Axial anomaly in three dimensions and planar fermions

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(Received 8 March 1984)

The origin and implications of the axial anomaly in three dimensions for fermions are discussed. Since the zero eigenvalue of the Dirac equation makes the partition function vanish, it is a singular point. The presence of the zero eigenvalue in the Dirac equation makes the functional space nontrivial topologically. An anomaly term is induced and a consistency condition emerges. The Hall conductivity is quantized and parity-violating light scattering occurs.

I. INTRODUCTION

In (2 + 1)-dimensional space-time, the physical properties of a system can be very different from those in ordinary (3 + 1)-dimensional space-time. The structure of the chiral anomaly is one of those properties which are sensitive to the dimensionality of space-time.

The chiral anomaly played an important role first in the calculation of the two-photon decay rate of the neutral π meson.[1] Later, its symmetry-breaking nature was realized by Adler[2] and by Bell and Jackiw.[3] These calculations were done in 3 + 1 dimensions.

In one of the author's recent papers,[4] it was shown that the origin of the anomaly is due to the path-dependent phase factor produced by the zero-eigenvalue solution of the Dirac equation. Nonrenormalization of the anomaly is easily understood.

In 2 + 1 dimensions, the anomaly exists in the ground-state energy.[5,6] This term violates parity invariance as well as time-reversal invariance in first order of the quantum-electrodynamic (QED) coupling α. The connection of this term with the quantized Hall effect observed in a two-dimensional metal was pointed out.[6] The observed quantized value of the conductivity agrees with the theory's prediction.

The real dimensionality of space-time is 3 + 1 dimensions. So an exact (2 + 1)-dimensional space-time cannot be built. It is achieved only approximately. There are many cases of approximate 2 + 1 dimensionality. They include the surface of a metal, semiconductor, etc.; generally, a boundary of two different materials has planar structure. In these cases, microscopically there may be an energy gap in an orthogonal direction to the surface. If the energy gap along the orthogonal direction is infinite, only one state, the ground state, in the orthogonal direction is allowed. Thus the system can be viewed as that of (2 + 1)-dimensional space-time. The finite energy gap in the orthogonal direction, on the other hand, causes the number of independent fields in (2 + 1)-dimensional space-time to be different from unity. This number, which is called the number of flavors in high-energy physics, depends on the value of the Fermi energy. As the Fermi energy becomes larger, the number of flavors becomes larger. This may be seen as the steplike structure of Hall conductivity and will be discussed later.

Since the effective Hamiltonian induced by the chiral anomaly not only breaks parity invariance, but also breaks time-reversal invariance in QED, the physical implications of this term may be tremendous.

In the low-energy region, much lower than the particle's mass, it is generally acceptable to use the Schrödinger equation for the equation of motion for fermions. But there is a special case where a Dirac-type equation is necessary to represent the system even in that energy region.

If two energy levels are involved in the motion, a linear term with respect to the momentum can exist in the system's energy. An energy-gap region of the two levels is an example of this. The equation of motion in this case becomes equivalent to the Dirac equation except that the energy-momentum dispersion is different from the relativistic form. The effective Dirac equation gives

$$E=(m^2+c^2P_x{}^2+d^2P_y{}^2+eP_xP_y)^{1/2}$$

with constant factors c, d, and e, contrary to the relativistic equation.

We study fermionic theory in the presence of external electromagnetic fields and its physical implication for surface phenomena. The existence of the zero-energy solution in the Dirac equation with a uniform external magnetic field and its nonexistence in the Klein-Gordon (Schrödinger) equation is shown in Sec. II. The second-quantized theory is discussed and the anomaly-induced interaction is obtained in Sec. III.

Physical implications of the quantized Hall effect, light emission and absorption by density fluctuations, and parity-violating light reflection and transmission by a surface are discussed in Sec. IV. Dimensional reduction from 3 + 1 dimensions to 2 + 1 dimensions is also made in this section. A summary is given in Sec. V.

