Topological invariance of the Witten index and related quantities

Marc D. Bernstein and Lowell S. Brown
Department of Physics FM-15, University of Washington, Seattle, Washington 98195
(Received 13 November 1984)

We demonstrate the topological invariance of the regulated Witten index in supersymmetric quantum mechanics. Our method applies to other quantities as well, such as the index of the Dirac operator on open spaces.

In supersymmetric quantum mechanics,[1] the Witten index[2] counts the difference between the number of zero-energy Bose and Fermi eigenstates. Witten[2] has given a demonstration that this index is a topological invariant: it is invariant under compact perturbations of the potential occurring in the supersymmetric Hamiltonian. In this Brief Report, we give a simple direct proof that the regulated version of the Witten index for an arbitrary supersymmetric Hamiltonian is also a topological invariant. Since the index itself is defined in an appropriate limit of the regulated index, this proves the invariance of the Witten index. In addition, recent proofs[3] of the Atiyah-Singer index theorem have been constructed based on supersymmetry. Thus, our method also covers the index of the Dirac operator in these cases. Even the open-space index theorems of Callias[4] and Bott and Seeley[5] are such that the regulated version of the index of the appropriate Dirac operator can be expressed in a form where our method is applicable.[6] Finally, Niemi[7] has shown that the fractional vacuum fermionic charge for certain Dirac Hamiltonians[8] may be expressed in terms of the regulated Witten index of a supersymmetric Hamiltonian. Our proof demonstrates the *a priori* topological invariance of this charge.

For any supersymmetric quantum-mechanical system the supercharge connects fermionic states with bosonic states and so has the form (for Hermitian charges)

$$Q=\begin{pmatrix} 0 & A^\dagger \\ A & 0 \end{pmatrix} , \tag{1}$$

where Q acts on vectors whose upper component is bosonic and whose lower component is fermionic. Thus, the Hamiltonian, which is given by $H=Q^2$, may be written in the form

$$H=\begin{pmatrix} A^\dagger A & 0 \\ 0 & AA^\dagger \end{pmatrix}=\begin{pmatrix} H_+ & 0 \\ 0 & H_- \end{pmatrix} , \tag{2}$$

where H_+ is the bosonic Hamiltonian and H_- the fermionic Hamiltonian. The regulated Witten index is then given by

$$\Delta(\beta)=\mathrm{Tr}[\exp(-\beta H_+)-\exp(-\beta H_-)] \ . \tag{3}$$

Our proof of the topological invariance of this index rests on the lemmas

$$\exp(-\beta AA^\dagger)A=A\exp(-\beta A^\dagger A) \ , \tag{4}$$

$$A^\dagger\exp(-\beta AA^\dagger)=\exp(-\beta A^\dagger A)A^\dagger \ , \tag{5}$$

which follow from the defining power-series expansion of the exponential. The operators A and $A^\dagger$ depend on the various potentials; denoting a compact deformation of these potentials by δ, we have, on account of the cyclic symmetry of the trace,

$$\begin{aligned}\delta\Delta(\beta)=-\beta\,\mathrm{Tr}\{&\exp(-\beta A^\dagger A)[(\delta A^\dagger)A+A^\dagger(\delta A)]\\ &-\exp(-\beta AA^\dagger)[(\delta A)A^\dagger+A(\delta A^\dagger)]\} \ .\end{aligned} \tag{6}$$

Using the lemmas above and again the cyclic symmetry of the trace, we see that effectively

$$\begin{aligned}\exp(-\beta AA^\dagger)(\delta A)A^\dagger&=A^\dagger\exp(-\beta AA^\dagger)\delta A\\ &=\exp(-\beta A^\dagger A)A^\dagger(\delta A) \ ,\end{aligned} \tag{7}$$

$$\begin{aligned}\exp(-\beta AA^\dagger)A(\delta A^\dagger)&=A\exp(-\beta A^\dagger A)\delta A^\dagger\\ &=\exp(-\beta A^\dagger A)(\delta A^\dagger)A \ ,\end{aligned} \tag{8}$$

and so

$$\delta\Delta(\beta)=0 \ . \tag{9}$$

We note that since this variation of the potential has, by assumption, compact support, the formal cyclic symmetry of the trace is valid (we can freely integrate by parts in the coordinate representation).

Since our method is directly applicable to the calculation of the regulated index of the Dirac operator on open spaces, we have also confirmed a conjecture of Callias[4] that this regulated index is a topological invariant; this conjecture was used recently in Ref. 6 to calculate the Witten index for a supersymmetric theory with an arbitrary number of degrees of freedom. Moreover, the lemmas (4) and (5) clearly generalize to any well-defined function F of $AA^\dagger$ and $A^\dagger A$, and so we conclude that in general $\mathrm{Tr}[F(AA^\dagger)-F(A^\dagger A)]$ is a topological invariant.

This research was supported in part by the U.S. Department of Energy through Contract No. DE-AC06-81ER-40048.

[1]E. Witten, Nucl. Phys. **B185**, 513 (1981).
[2]E. Witten, Nucl. Phys. **B202**, 253 (1982).
[3]L. Alvarez-Gaumé, Commun. Math. Phys. **90**, 161 (1983); J. Phys. A **16**, 4177 (1983); D. Friedan and P. Windey, Nucl. Phys. **B235**, 395 (1984); B. Zumino, University of California, Berkeley, Report No. LBL-17972, 1984 (unpublished).
[4]C. Callias, Commun. Math. Phys. **62**, 213 (1978); M. Hirayama, Prog. Theor. Phys. **70**, 1444 (1983).
[5]R. Bott and R. Seeley, Commun. Math. Phys. **62**, 235 (1978).
[6]C. Imbimbo and S. Mukhi, Nucl. Phys. **B242**, 81 (1984).
[7]A. J. Niemi, Phys. Lett. **146B**, 213 (1984).
[8]F. Wilczek and J. Goldstone, Phys. Rev. Lett. **47**, 968 (1981); R. Jackiw and C. Rebbi, Phys. Rev. D **13**, 3398 (1976).

