Brief Reports

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Quadratic Lagrangian for general relativity theory

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A quadratic Lagrangian for the Einstein theory of gravitation is proposed. It gives all that is obtained from Hilbert's (linear) Lagrangian and, in addition, the tracelessness of the Weyl conformal tensor.

In contrast to other field theories, and in spite of the great effort invested in it, Einstein's gravitational theory seems to resist attempts to quantize it. Perhaps the most conspicuous difference between Einstein's and the other field theories is the fact that the first is obtained from a linear (in the Riemann tensor) Lagrangian. It is therefore important to have a quadratic Lagrangian for the gravitational field with similar structure to those of the other fields.

Carmeli[1-3] has provided a quadratic Lagrangian for his $SL(2,C)$ gauge theory of gravitation. This theory, as is known, is equivalent to Einstein's general relativity. Its quadratic Lagrangian is invariant under the $SL(2,C)$ group and gives the gravitational field equations in the Newman-Penrose[4] form. Herrera[5,6] has introduced some modifications to Carmeli's Lagrangian.

In this paper we propose a quadratic Lagrangian, inspired by Carmeli's work, which gives the Einstein field equations in their traditional tensorial form.

Let us have a Riemannian four-dimensional space with metric $g^{\mu\nu}$, affine connections $\Gamma^{\alpha}_{\beta\gamma}=\Gamma^{\alpha}_{\gamma\beta}$, a geometrical tensor satisfying $C_{\alpha\beta\gamma\delta}=-C_{\alpha\beta\delta\gamma}$, and a matter tensor $T_{\alpha\beta\gamma\delta}$ defined by

$$T_{\alpha\beta\gamma\delta}=\tfrac{1}{2}(g_{\alpha\gamma}T_{\beta\delta}-g_{\alpha\delta}T_{\beta\gamma}-g_{\beta\gamma}T_{\alpha\delta}+g_{\beta\delta}T_{\alpha\gamma}) + \tfrac{1}{3}(g_{\beta\gamma}g_{\alpha\delta}-g_{\alpha\gamma}g_{\beta\delta})T, \tag{1}$$

where $T_{\alpha\beta}$ is the energy-momentum tensor and T is its trace. *A priori* no relations are assumed to exist among these four fundamental elements of the theory. As usual, the curvature tensor is defined by means of the affine connections and their first derivatives,

$$R^{\alpha}{}_{\beta\gamma\delta}=\Gamma^{\alpha}_{\beta\delta,\gamma}-\Gamma^{\alpha}_{\beta\gamma,\delta}+\Gamma^{\sigma}_{\beta\delta}\Gamma^{\alpha}_{\sigma\gamma}-\Gamma^{\sigma}_{\beta\gamma}\Gamma^{\alpha}_{\sigma\delta} \ . \tag{2}$$

Several scalar densities can now be defined from those four tensors. We define six such densities to build up our Lagrangian:

$$\mathscr{L}_{\rm I}=\sqrt{-g}\,C_{\alpha\beta\gamma\delta}C_{\mu\nu\rho\sigma}g^{\alpha\gamma}g^{\mu\rho}g^{\beta\nu}g^{\delta\sigma} \ , \tag{3}$$

$$\mathscr{L}_{\rm II}=\tfrac{1}{4}\,\varepsilon^{\gamma\delta\rho\sigma}C_{\alpha\beta\gamma\delta}C_{\mu\nu\rho\sigma}g^{\alpha\nu}g^{\beta\mu} \ , \tag{4}$$

$$\mathscr{L}_{\rm III}=-\tfrac{1}{2}\,\varepsilon^{\gamma\delta\rho\sigma}C_{\alpha\beta\gamma\delta}R^{\beta}{}_{\nu\rho\sigma}g^{\alpha\nu} \ , \tag{5}$$

$$\mathscr{L}_{\rm IV}=\frac{\kappa^2}{4}\,\varepsilon^{\gamma\delta\rho\sigma}T_{\alpha\beta\gamma\delta}T_{\mu\nu\rho\sigma}g^{\alpha\nu}g^{\beta\mu} \ , \tag{6}$$

$$\mathscr{L}_{\rm V}=\frac{\kappa}{2}\,\varepsilon^{\gamma\delta\rho\sigma}T_{\alpha\beta\gamma\delta}C_{\mu\nu\rho\sigma}g^{\alpha\nu}g^{\beta\mu} \ , \tag{7}$$

$$\mathscr{L}_{\rm VI}=-\frac{\kappa}{2}\,\varepsilon^{\gamma\delta\rho\sigma}T_{\alpha\beta\gamma\delta}R^{\beta}{}_{\nu\rho\sigma}g^{\alpha\nu} \ . \tag{8}$$

Here $\varepsilon^{\gamma\delta\rho\sigma}$ is the Levi-Civita skew-symmetric tensor density with values $+1$ and -1, depending upon whether $\gamma\delta\rho\sigma$ is an even or an odd permutation of 0123 and zero otherwise, and κ is the Einstein gravitational constant. In Eqs. (5) and (8) $R^{\alpha}{}_{\beta\gamma\delta}$ is understood to be given in terms of $\Gamma^{\mu}_{\alpha\beta}$ and $\Gamma^{\mu}_{\alpha\beta,\gamma}$ according to Eq. (2).

