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Chiral anomaly in the Schwinger-Symanzik formalism

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It is shown how chiral anomalies appear nonperturbatively in the Schwinger-Symanzik functional formalism of quantum field theory.

In this Brief Report we argue that the so-called path-integral approach to chiral anomalies pioneered by Fujikawa[1] has *a priori* nothing to do with path integrals, but is a nonperturbative or, more fitting, a nondiagrammatic method of treating the variation of the fermionic determinant under chiral transformations. The interpretation of the anomalies hereby appearing as Jacobians is specific to the formulation of field theory via path integrals, which, of course, is by no means a necessity. To illustrate this point, we derive the Adler-Bell-Jackiw anomaly[2] of QED (the generalization to more complicated theories would be straightforward) within the framework of the Schwinger-Symanzik formalism,[3,4] whose basic ingredients are functional differential equations for the generating functional of the n-point functions instead of the path integrals preferably used nowadays.

We start from the Lagrangian

$$\mathscr{L}=\bar{\psi}[i\not{D}(A)-m]\psi-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}+\bar{\eta}\psi+\bar{\psi}\eta+j_\mu A^\mu \tag{1}$$

with

$$D_\mu(A)\equiv\partial_\mu+ieA_\mu \tag{2}$$

and where the fermion and the photon fields are coupled to external sources $\bar{\eta}$, η, and j_μ, respectively. Next we perform an infinitesimal chiral transformation

$$\begin{aligned}\psi(x)&\to\tilde{\psi}(x)=e^{i\alpha(x)\gamma_5}\psi(x)\ ,\\ \bar{\psi}(x)&\to\tilde{\bar{\psi}}(x)=\bar{\psi}(x)e^{i\alpha(x)\gamma_5}\end{aligned} \tag{3}$$

on the spinor field and, thus, get a theory containing additional couplings to the external (infinitesimal) pseudoscalar field $\alpha(x)$:

$$\begin{aligned}\mathscr{L}_\alpha=&\ \bar{\psi}[i\not{D}(A)-m]\psi-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}-(\partial^\mu\alpha)\bar{\psi}\gamma_\mu\gamma_5\psi\\ &-2im\,\alpha\bar{\psi}\gamma_5\psi+\tilde{\bar{\eta}}\psi+\bar{\psi}\tilde{\eta}+j_\mu A^\mu+O(\alpha^2)\ ,\end{aligned} \tag{4}$$

where

$$\begin{aligned}\tilde{\eta}(x)&\equiv e^{i\alpha(x)\gamma_5}\eta(x)\ ,\\ \tilde{\bar{\eta}}(x)&\equiv\bar{\eta}(x)e^{i\alpha(x)\gamma_5}\ .\end{aligned} \tag{5}$$

The generating functional for the n-point functions of the theory described by $\mathscr{L}_\alpha$ is defined by

$$\hat{Z}[\eta,\bar{\eta},j_\mu;\alpha]=\left\langle 0\left|T\exp\left(i\int d^4x\,[\tilde{\bar{\eta}}\psi+\bar{\psi}\tilde{\eta}+j_\mu A^\mu-(\partial^\mu\alpha)\bar{\psi}\gamma_\mu\gamma_5\psi-2im\,\alpha\bar{\psi}\gamma_5\psi]\right)\right|0\right\rangle\ . \tag{6}$$

We now use $\hat{Z}$ to calculate the variation of transition matrix elements $\langle a|b\rangle^\alpha$ for arbitrary in states $|b\rangle$ and out states $|a\rangle$ under the change of the external parameter $\alpha(x)$; using the Lehmann-Symanzik-Zimmermann (LSZ) reduction formulas, one has in an obvious notation[5]

$$\begin{aligned}\frac{1}{i}\frac{\delta\langle a|b\rangle^\alpha}{\delta\alpha(w)}\bigg|_{\alpha=0}=&\ (\text{const})\int d^4x\cdots e^{ik\cdot x}\cdots\bar{u}(i\overrightarrow{\not{\partial}}-m)\cdots\bar{v}(i\overrightarrow{\not{\partial}}-m)\cdots\epsilon\overrightarrow{\Box}\cdots\\ &\times\frac{1}{i}\frac{\delta}{\delta j^\mu}\cdots\frac{-1}{i}\frac{\delta}{\delta\eta}\cdots\frac{1}{i}\frac{\delta}{\delta\bar{\eta}}\cdots\frac{1}{i}\frac{\delta\hat{Z}[\eta,\bar{\eta},j_\mu;\alpha]}{\delta\alpha(w)}\bigg|_{\alpha=0,\,\eta=\bar{\eta}=j_\mu=0}\\ &\times(-i\overleftarrow{\not{\partial}}-m)u\cdots(-i\overleftarrow{\not{\partial}}-m)v\cdots\overleftarrow{\Box}\epsilon\cdots\ .\end{aligned} \tag{7}$$

