Gluon propagator and transverse vertices

R. B. Zhang
Department of Physics, University of Tasmania, Hobart, Tasmania, Australia 7001
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It is shown explicitly that in the axial gauge the Slavnov-Taylor identity contains no useful information for determining the gluon propagator through the Dyson-Schwinger equation; rather, it is the transverse vertex which plays the crucial role.

With the aim of investigating the color-confinement problem of QCD, people have studied the infrared properties of the gluon propagator for some years. The method adopted is similar to the so-called "gauge technique" in QED,[1] which successfully produces the infrared behaviors of charged particle propagators. However, the situation in QCD is not so clear as the results obtained by different people often disagree with one another.[2,3] Here we want to show that there exist intrinsic problems with the approach for QCD.

The essence of this method is to analyze the Dyson-Schwinger equations with the aid of the gauge identities. By "solving" Ward identities, *Ansätze* for vertices are derived and then applied to truncate the Dyson-Schwinger equations which relate the propagators with the vertices. This results in self-consistent equations for propagators (or the spectral functions of propagators).

As noted many times, solutions of Ward identities are by no means unique and the existence of arbitrary transverse pieces cannot be avoided altogether. But in QED, if we are only concerned about charged particle propagators, the lowest-order *Ansätze* dominate the transverse parts: hence, the latter can be neglected without affecting final results, certainly in the infrared region.[4] The situation changes when we go to QCD.

Instead of studying charged particle (quark) propagators, people choose to work in pure Yang-Mills theory (that is, with the quark-gluon interaction neglected), and evaluate the gluon propagator as an indicator of color confinement. We will show explicitly that not only do the longitudinal vertices no longer dominate the transverse parts, they do not even contribute.

In axial-gauge QCD the Dyson-Schwinger equation reads

$$\begin{aligned}\Pi_{\mu\nu}(q) = & -q^2(g_{\mu\nu}-q_\mu q_\nu/q^2) \\ & -i\tfrac{1}{2}g_0{}^2\int đ^4k\,\Gamma^{(0)}_{\mu\lambda\lambda'}(q,-k,-k')\Delta^{\lambda\sigma}(k) \\ & \qquad \times\Delta^{\lambda'\sigma'}(k')\Gamma_{\sigma\sigma'\nu}(k,k',-q) \\ & +i\tfrac{1}{2}g_0{}^2\int đ^4k\,\Gamma^{(0)}_{\mu\lambda\sigma\nu}(q,k,-k,-q)\Delta^{\lambda\sigma}(k) \\ & +\text{ four-gluon terms} \qquad (1)\end{aligned}$$

upon neglecting quarks, where $đ^4k=[N_c/(2\pi)^4]d^4k$ with N_c the number of colors. In (1) we have suppressed the color indices, and $\Delta_{\mu\nu}(q)$ and $\Pi_{\mu\nu}(q)$ are the gluon propagator and self-energy, respectively, which are related to each other by

$$\Pi^\lambda_\mu(q)\Delta_{\lambda\nu}(q)=g_{\mu\nu}-n_\mu q_\nu/n\cdot q \ . \qquad (2)$$

We display (1) diagrammatically in Fig. 1. There it is clear that we have used $\Gamma_{\mu\nu\rho}$ and $\Gamma_{\mu\nu\rho\sigma}$ to represent the three- and four-gluon vertices, respectively, with $\Gamma^{(0)}_{\mu\nu\rho}$ and $\Gamma^{(0)}_{\mu\nu\rho\sigma}$ their corresponding bare ones.

In the axial gauge the gauge identity adopts the naive form of the QED Ward identity

$$\Gamma_{\sigma\sigma'\nu}(k,k',-q)(-q)^\nu=\Pi_{\sigma\sigma'}(k)-\Pi_{\sigma\sigma'}(k') \ . \qquad (3)$$