II. SOLVING THE DIRAC AND SCHRÖDINGER EQUATIONS WITH A UNIFORM MAGNETIC FIELD

We solve the equations of motion for a fermion in (2 + 1)-dimensional space-time in the presence of a uniform magnetic field.[7] The Dirac equation and the Klein-Gordon (Schrödinger) equation are studied and the two results are compared. A remarkable difference in the eigenvalues will be shown to exist. The Dirac equation has a zero eigenvalue, but the Klein-Gordon (Schrödinger) equation does not have such a zero-energy eigenvalue.

The existence of the normalized zero eigenvalue solution to the Euclidean Dirac equation

$$\gamma^0\left[\frac{\partial}{\partial t}-h\right]\psi=0\ ,\qquad h=eA_0+\gamma^0\gamma^i(i\partial_i+eA_i) \tag{2.1}$$

says that the eigenvalue of the Hamiltonian h should change sign. Conversely, if h has an eigenvalue that changes sign in the adiabatic approximation, there could be a normalized zero-eigenvalue solution to the Euclidean Dirac equation. Thus it is important to study the zero-energy solution of the Dirac Hamiltonian.

Let us first study the following Hamiltonian Dirac equation:[8]

$$\gamma^j(i\hbar\partial_j+eA_j)\psi=\gamma^0E\psi\ ,\qquad \{\gamma^\mu,\gamma^\nu\}=2g^{\mu\nu}\ . \tag{2.2}$$

A representation for the γ^μ matrices in 2 + 1 dimensions is

$$\gamma^0=\sigma^3\ ,\quad \gamma^1=i\sigma^1\ ,\quad \gamma^2=i\sigma^2\ , \tag{2.3}$$

and satisfies

$$\gamma^0=-i\gamma^1\gamma^2\ ,\quad \gamma^1=i\gamma^2\gamma^0,\quad \gamma^2=-i\gamma^0\gamma^1\ . \tag{2.4}$$

We use the following simple form of the vector potential which gives a constant magnetic field:

$$A_1=yH\ ,\quad A_2=0\ . \tag{2.5}$$

Then the Hamiltonian Dirac equation (2.2) becomes

$$\begin{bmatrix} 0 & -\hbar\partial_1+i\hbar\partial_2+ieA_1+eA_2 \\ \hbar\partial_1+i\hbar\partial_2-ieA_1+eA_2 & 0 \end{bmatrix}=E\psi\ . \tag{2.6}$$

Now it is easy to see that the solutions of the equation are

$$\psi_0=N_0e^{-\alpha^2(y-y_0)^2/2}H_0(\alpha(y-y_0))\begin{bmatrix}1\\0\end{bmatrix}e^{ikx}\ , \tag{2.7a}$$

$$\psi_+=e^{-\alpha^2(y-y_0)^2/2}\begin{bmatrix}N_nH_n(\alpha(y-y_0))\\ iN_{n-1}H_{n-1}(\alpha(y-y_0))\end{bmatrix}e^{ikx}\ , \tag{2.7b}$$

$$\psi_-=e^{-\alpha^2(y-y_0)^2/2}\begin{bmatrix}N_nH_n(\alpha(y-y_0))\\ -iN_{n-1}H_{n-1}(\alpha(y-y_0))\end{bmatrix}e^{ikx}\ , \tag{2.7c}$$

$$y_0=\hbar k/eH\ ,$$

where N_n are constants and $\alpha=\sqrt{eH}$. ψ_0, ψ_+, and ψ_- have the discrete eigenvalues

$$E_0=0\ , \tag{2.8a}$$

$$E_{+,n}=(2\hbar eHn)^{1/2} \tag{2.8b}$$

$$(n\geq 1)\ .$$

$$E_{-,n}=-(2\hbar eHn)^{1/2} \tag{2.8c}$$

The states are generally called Landau levels.