To construct $\hat{Z}$ explicitly, either the Schwinger or the Symanzik approach (for a review, see Fried[4]) can be used and in both cases the result reads (in matrix notation)

$$\begin{aligned}\hat{Z}[\eta,\bar{\eta},j_\mu;\alpha]=&\exp\left(i\frac{\delta}{\delta\tilde{\eta}}(-\partial^\mu\alpha\gamma_\mu\gamma_5-2im\,\alpha\gamma_5)\frac{\delta}{\delta\tilde{\bar{\eta}}}\right)\\ &\times\exp\left[-\frac{\delta}{\delta\tilde{\eta}}\left(e\gamma^\mu\frac{\delta}{\delta j^\mu}\right)\frac{\delta}{\delta\tilde{\bar{\eta}}}\right]\\ &\times Z_0[\tilde{\eta},\tilde{\bar{\eta}},j_\mu]\Big|_{\tilde{\eta}=e^{i\alpha\gamma_5}\eta,\ \tilde{\bar{\eta}}=\bar{\eta}e^{i\alpha\gamma_5}}\end{aligned} \tag{8}$$

with Z_0 the generating functional of the free $(e=0,\alpha=0)$ theory given by

$$Z_0[\tilde{\eta},\tilde{\bar{\eta}},j_\mu]=\exp\left(\frac{i}{2}jD_+j\right)\times\exp[-i\tilde{\bar{\eta}}(i\not{\partial}-m)^{-1}\tilde{\eta}]\det(i\not{\partial}-m)\ , \tag{9}$$

with D_+ denoting the photon propagator. Applying only the second exponential operator in (8) to Z_0 yields the usual QED functional Z_{QED}, which, by exploiting the identity[4]

$$\exp\left(-i\frac{\delta}{\delta\eta}A\frac{\delta}{\delta\bar{\eta}}\right)\exp(i\bar{\eta}B\eta)=\exp[i\bar{\eta}B(1+AB)^{-1}\eta]\times\det(1+AB) \tag{10}$$

can be written as

$$Z_{\rm QED}[\tilde{\eta},\tilde{\bar{\eta}},j_\mu]=\exp\left\{-i\tilde{\bar{\eta}}\left[i\not{D}\left(\frac{1}{i}\frac{\delta}{\delta j}\right)-m\right]^{-1}\tilde{\eta}\right\}\times\det\left[i\not{D}\left(\frac{1}{i}\frac{\delta}{\delta j}\right)-m\right]\exp\left(\frac{i}{2}jD_+j\right) . \quad (11)$$

Now the $A^\mu(x)$ field appearing in the covariant derivatives is replaced by the operator $(1/i)\delta/\delta j_\mu(x)$ acting on the last exponential. A more useful representation of $Z_{\rm QED}$ can be obtained by using another identity,[4] viz.,

$$F\left[\frac{1}{i}\frac{\delta}{\delta j}\right]\exp\left(\frac{i}{2}j\Delta j\right)=\exp\left(\frac{i}{2}j\Delta j\right)\exp\left(-\frac{i}{2}\frac{\delta}{\delta A}\Delta\frac{\delta}{\delta A}\right)F[A]\Big|_{A=\Delta j} , \quad (12)$$

which is valid for any sufficiently differentiable functional F. Equation (11) then reads

$$Z_{\rm QED}[\tilde{\eta},\tilde{\bar{\eta}},j_\mu]=\exp\left(\frac{i}{2}jD_+j\right)\exp\left(-\frac{i}{2}\frac{\delta}{\delta A}D_+\frac{\delta}{\delta A}\right)\times\exp\{-i\tilde{\bar{\eta}}[i\not{D}(A)-m]^{-1}\tilde{\eta}\}\times\det[i\not{D}(A)-m] , \quad (13)$$

with A^μ now being defined as $A=D_+j$. Inserting this in (8) and applying the first exponential operator with the help of (10) leads to the following form of $\hat{Z}$:

$$\hat{Z}[\eta,\bar{\eta},j_\mu;\alpha]=\exp\left(\frac{i}{2}jD_+j\right)\exp\left(-\frac{i}{2}\frac{\delta}{\delta A}D_+\frac{\delta}{\delta A}\right)\exp\{-i\tilde{\bar{\eta}}e^{-i\alpha\gamma_5}[i\not{D}(A)-m]^{-1}e^{-i\alpha\gamma_5}\tilde{\eta}\}\times\det[i\not{D}(A)-m-\partial^\mu\alpha\gamma_\mu\gamma_5-2im\alpha\gamma_5]\Big|_{\tilde{\eta}=e^{i\alpha\gamma_5}\eta,\ \tilde{\bar{\eta}}=\bar{\eta}e^{i\alpha\gamma_5}} . \quad (14)$$