Needless to say (3) is also an exact relation, like the Dyson-Schwinger equation (1). To study (3), we decompose $\Gamma_{\sigma\sigma'\nu}$ into two pieces, of which one is purely longitudinal to q^ν through index ν, and the other is transverse. They are *defined* by

$$\Gamma^{(L)}_{\sigma\sigma'\nu}(k,k',-q)=\Gamma_{\sigma\sigma'\nu'}(k,k',-q)\frac{(-q)^{\nu'}(-q)_\nu}{q^2} \ , \qquad (4a)$$

$$\Gamma^{(T)}_{\sigma\sigma'\nu}(k,k',-q)=\Gamma_{\sigma\sigma'\nu}(k,k',-q)-\Gamma^{(L)}_{\sigma\sigma'\nu}(k,k',-q) \ . \qquad (4b)$$

This decomposition seems to introduce singularities in $\Gamma^{(T)}_{\sigma\sigma'\nu}$ and $\Gamma^{(L)}_{\sigma\sigma'\nu}$ at $q^2=0$, but when we apply it to the Dyson-Schwinger equation, the singularities do not matter. As it is quite obvious that $\Gamma^{(T)}_{\sigma\sigma'\nu}$ is completely irrelevant to (3), we cannot obtain any information for it from (3). Applying

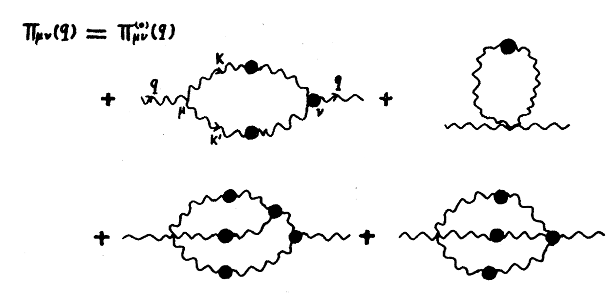

FIG. 1. Diagrammatic representation of Eq. (1).

(4a) and (4b) to (1), we arrive at

$$\begin{aligned}\Pi_{\mu\nu}(q) = &-q^2(g_{\mu\nu}-q_\mu q_\nu/q^2)\\ &-i\tfrac{1}{2}g_0{}^2\int d^4k\,\Gamma^{(0)}_{\mu\lambda\lambda'}(q,-k,-k')\Delta^{\lambda\sigma}(k)\Delta^{\lambda'\sigma'}(k')\Gamma^{(T)}_{\sigma\sigma'\nu}(k,k',-q)\\ &-i\tfrac{1}{2}g_0{}^2\int d^4k\,\Gamma^{(0)}_{\mu\lambda\lambda'}(q,-k,-k')[\Delta^{\lambda\lambda'}(k')-\Delta^{\lambda\lambda}(k)]\left(-\frac{q_\nu}{q^2}\right)\\ &+i\tfrac{1}{2}g_0{}^2\int d^4k\,\Gamma^{(0)}_{\mu\lambda\sigma\nu}(q,k,-k,-q)\Delta^{\lambda\sigma}(k)+\text{four-gluon terms}\ .\end{aligned} \tag{5}$$

In deriving (5) we combined (3) with (2) and utilized the property of $\Delta_{\mu\nu}(k)$ that

$$n^\mu\Delta_{\mu\nu}(k)=\Delta_{\nu\mu}(k)n^\mu=0\ . \tag{6}$$

By recalling

$$\Gamma^{(0)}_{\mu\lambda\lambda'}(q,-k,-k')=(q+k)_{\lambda'}g_{\mu\nu}+(k'-k)_\mu g_{\lambda\lambda'}-(q+k')_\lambda g_{\mu\lambda'} \tag{7}$$

and

$$\Gamma^{(0)}_{\mu\lambda\sigma\nu}(q,k,-k,-q)=2(g_{\mu\nu}g_{\sigma\lambda}-g_{\mu\sigma}g_{\nu\lambda})\ , \tag{8}$$

the third and fourth terms on the right-hand side of (5) can be simplified to

$$-ig_0{}^2\int d^4k\left(g_\nu^{\nu'}-\frac{q_\nu q^{\nu'}}{q^2}\right)[\Delta_{\mu\nu'}(k)-g_{\mu\nu'}\Delta^\lambda_\lambda(k)]\ , \tag{9}$$

