our wave functions are of the form

$$\psi^m_{nl}(\vec{r})=\sum_{k=0}^{K} a_{L,nl}\left[\frac{r}{R}\right]^L e^{-r/R}Y^m_l(\Omega_{\vec{r}}),\quad L=k+l , \tag{2.2}$$

with the Lth component

$$\psi_L(\vec{r})=\left[\frac{r}{R}\right]^L e^{-r/R}Y^m_l(\Omega_{\vec{r}}) . \tag{2.3}$$

The coefficients $a_{L,nl}$ will be determined by the variational technique of minimizing the expectation value of the unperturbed Hamiltonian

$$\mathscr{H}_0=2(m^2+\vec{p}^{\,2})^{1/2} -\frac{4\alpha_s}{3r}\left[1-\frac{3\alpha_s}{2\pi}+\frac{\alpha_s}{6\pi}(33-2n_f)[\ln(\mu r)+\gamma_E]\right] +Ar+C , \tag{2.4}$$

while the optimum value of the parameter R will be determined by satisfying the virial theorem

$$\langle p\, d\mathscr{H}_0/dp\rangle-\langle r\, d\mathscr{H}_0/dr\rangle=0 . \tag{2.5}$$

Finally, the contribution of $\mathscr{H}'=\mathscr{H}-\mathscr{H}_0$ to the energy levels will be included by first-order perturbation theory.

The main complication in the above procedure arises from the matrix elements

$$\langle L\,|\,(m^2+\vec{p}^{\,2})^{1/2}\,|\,L'\rangle=\int d\vec{r}\,\psi^*_L(\vec{r})(m^2+\vec{p}^{\,2})^{1/2}\psi_{L'}(\vec{r}) \tag{2.6}$$

and

$$\langle L\,|\,\vec{p}^{\,2}(m^2+\vec{p}^{\,2})^{-1/2}\,|\,L'\rangle =\int d\vec{r}\,\psi^*_L(\vec{r})\vec{p}^{\,2}(m^2+\vec{p}^{\,2})^{-1/2}\psi_{L'}(\vec{r}) , \tag{2.7}$$

which appear in the expectation value of $\mathscr{H}_0$ and the virial theorem, respectively, and we shall describe our procedure for their evaluation.

A. Fourier transformation

The Fourier transform of (2.3) is

$$\phi_L(\vec{p})=\frac{1}{(2\pi)^{3/2}}\int d\vec{r}\,e^{-i\vec{p}\cdot\vec{r}}\psi_L(r)\ . \tag{2.8}$$

Upon expansion of the exponential as

$$e^{-i\vec{p}\cdot\vec{r}}=4\pi\sum_{l'=0}^{\infty}(-i)^{l'}j_{l'}(pr)\sum_{m'=-l'}^{l'}Y_{l'}^{m'*}(\Omega_{\vec{r}})Y_{l'}^{m'}(\Omega_{\vec{p}})\ ,$$

and use of the orthogonality relation

$$\int d\Omega_{\vec{r}}Y_{l'}^{m'*}(\Omega_{\vec{r}})Y_l^m(\Omega_{\vec{r}})=\delta_{l'l}\delta_{m'm}\ ,$$

(2.8) becomes

$$\phi_L(\vec{p})=\frac{2(-i)^l}{(2\pi)^{1/2}}\int_0^{\infty}r^2dr\,j_l(pr)\left[\frac{r}{R}\right]^L e^{-r/R}Y_l^m(\Omega_{\vec{p}})\ .$$

Then, using the representation

$$j_l(z)=\frac{(-i)^l}{2}\int_{-1}^{1}dx\,P_l(x)e^{izx}\ ,$$

and carrying out integration over r, we find

$$\phi_L(\vec{p})=\frac{(-i)^{2l}}{(2\pi)^{1/2}}Y_l^m(\Omega_{\vec{p}})\frac{\Gamma(L+3)}{R^L}\times\int_{-1}^{1}dx\frac{P_l(x)}{(R^{-1}-ipx)^{L+3}}\ . \tag{2.9}$$

The matrix elements (2.6) and (2.7) can be expressed in terms of (2.9) as

$$\langle L\,|\,(m^2+\vec{p}^{\,2})^{1/2}\,|\,L'\rangle=\int d\vec{p}\,\phi_L^*(\vec{p})(m^2+\vec{p}^{\,2})^{1/2}\phi_{L'}(\vec{p})\ , \tag{2.10}$$

$$\langle L\,|\,\vec{p}^{\,2}(m^2+\vec{p}^{\,2})^{-1/2}\,|\,L'\rangle=\int d\vec{p}\,\phi_L^*(\vec{p})\vec{p}^{\,2}(m^2+\vec{p}^{\,2})^{-1/2}\phi_{L'}(\vec{p})\ , \tag{2.11}$$

and simplified for various values of l.