Now the important point comes in. As we shall prove below, it holds to first order in α that ($^*F^{\mu\nu}$ is the dual field strength tensor)

$$\det[i\not{D}(A)-m-\partial^\mu\alpha\gamma_\mu\gamma_5-2im\,\alpha\gamma_5]=\exp\left(i\int d^4x\,\alpha(x)\frac{e^2}{8\pi^2}F_{\mu\nu}{}^*F^{\mu\nu}\right)\det[i\not{D}(A)-m] , \quad (15)$$

so that we may write

$$\begin{aligned}\hat{Z}[\eta,\bar{\eta},j_\mu;\alpha]&=\exp\left(\frac{i}{2}jD_+j\right)\exp\left(-\frac{i}{2}\frac{\delta}{\delta A}D_+\frac{\delta}{\delta A}\right)\exp\{-i\bar{\eta}[i\not{D}(A)-m]^{-1}\eta\}\\&\quad\times\exp\left(i\int d^4x\,\alpha(x)\frac{e^2}{8\pi^2}(F_{\mu\nu}{}^*F^{\mu\nu})[A(x)]\right)\det[i\not{D}(A)-m]\\&=\exp\left[i\int d^4x\,\alpha(x)\frac{e^2}{8\pi^2}(F_{\mu\nu}{}^*F^{\mu\nu})\left(\frac{1}{i}\frac{\delta}{\delta j(x)}\right)\right]\hat{Z}[\eta,\bar{\eta},j_\mu;\alpha=0] . \end{aligned}\quad (16)$$

To obtain the last line, (12) was used again. Now the derivative with respect to α is trivial to perform:

$$\frac{1}{i}\frac{\delta\hat{Z}[\eta,\bar{\eta},j^\mu;\alpha]}{\delta\alpha(w)}\Big|_{\alpha=0}=\frac{e^2}{8\pi^2}(F_{\mu\nu}{}^*F^{\mu\nu})\left(\frac{1}{i}\frac{\delta}{\delta j(w)}\right)\times\hat{Z}[\eta,\bar{\eta},j_\mu;\alpha=0] . \quad (17)$$

Inserting this in (7) yields

$$\frac{1}{i}\frac{\delta\langle a|b\rangle^\alpha}{\delta\alpha(w)}\Big|_{\alpha=0}=\left\langle a\left|\frac{e^2}{8\pi^2}(F_{\mu\nu}{}^*F^{\mu\nu})[A(w)]\right|b\right\rangle , \quad (18)$$

with the original meaning of $A_\mu(x)$ as an operator field. On the other hand, by explicitly differentiating Eq. (6) with respect to α, we observe that $(1/i)\delta/\delta\alpha(w)$ causes an insertion of the operator (after an integration by parts) $\partial^\mu(\bar{\psi}\gamma_\mu\gamma_5\psi)-2im(\bar{\psi}\gamma_5\psi)$ in any n-point function calculated from $\hat{Z}$. By LSZ reduction, one then gets

$$\frac{1}{i}\frac{\delta\langle a|b\rangle^\alpha}{\delta\alpha(w)}\Big|_{\alpha=0}=\langle a|\{\partial^\mu[\bar{\psi}(w)\gamma_\mu\gamma_5\psi(w)]-2im\bar{\psi}(w)\gamma_5\psi(w)\}|b\rangle . \quad (19)$$

Equating (18) and (19), we thus deduce that the operator relation

$$\partial^\mu(\bar{\psi}\gamma_\mu\gamma_5\psi)=2im\bar{\psi}\gamma_5\psi+\frac{e^2}{8\pi^2}F_{\mu\nu}{}^*F^{\mu\nu} \quad (20)$$

is satisfied between arbitrary states. This nonconservation law is one way to state the γ_5 anomaly; another one is the corresponding Ward-Takahashi identity, which could be derived by also taking derivatives with respect to η and $\bar{\eta}$ in (6).

Obviously, the only nontrivial part of the above derivation is Eq. (15), which we are now going to prove. It is clear that when representing the determinants appearing there as Gauss-type Grassmann integrals, the anomaly factor on the right-hand side corresponds to the Jacobian of the transformations (3), where ψ and $\bar{\psi}$ now denote independent variables of integration. In the present approach, however, this interpretation is not necessary.