Then the Lagrangian density is defined by

$$\mathscr{L}(g^{\alpha\beta};\Gamma^{\mu}_{\alpha\beta},\Gamma^{\mu}_{\alpha\beta,\gamma};C_{\alpha\beta\gamma\delta};T_{\alpha\beta\gamma\delta})=\sum_{i={\rm I}}^{\rm VI}\mathscr{L}_i \ , \tag{9}$$

and is considered as a function of the independent dynamical variables $g^{\alpha\beta}$, $\Gamma^{\mu}_{\alpha\beta}$, $\Gamma^{\mu}_{\alpha\beta,\gamma}$, $C_{\alpha\beta\gamma\delta}$, and $T_{\alpha\beta\gamma\delta}$. It is built of three parts. (i) The first three terms constitute the matter-free Lagrangian; (ii) the fourth represents the material term; and (iii) the last two are the interaction between gravitation and matter with the Einstein gravitational constant κ as the coupling constant.

Now, performing the usual variational procedure with respect to the components of the matter tensor $T_{\alpha\beta\gamma\delta}$, we obtain

$$C^{\alpha}{}_{\beta\gamma\delta}+\kappa T^{\alpha}{}_{\beta\gamma\delta}=R^{\alpha}{}_{\beta\gamma\delta} \ , \tag{10}$$

and, from the variation of the components of the tensor

$C_{\alpha\beta\gamma\delta}$ and using Eq. (10), it follows that

$$C^{\alpha}{}_{\beta\alpha\delta}=0 \ . \tag{11}$$

Therefore, using the definition (1), we find

$$R_{\alpha\beta}=\kappa(T_{\alpha\beta}-\tfrac{1}{2}g_{\alpha\beta}T) \ , \tag{12}$$

where $R_{\alpha\beta}=R^{\rho}{}_{\alpha\rho\beta}$ is the Ricci tensor. Furthermore, one sees from Eqs. (10) and (11) that $C^{\alpha}{}_{\beta\gamma\delta}$ is the Weyl tensor.

From the variation of the components of the affine connections $\Gamma^{\alpha}_{\beta\gamma}$ one has

$$\varepsilon^{\alpha\beta\gamma\delta}(R_{\mu\nu\alpha\beta}g^{\mu\rho})_{;\delta}=0 \ . \tag{13}$$

But, as is well known, from the definition (2) of the curvature tensor it follows that

$$\varepsilon^{\alpha\beta\gamma\delta}R_{\mu\nu\alpha\beta;\delta}=0 \ , \tag{14}$$

which are the Bianchi identities. These identities, along with Eq. (13), give

$$\varepsilon^{\alpha\beta\gamma\delta}R_{\mu\nu\alpha\beta}g^{\mu\rho}{}_{;\delta}=0 \ . \tag{15}$$

Hence, for any given ρ we have a set of 16 homogeneous equations for the same number of quantities $g^{\mu\rho}{}_{;\delta}$. Its coefficient determinant is a homogeneous expression of the 16th order in the components of the Riemann tensor which, in general, is not equal to zero. Consequently,

$$g^{\mu\rho}{}_{;\delta}=0 \ , \tag{16}$$

and

$$\Gamma^{\alpha}_{\beta\gamma}=\tfrac{1}{2}g^{\alpha\rho}(g_{\rho\beta,\gamma}+g_{\rho\gamma,\beta}-g_{\beta\gamma,\rho}) \ . \tag{17}$$

Using Eq. (17) in Eq. (12) we obtain the standard differential form of the Einstein field equations.

No additional information is obtained from the variation of the metric tensor components $g^{\mu\nu}$. This is so because, through Eqs. (10) and (11), our Lagrangian becomes

$$\mathscr{L}=-\tfrac{1}{4}\varepsilon^{\alpha\beta\gamma\delta}R^{\nu}{}_{\mu\alpha\beta}R^{\mu}{}_{\nu\gamma\delta} \ , \tag{18}$$

and as a consequence of Eq. (17), no dynamical law can be derived from it.[7]

In conclusion, the Lagrangian density (9) gives all that is available from that of Hilbert. In addition, the traceless nature of the Weyl conformal tensor $C_{\alpha\beta\gamma\delta}$ is also obtained from it. Our Lagrangian has the advantage that it is more similar, in its structure, to the Lagrangians of the other fields, such as the electromagnetic field. It is worthwhile noticing that it is, in its extremum value, given by Eq. (18), invariant under a scale change,[7] namely, under the substitution

$$g^{\mu\nu}\to g'^{\mu\nu}=\lambda g^{\mu\nu} \ , \tag{19}$$

with a constant λ.

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