It should be remarked that there is a zero-energy solution (2.7a) to Eq. (2.2) for arbitrary values of H. This zero-energy solution is responsible for the phenomena which is related to the degeneracy. The anomaly is one of those interesting things.

Second, we obtain energy eigenvalues for a scalar field. The equation (2.6) in this case is replaced by

$$[-\hbar^2\partial_y{}^2+e^2H^2(y-y_0)^2]\phi=E^2\phi\ . \tag{2.9}$$

The eigenvalues of the above equation are

$$E^2=\hbar eH(2n+1)\ . \tag{2.10}$$

There is a positive-definite zero-point oscillation eH, and no zero-eigenvalue solution. Degeneracy of the states does not occur.

So far massless particles have been discussed. The massive case is slightly different, since a nonzero energy gap exists between the positive-energy values and the negative-energy values. The Hamiltonian Dirac equation (2.6), for the massive case, becomes

$$\begin{bmatrix} m & -\hbar\partial_1+i\hbar\partial_2+ieA_1+eA_2 \\ \hbar\partial_1+i\hbar\partial_2-ieA_1+eA_2 & m \end{bmatrix}\psi=E\psi\ . \tag{2.11}$$

The y-dependent part of the eigenfunctions and their eigenvalues are

$$\psi_0=\begin{bmatrix}|0\rangle\\0\end{bmatrix},\quad E_0=m\ , \tag{2.12a}$$

$$\psi_+=N\begin{bmatrix}|n\rangle\\ \dfrac{i(2e\hbar Hn)^{1/2}}{(m^2+2e\hbar Hn)^{1/2}+m}\,|n-1\rangle\end{bmatrix},\quad E_+=(m^2+2\hbar neH)^{1/2}\ , \tag{2.12b}$$

$$\psi_-=N'\begin{bmatrix} |n\rangle \\ \dfrac{i(2e\hbar Hn)^{1/2}}{m-(m^2+2e\hbar Hn)^{1/2}}\,|n-1\rangle \end{bmatrix},\quad E_-=-(m^2+2\hbar neH)^{1/2}\,, \tag{2.12c}$$

where n is a positive integer. N and N' are normalization factors, and $|n\rangle$ are harmonic-oscillator eigenstates.

The zero-energy solution ψ_0 to the massless equation is also interpreted as the eigenfunction of the massive Dirac Hamiltonian with the minimum positive-energy eigenvalue m, since ψ_0 is the eigenstate of γ_0 with positive eigenvalue 1. There is no counterpart to this solution which possesses the negative value $-m$. Thus there is an asymmetry in the energy spectrum. For other eigenvalues, solutions are always paired and there is no asymmetry between positive and negative energies.

The zero-energy solution ψ_0, which was responsible for the anomaly in the divergence of the axial-vector current in two dimensions now gives the asymmetry of the spectrum. This asymmetry will be shown to be responsible for the nonzero-induced current in 2 + 1 dimensions.

III. SECOND-QUANTIZED THEORY AND CHIRAL ANOMALY

In this section, we study many-body problems. The fields are quantized by the path-integral method. The canonical method is also convenient for a fermion theory and is used sometimes.

We investigate the theory described by the following Lagrangian density in 2 + 1 dimension with external electromagnetic fields:

$$\mathscr{L}=\bar\psi\gamma^\mu(i\hbar\partial_\mu+eA_\mu)\psi-m\bar\psi\psi\,. \tag{3.1}$$

In the presence of a nonzero fermion density, a chemical potential term $\mu\psi^\dagger\psi$ is added to the Lagrangian density. The vector potential eA_0 in Eq. (3.1) is replaced with $eA_0+\mu$ then. Thus it is sufficient to study Eq. (3.1) regardless of whether the density is finite or zero.

Both quantum field theory and statistical mechanics are formulated with Feynman's path-integral method. The metric is Minkowski in quantum field theory and is Euclidean in statistical mechanics. It is well accepted that analytic continuation from one to the other can be made without any problem. Although our argument is mainly for quantum field theory, it may be applied also to statistical mechanics.