To compute

$$J[\alpha]\equiv\frac{\det(i\not{D}-m)}{\det(i\not{D}-m-\partial^\mu\alpha\gamma_\mu\gamma_5-2im\,\alpha\gamma_5)} , \quad (21)$$

we note that $\det(\cdots)=\det[\gamma_5(\cdots)\gamma_5]$, so that (21) is equal to

$$J[\alpha]=\left\{\frac{\det(i\not{D}-m)\det(-i\not{D}-m)}{\det(i\not{D}-m-\partial^\mu\alpha\gamma_\mu\gamma_5-2im\alpha\gamma_5)\det(-i\not{D}-m+\partial^\mu\alpha\gamma_\mu\gamma_5-2im\alpha\gamma_5)}\right\}^{1/2} . \qquad (22)$$

Hence, one gets

$$\ln J[\alpha]=-\tfrac{1}{2}[\ln\det\Omega-\ln\det(\not{D}^2+m^2)] \qquad (23)$$

with

$$\Omega_z\equiv\not{D}_z^{\,2}+m^2+i\not{D}_z[\partial_z^\mu\alpha(z)\gamma_\mu\gamma_5-2im\alpha(z)\gamma_5]$$
$$+i[\partial_z^\mu\alpha(z)\gamma_\mu\gamma_5+2im\alpha(z)\gamma_5]\not{D}_z$$
$$+4im^2\alpha(z)+O(\alpha^2) . \qquad (24)$$

Because we are working only to first order in α, (23) can be replaced by [note $\Omega(\alpha=0)=\not{D}^2+m^2$]

$$\ln J[\alpha]=-\tfrac{1}{2}\int d^4w\,\alpha(w)\frac{\delta}{\delta\alpha(w)}\ln\det\Omega\bigg|_{\alpha=0}+O(\alpha^2) . \qquad (25)$$

The standard evaluation[1] of (25) is achieved by performing a Wick rotation to Euclidean space and introducing a complete set $\{\phi_n\}$ of eigenfunctions of the Hermitian operator $\not{D}^2+m^2$ in order to write

$$\frac{\delta}{\delta\alpha(w)}\ln\det\Omega\bigg|_{\alpha=0}=\frac{\delta}{\delta\alpha(w)}\operatorname{Tr}\ln\Omega\bigg|_{\alpha=0}$$
$$=\operatorname{Tr}\left[\Omega^{-1}\frac{\delta\Omega}{\delta\alpha(w)}\right]_{\alpha=0}$$
$$=\sum_n\int d^4z\,\phi_n^\dagger(z)\Omega_z^{-1}\frac{\delta\Omega_z}{\delta\alpha(w)}\bigg|_{\alpha=0}\phi_n(z)$$
$$=\sum_n\lambda_n^{-1}\int d^4z\,\phi_n^\dagger(z)\frac{\delta\Omega_z}{\delta\alpha(w)}\bigg|_{\alpha=0}\phi_n(z), \qquad (26)$$

where λ_n is the eigenvalue belonging to ϕ_n. Making use of the explicit form of Ω_z, this leads to

$$\ln J[\alpha]=-2i\int d^4w\,\alpha(w)\sum_n\phi_n^\dagger(w)\gamma_5\phi_n(w) . \qquad (27)$$

Apart from the fact that in our case the $\{\phi_n\}$ are eigenfunctions of $\not{D}^2+m^2$ (as opposed to $\not{D}$ in Ref. 1), (27) is a divergent expression of the same kind as in the works of Fujikawa;[1] if we regularize it by substituting

$$\sum_n\phi_n^\dagger\gamma_5\phi_n\rightarrow\lim_{M\rightarrow\infty}\sum_n\phi_n^\dagger\gamma_5e^{-\lambda_n/M^2}\phi_n \qquad (28)$$

we obtain in the well-known manner[1]

$$\ln J[\alpha]=-i\int d^4w\,\alpha(w)\frac{e^2}{8\pi^2}F_{\mu\nu}{}^*F^{\mu\nu} , \qquad (29)$$

which establishes (15).

A better way to proceed from (25) is to express the determinant of Ω in terms of its zeta function.[6] This method leads to the same anomaly (29) but produces finite results at every intermediate stage of the calculation and, thus, avoids artificial modifications like (28). (For this procedure to be applicable, the basic operator has to be positive for $\alpha=0$; that is why we work with $\not{D}^2+m^2$ instead of $\not{D}$.)

This completes the argument that the nonperturbative anomaly proof of Fujikawa is intrinsically a regularizing scheme for the fermionic determinant, which also appears in the older language of quantum field theory as originated by Schwinger and Symanzik.

It is, perhaps, not unfair to say that the regularization scheme used in the present paper—which is essentially also Fujikawa's—is similar to the proper-time method plus point splitting as introduced by Schwinger more than three decades ago[7] to derive the axial-vector anomaly. In this context it is also fitting to mention the dissident view of the authors of Ref. 8 regarding second-order radiative corrections to the triangle graph.

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