and (5) can be rewritten as

$$\begin{aligned}\Pi_{\mu\nu}(q)= &-q^2(g_{\mu\nu}-q_\mu q_\nu/q^2)\\ &-i\tfrac{1}{2}g_0{}^2\int d^4k\,\Gamma^{(0)}_{\mu\lambda\lambda'}(q,-k,-k')\\ &\qquad\times\Delta^{\lambda\sigma}(k)\Delta^{\lambda'\sigma'}(k')\Gamma^{(T)}_{\sigma\sigma'\nu}(k,k',-q)\\ &-ig_0{}^2\int d^4k\,(g_\nu^{\nu'}-q_\nu q^{\nu'}/q^2)[\Delta_{\mu\nu'}(k)-g_{\mu\nu'}\Delta^\lambda_\lambda(k)]\\ &+\text{four-gluon terms}\ .\end{aligned} \tag{10}$$

Basically what we have done is to absorb the $\Gamma^{(L)}_{\sigma\sigma'}$ part of the original equation into a tadpole term. Upon multiplying (10) with n^μ, the four-gluon terms are eliminated,[5] and we arrive at

$$\begin{aligned}n^\mu\Pi_{\mu\nu}(q)= &-q^2(n_\nu-q_\nu n\cdot q/q^2)\\ &+i\tfrac{1}{2}g_0{}^2\int d^4k\,n\cdot(k-k')\Delta^\sigma_\lambda(k)\\ &\qquad\times\Delta^{\lambda\sigma'}(k')\Gamma^{(T)}_{\sigma\sigma'\nu}(k,k',-q)\\ &+ig_0{}^2\int d^4k\,\Delta^\lambda_\lambda(k)(n_\nu-q_\nu n\cdot q/q^2)\ .\end{aligned} \tag{11}$$

In (11), even if we suppose the tadpole integration does not vanish, the third term cannot possibly dominate, because it must be cancelled away by certain contributions of the second term, in order for $\Pi_{\mu\nu}(q)$ to satisfy the requirement

$$\Pi_{\mu\nu}(q)\to 0 \text{ as } q\to 0\ .$$

By decomposing $\Gamma^{(T)}_{\sigma\sigma'\nu}$ into two pieces

$$\Gamma^{(T)}_{\sigma\sigma'\nu}(k,k',-q)=\Gamma^{(T')}_{\sigma\sigma'\nu}(k,k',-q)+\Gamma^{(T'')}_{\sigma\sigma'\nu}(k,k',-q) \tag{12}$$

with

$$\begin{aligned}n\cdot(k-k')\Gamma^{(T'')}_{\sigma\sigma'\nu}(k,k',-q)\\ =[\Pi_{\sigma\sigma'}(k)+\Pi_{\sigma\sigma'}(k')](n_\nu-q_\nu n\cdot q/q^2)\ ,\end{aligned} \tag{13}$$

the tadpole term is removed and (11) is altered to

$$\begin{aligned}n^\mu\Pi_{\mu\nu}(q)= &-q^2(n_\nu-q_\nu n\cdot q/q^2)\\ &+i\tfrac{1}{2}g_0{}^2\int d^4k\,n\cdot(k-k')\Delta^\sigma_\lambda(k)\\ &\qquad\times\Delta^{\lambda\sigma'}(k')\Gamma^{(T')}_{\sigma\sigma'\nu}(k,k',-q)\ .\end{aligned} \tag{14}$$

Like $\Gamma^{(T)}_{\sigma\sigma'\nu}$ itself, $\Gamma^{(T')}_{\sigma\sigma\nu}$ is purely transverse to q through ν: therefore it does not arise in the identity (3).

The point we are trying to stress is that, contrary to common belief, the purely longitudinal three-gluon vertex does not contribute to the self-energy equation; rather the radiation corrections to the gluon propagator lie solely in the contribution of a transverse part of the vertex, which is completely irrelevant to the Slavnov-Taylor identity. Thus *in the axial gauge* the Slavnov-Taylor identity contains no useful information for determining the gluon propagator through the Dyson-Schwinger equation and we have to look for other effective methods which can determine the transverse vertex at least in the infrared region.

Multiplying (14) with n_ν/n^2 results in an equation entirely equivalent to that of Baker, Ball, and Zachariasen.[2] Therefore, the longitudinal part of the vertex is not present there either.