The path-integral representation of the Green's function (or of expectation values) in Euclidean space is given by[9]

$$\begin{aligned}&\frac{1}{Z}\int\prod d\psi\, d\bar\psi\, e^{-\int\mathscr{L}\,d\mathbf{x}\,dt}O(\psi,\bar\psi)\,,\\ &Z=\int\prod d\psi\, d\bar\psi\, e^{-\int\mathscr{L}\,d\mathbf{x}\,dt}\,,\end{aligned} \tag{3.2}$$

where $O(\psi,\bar\psi)$ is a function of ψ and $\bar\psi$. To know the property of

$$Z=\int\prod d\psi\, d\bar\psi\, e^{-\int\mathscr{L}\,d\mathbf{x}\,dt} \tag{3.3}$$

is important. The corresponding quantity and its logarithm are called, respectively, the partition function and the free energy in statistical mechanics. Let us use these terms here.

It is useful to expand the field in terms of a complete set of functions:

$$\begin{aligned}\psi&=\sum_n a_n(t)\phi_n(\mathbf{x})\,,\\ \bar\psi&=\sum_n b_n(t)\phi_n^\dagger(\mathbf{x})\,,\end{aligned} \tag{3.4}$$

where

$$\begin{aligned}\not{D}\phi_n&\equiv[\gamma^0A^0+\gamma^j(i\hbar\partial_j+eA_j)]\phi_n(\mathbf{x})\\ &=i\lambda_n\phi_n(\mathbf{x})\,,\\ \phi_n^\dagger(\mathbf{x})&[\gamma^0A^0+\gamma^j(-i\hbar\overleftarrow{\partial}^j+eA_j)]=i\lambda_n\phi_n^\dagger(\mathbf{x})\,.\end{aligned} \tag{3.5}$$

We assume that A_μ do not depend on t, and derive a consistency condition for a magnetic flux. Using the above expansion, Z is written as

$$Z=\int\prod da_n(t)db_m(t)\times\exp\left[-\int dt\sum(b_nc_{nm}a_m-i\lambda_nb_na_n-mb_na_n)\right]\,, \tag{3.6}$$

where c_{nm} is defined by

$$c_{nm}=\int d^2x\,\phi_n^\dagger\gamma_0\phi_m\,. \tag{3.7}$$

The variables $a_n(t)$, $b_n(t)$ are decomposed further by a complete set of the variable t,

$$\begin{aligned}a_{n(t)}&=\sum_m a_n{}^mN_me^{iw_mt}\,,\\ b_n(t)&=\sum_m b_n{}^mN_me^{-iw_mt}\,,\end{aligned}$$

where N_m are the normalization factors and w_n are determined by a boundary condition. W_0 is set at zero.

Now let us study a transformation

$$\begin{aligned}&\int dt\,\psi(\mathbf{x},t)\to e^{i\gamma_0\alpha(\mathbf{x})}\int dt\,\psi(\mathbf{x},t)\,,\\ &\int dt\,\bar\psi(\mathbf{x},t)\to\int dt\,\bar\psi(\mathbf{x},t)e^{i\gamma_0\alpha(\mathbf{x})}\,,\end{aligned} \tag{3.8}$$

and obtain the anomaly term which is connected with this transformation. The time-independent components in $(a_n(t),\,b_n(t))$, $(\bar a_n,\bar b_n)[=(a_n^0,b_n^0)]$ are transformed as

$$\begin{aligned}\bar a_n&\to\sum_m d_{nm}\bar a_m\,,\\ \bar b_n&\to\sum_m d_{nm}\bar b_m\,,\end{aligned}$$

where

$$d_{nm}=(\phi_n^\dagger e^{i\gamma_0\alpha(\mathbf{x})}\phi_m)\,.$$

The Z in Eq. (3.6) is decomposed into

$$Z = Z_0 Z' , \tag{3.9}$$

where

$$Z_0 = \int \prod d\bar{a}_n d\bar{b}_m \exp\left[\sum (i\lambda_n \bar{b}_n \bar{a}_n + m\bar{b}_n \bar{a}_m)\right] , \tag{3.10}$$

and Z' does not depend on $(\bar{b}_n, \bar{a}_n)$.