B. S states

Since $P_l(x)=1$ for $l=0$, Eq. (2.9) gives, upon integration over x,

$$\phi_L^S(\vec{p})=\frac{1}{(2\pi)^{1/2}}Y_0^0(\Omega_{\vec{p}})\times\frac{\Gamma(L+2)}{ipR^L}\left[\frac{1}{(R^{-1}-ip)^{L+2}}-\frac{1}{(R^{-1}+ip)^{L+2}}\right]\ . \tag{2.12}$$

It is convenient to define

$$\tan\theta=pR\ , \tag{2.13}$$

so that

$$\sin\theta=\frac{p}{(R^{-2}+p^2)^{1/2}}\ ,\qquad \cos\theta=\frac{1}{R(R^{-2}+p^2)^{1/2}}\ , \tag{2.14}$$

and (2.12) is expressible as

$$\phi_L^S(\vec{p})=\frac{1}{(2\pi)^{1/2}}Y_0^0(\Omega_{\vec{p}})\frac{2R^2\Gamma(L+2)}{p}\times(\cos\theta)^{L+2}\sin[(L+2)\theta]\ . \tag{2.15}$$

Substituting (2.15) into (2.10) and (2.11), and changing the variable of integration to θ with the use of (2.13), we obtain for the matrix elements

$$\langle L\,|\,(m^2+\vec{p}^{\,2})^{1/2}\,|\,L'\rangle^S=\frac{2}{\pi}\Gamma(L+2)\Gamma(L'+2)R^2\int_0^{\pi/2}d\theta(\cos\theta)^{L+L'+1}\times(\sin^2\theta+m^2R^2\cos^2\theta)^{1/2}\sin[(L+2)\theta]\sin[(L'+2)\theta]\ , \tag{2.16}$$

$$\langle L\,|\,\vec{p}^{\,2}(m^2+\vec{p}^{\,2})^{-1/2}\,|\,L'\rangle^S=\frac{2}{\pi}\Gamma(L+2)\Gamma(L'+2)R^2\int_0^{\pi/2}d\theta(\cos\theta)^{L+L'+1}\times\frac{\sin^2\theta}{(\sin^2\theta+m^2R^2\cos^2\theta)^{1/2}}\sin[(L+2)\theta]\sin[(L'+2)\theta]\ . \tag{2.17}$$

C. P states

For $l=1$, $P_l(x)=x$, and (2.9) can be integrated to yield

$$\phi_L^P(\vec{p})=-\frac{1}{(2\pi)^{1/2}}Y_l^m(\Omega_{\vec{p}})\left[\frac{\Gamma(L+2)}{ipR^L}\left[\frac{1}{(R^{-1}-ip)^{L+2}}+\frac{1}{(R^{-1}+ip)^{L+2}}\right]+\frac{\Gamma(L+1)}{p^2R^L}\left[\frac{1}{(R^{-1}-ip)^{L+1}}-\frac{1}{(R^{-1}+ip)^{L+1}}\right]\right]\ ,$$

which can be expressed as

$$\phi_L^P(\vec{p})=\frac{i}{(2\pi)^{1/2}}Y_l^m(\Omega_{\vec{p}})\frac{2R^2\Gamma(L+1)}{p}(\cos\theta)^{L+2}\left[(L+2)\cos[(L+2)\theta]-\frac{\sin[(L+2)\theta]}{pR}\right]. \tag{2.18}$$

Substituting (2.18) into (2.10) and (2.11), changing the variable of integration to θ, and simplifying by means of integration by parts, we arrive at the results

$$\begin{aligned}\langle L\,|\,(m^2+\vec{p}^{\,2})^{1/2}\,|\,L'\rangle^P&=\frac{2}{\pi}\Gamma(L+1)\Gamma(L'+1)R^2\int_0^{\pi/2}d\theta(\cos\theta)^{L+L'+1}\\&\times\left[(\sin^2\theta+m^2R^2\cos^2\theta)^{1/2}\{(L+2)(L'+2)\cos[(L+2)\theta]\cos[(L'+2)\theta]\right.\\&\quad-(L+L'+3)\sin[(L+2)\theta]\sin[(L'+2)\theta]\}\\&\left.+\frac{1}{(\sin^2\theta+m^2R^2\cos^2\theta)^{1/2}}\sin[(L+2)\theta]\sin[(L'+2)\theta]\right],\end{aligned} \tag{2.19}$$

$$\begin{aligned}\langle L\,|\,\vec{p}^{\,2}(m^2+\vec{p}^{\,2})^{-1/2}\,|\,L'\rangle^P&=\frac{2}{\pi}\Gamma(L+1)\Gamma(L'+1)R^2\int_0^{\pi/2}d\theta(\cos\theta)^{L+L'+1}\\&\times\left[\frac{\sin^2\theta}{(\sin^2\theta+m^2R^2\cos^2\theta)^{1/2}}\{(L+2)(L'+2)\cos[(L+2)\theta]\cos[(L'+2)\theta]\right.\\&\quad-(L+L'+3)\sin[(L+2)\theta]\sin[(L'+2)\theta]\}\\&\left.+\frac{\sin^2\theta+2m^2R^2\cos^2\theta}{(\sin^2\theta+m^2R^2\cos^2\theta)^{3/2}}\sin[(L+2)\theta]\sin[(L'+2)\theta]\right].\end{aligned} \tag{2.20}$$

The above procedure for evaluation of the matrix elements (2.6) and (2.7) for S and P states can be used for any value of l.