People have also studied (5) in another slightly different way.[6] Instead of making an *Ansatz* for $\Gamma_{\sigma\sigma'\nu}(k,k',-q)$ which satisfies the Slavnov-Taylor identity, they have tried to obtain an expression for

$$[\Delta(k)\Delta(k')\Delta(-q)\Gamma(k,k',-q)]_{\mu\nu\rho}$$

which satisfies

$$[\Delta(k)\Delta(k')\Gamma(k,k',-q)]_{\mu\nu\rho}(-q)^\rho=\Delta_{\mu\nu}(k')-\Delta_{\mu\nu}(k)\ . \tag{15}$$

This does not change the problem in any essential way, for although they are able to introduce a transverse piece into $\Gamma_{\sigma\sigma'\nu}$, it is inherently nonunique; hence, the resultant equation obtained by applying their *Ansatz* to (5) is still arbitrary,[7] although they do attempt to incorporate lowest-order perturbation theory in their work.

The same thing happens to QED when we apply the gauge technique to study the photon propagator. After the lowest-order *Ansatz*

$$S(p')\Gamma_\mu(p',p)S(p)=\int dw\,\rho(w)\frac{1}{\not p'-w}\gamma_\mu\frac{1}{p-w} \tag{16}$$

is used, the vacuum polarization tensor adopts the form

$$\Pi_{\mu\nu}(k)=\int dw\,\rho(w)\left[ie^2\int dp\ \mathrm{tr}\left(\frac{1}{\not p-\not k-w}\gamma_\mu\frac{1}{\not p-w}\gamma_\nu\right)\right]. \tag{17}$$

If we regularize $\Pi_{\mu\nu}(k)$ by the dimensional method we can confidently change (14) to

$$\Pi_{\mu\nu}(k)=\int dw\rho(w)\left\{ie^2\int dp\ \mathrm{tr}\left[\frac{1}{p-k-w}\left(\gamma_\mu-\frac{k_\mu k}{k^2}\right)\frac{1}{p-w}\gamma_\nu\right]\right\}. \tag{17'}$$

Obviously the purely transverse piece of (13) determines $\Pi_{\mu\nu}(k)$. It has been recently shown that the lowest-order *Ansatz* (12) is unique.[8] Hence, we can expect (14) to give the proper result. But one thing should be emphasized, the Ward identity

$$S(p')\Gamma_\mu(p',p)S(p)(p'-p)^\mu=S(p)-S(p') \tag{18}$$

only determines the longitudinal part of the vertex, which is irrelevant to the determination of $\Pi_{\mu\nu}(k)$; the purely transverse piece is introduced through other considerations.[9]

We conclude that the common belief that the longitudinal three-gluon vertex gives the dominant contribution to the gluon propagator through Dyson-Schwinger equation is not true; the lowest-order gauge technique is in principle useless for QCD.

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[4]R. Delbourgo and P. West, Phys. Lett. **72B**, 97 (1977).

[5]Consider the four-gluon terms. The last diagram in Fig. 1 is $\int\Gamma_4^{(0)}\Delta\Delta\Delta\Gamma_4$. With $\Gamma_4^{(0)}$, (6) gives us $\int n\Gamma_4^{(0)}\Delta\Delta\Delta\Gamma_4=0$. The same argument applies to the other four-gluon diagram.

[6]For this method we refer to the first two references of Ref. 3.

[7]B. K. Jennings and R. M. Woloshyn, J. Phys. G **9**, 997 (1983).

[8]To the extent that the representation

$$S(p')\Gamma_\mu(p',p)S(p)=\int dw\rho(w)g_\mu(p',p\,|\,w)$$

with

$$g_\mu(p',p\,|\,w)=g_\mu^{(0)}(p',p\,|\,w)+e^2g_\mu^{(2)}(p',p\,|\,w)+O(e^4)$$

gives the unique expression

$$g_\mu^{(0)}(p',p\,|\,w)=\frac{1}{\not p'-w}\gamma_\mu\frac{1}{\not p-w}.$$

[9]R. Delbourgo and R. Zhang, J. Phys. G (to be published).