We rewrite Z_0, using transformed variables as

$$\begin{aligned} Z_0 &= \int \prod d\bar{a}'_n d\bar{b}'_m \exp\left[\sum (i\lambda_n \bar{b}'_n \bar{a}'_n + \bar{m}\, \bar{b}'_n \bar{a}'_n)\right] \\ &= \int \prod d\bar{a}_n d\bar{b}_m (\det d^2) \exp\Big[\sum (i\bar{b}_{n'} d_{n'n} \lambda_n d_{nm'} \bar{a}_{m'} \\ &\qquad + m\bar{b}_m \bar{a}_n)\Big] \\ &= \int \prod d\bar{a}_n d\bar{b}_m (\det d^2) \exp\Big[\sum [\bar{b}_{n'} (\phi^\dagger_{n'} e^{i\gamma_0 \alpha} \not{D} e^{i\gamma_0 x} \phi_{m'}) a_{m'} \\ &\qquad + m\bar{b}_m \bar{a}_n]\Big] . \end{aligned} \tag{3.11}$$

We make another transformation from $(\bar{a}_n, \bar{b}_n)$ to $(\bar{a}''_n, \bar{b}''_n)$ in such a way that the path dependence separates in Z_0 in the following way:

$$\begin{aligned} \sum \bar{a}_n \phi_n &= \sum \bar{a}''_n \phi''_n , \\ \sum \bar{b}_n \phi^\dagger_n &= \sum \bar{b}''_n \phi^{\dagger\prime\prime}_n , \end{aligned} \tag{3.12}$$

where

$$\begin{aligned} e^{i\gamma_0 \alpha} \not{D} e^{i\gamma_0 \alpha} \phi''_n &= i\lambda''_n \phi''_n , \\ \phi^{\dagger\prime\prime}_n e^{i\gamma_0 \alpha} \overleftarrow{\not{D}} e^{i\gamma_0 \alpha} &= i\lambda''_n \phi^{\dagger\prime\prime}_n . \end{aligned} \tag{3.13}$$

Then Z_0 becomes

$$\begin{aligned} Z_0 = \int \prod d\bar{a}''_n d\bar{b}''_m \det d^2 \det d' d'' \\ \times \exp\left[\sum (\bar{b}''_n i\lambda''_n \bar{a}''_n + m\bar{b}''_n \bar{a}''_n)\right] , \end{aligned} \tag{3.14}$$

where

$$(d')_{nm} = \int d^2x\, \phi^{\dagger\prime\prime}_m \phi_m , \quad (d'')_{nm} = \int d^2x\, \phi^\dagger_n \phi''_m .$$

For infinitesimal $\alpha(x)$, we have

$$\begin{aligned} \det d^2 &= 1 + 2i \sum_n \phi^{\dagger\prime}_n \gamma_0 \alpha(x) \phi'_n , \\ \det d' d'' &= 1 . \end{aligned} \tag{3.15}$$

Thus we have

$$\begin{aligned} Z_0 &= \exp\left[2i \sum_n [\phi^{\dagger\prime}_n \gamma_0 \alpha(x) \phi'_n]\right] \\ &\quad \times \int \prod d\bar{a}''_n d\bar{b}''_n \exp\left[\sum (i\bar{b}''_n \lambda''_n \bar{a}''_n + m\bar{b}''_n \bar{a}''_n)\right] , \\ W_0 &= -2i \sum_n [\phi^{\dagger\prime}_n \gamma_0 \alpha(x) \phi'_n] + W'_0 , \end{aligned} \tag{3.16}$$

where the following definition was used:

$$W'_0 = -\ln \int \prod d\bar{a}''_n d\bar{b}''_n \times \exp\left[\sum (i\bar{b}''_n \lambda''_n \bar{a}''_n + m\bar{b}''_n \bar{a}''_n)\right] . \tag{3.17}$$