III. $c\bar{c}$ SPECTRUM

Following the same procedure as in Ref. 1, together with the mathematical formalism described in Sec. II, we have obtained the wave functions and energy levels of $c\bar{c}$. The value of the renormalization-scale parameter μ was also chosen by applying the criterion discussed in Ref. 1.

Our values for the energy levels below the charm threshold as well as the values of the parameters are given in Table I. The splittings of the energy levels are

$$\begin{aligned}&M(\psi')-M(\psi)=588\text{ MeV},\\&M(\psi)-M(\eta_c)=109\text{ MeV},\\&M(\psi')-M(\eta_c')=96\text{ MeV},\\&M(\chi_{\text{c.o.g.}})-M(\psi)=430\text{ MeV},\\&M(\chi_1)-m(\chi_0)=94\text{ MeV},\\&M(\chi_2)-M(\chi_1)=45\text{ MeV},\end{aligned} \tag{3.1}$$

which are in closer agreement with the experimental results[4] than the earlier nonrelativistic values.

In Ref. 1, we were able to obtain reasonable results for the leptonic widths with the use of the Van Royen—Weisskopf formula[5]

$$\Gamma_{ee}=\frac{16\pi\alpha^2e_Q^{\,2}}{M^2(Q\bar{Q})}\,|\,\psi(0)\,|^2. \tag{3.2}$$

TABLE I. Charmonium spectrum with $m_c=1.55$ GeV, $\mu=1.98$ GeV, $\alpha_s=0.3475$, and $A=0.205$ GeV^2.

State	Mass (GeV)	State	Mass (GeV)
$1\,^3S_1$ (ψ)	3.097	$1\,^3P_2$ (χ_2)	3.558
$1\,^1S_0$ (η_c)	2.988	$1\,^3P_1$ (χ_1)	3.513
		$1\,^3P_0$ (χ_0)	3.419
$2\,^3S_1$ (ψ')	3.685	$1\,^1P_1$	3.529
$2\,^1S_0$ (η_c')	3.589		

TABLE II. Leptonic widths in keV. Γ_{ee} corresponds to the Van Royen—Weisskopf formula, while Γ_{ee}' and Γ_{ee}'' include the relativistic and radiative corrections, respectively.

Transition	Γ_{ee}	Γ_{ee}'	Γ_{ee}''	Γ_{ee} (expt)
$\psi\to e^+e^-$	10.27	8.05	4.21	4.66±1.5
$\psi'\to e^+e^-$	6.19	4.67	2.54	1.94±0.6

TABLE III. $E1$ transition widths in keV. Theoretical results for both the nonrelativistic and the semirelativistic models are given.

Transition	J	Nonrelativistic widths	Semirelativistic widths	Experimental values
$\psi' \to \chi_J + \gamma$	2	33	24	17±5
	1	49	35	19±5
	0	64	44	21±6
$\chi_J \to \psi + \gamma$	2	753	502	330±170
	1	562	369	<700
	0	253	171	97±38

But, the semirelativistic model yields quite different leptonic widths, and in Table II we give the results obtained with the use of (3.2) as well as the modified formulas

$$\Gamma'_{ee} = \Gamma_{ee} \left[\frac{2}{3} + \frac{2m^2}{3\pi} \int_0^\infty dr\, r K_1(mr) \frac{\psi(r)}{\psi(0)} \right]^2 \tag{3.3}$$

and

$$\Gamma''_{ee} = \Gamma_{ee}(1 - 16\alpha_s/3\pi) , \tag{3.4}$$

which include relativistic[6] and radiative[7] corrections, respectively. Although the theoretical results with radiative correction are in reasonable agreement with the experimental data,[4] we consider the use of the formula (3.4) questionable here because radiative correction to the quark-antiquark interaction has already been included in our potential.

In Table III, the $E1$ transition widths for both the nonrelativistic and the semirelativistic models are given, and compared with the experimental data.[8] The semirelativistic model yields distinctly better results, but fails to resolve the discrepancy between the theoretical and experimental values.

It should be noted that while our energy levels include the effect of perturbation, our widths have been obtained with the use of unperturbed wave functions. It is, therefore, not surprising that the theoretical energy levels are in excellent agreement with experiments, but the theoretical leptonic and $E1$ transition widths are less satisfactory.[9]

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