The term

$$\sum_n [\phi^{\dagger\prime}_n \gamma_0 \alpha(x) \phi'_n]$$

is calculated by the prescription given by Fujikawa,[10]

$$\begin{aligned} \sum_n (\phi^{\dagger\prime}_n \gamma_0 \alpha(x) \phi'_n) &= \lim_{\text{reg } M\to\infty} \sum_n (\phi^{\dagger\prime}_n \gamma_0 \alpha(x) e^{-\lambda_n{}^2/M^2} \phi'_n) \\ &= \lim_{M\to\infty} \sum_n (\phi^{\dagger\prime}_n \gamma_0 \alpha(x) e^{-\not{D}^2/M^2} \phi'_n) \\ &= \lim_{M\to\infty} \sum_r (e^{-ikx} \gamma_0 \alpha(x) e^{-\not{D}^2/M^2} e^{ikx}) \\ &= \lim_{M\to\infty} \frac{1}{(2\pi)^2} \int d^2k\, e^{-(k+A)^2/M^2} \mathrm{Tr}\left\{\left[1 + \tfrac{1}{2}[\gamma_1 \gamma_2] \frac{F_{12}}{M^2} + O\left[\frac{1}{M^4}\right]\right] \gamma_0 \alpha(x)\right\} \\ &= \int d\mathbf{x}\, \alpha(x) \frac{e}{h} F_{12}(\mathbf{x}) . \end{aligned} \tag{3.18}$$

In Eq. (3.16), W'_0 is a local functional of variables but the first term is not. We assign the two nearby points P and P' in Fig. 1 to the position of variables before the infinitesimal transformation and after the transformation specified by Eq. (3.16). W'_0 is defined locally on P', but the first term is defined on both points P and P'. Thus the first term depends on the path.

A finite transformation is expressed by a multiplication of the successive infinitesimal transformations. Then we have

$$W_0 = -2i \int d\mathbf{x}\, \alpha(\mathbf{x}) \frac{e}{h} F_{12}(\mathbf{x}) + W'_0 , \tag{3.19}$$

where $\alpha(x)$ is finite and W'_0 is a local functional. By a transformation along a closed path, which is described in Fig. 1, $\alpha(x)$ becomes 2π and Z'_0 must agree to Z_0. Otherwise Z_0 vanishes due to destructive interference. Thus the path-dependent term in Eq. (3.30) is

$$\begin{aligned} 2\int d\mathbf{x}\, \alpha(\mathbf{x}) \frac{e}{h} F_{12}(\mathbf{x}) \big|_{\alpha(\mathbf{x})=2\pi} &= 2\pi \times \text{integer} , \\ \int d\mathbf{x}\, F_{12}(\mathbf{x}) &= \frac{1}{2} \frac{h}{e} \times \text{integer} . \end{aligned} \tag{3.20}$$

The singularities inside the closed path, which are described by crosses in Fig. 1, give these nonzero values. Equation (3.20) is a condition of a magnetic-flux quantization.[11]

Next we study a change of action which is produced by an adiabatic change of A_μ. It is convenient to calculate this in the Euclidean metric space. Using the Euclidean Dirac operator

$$\not{D}=\gamma^0(i\hbar\partial_0+eA_0)+\gamma^i(i\hbar\partial_i+eA_i)\ , \tag{3.21}$$

which is anti-Hermitian, a complete set is defined as

$$\not{D}\phi_n=i\lambda_n\phi_n\ . \tag{3.22}$$

Then we can represent $\not{D}+m$ as a matrix:

$$(\not{D}+m)_{lm}=(i\lambda_l+m)\delta_{lm}\ . \tag{3.23}$$

Then Z becomes

$$Z=\mathrm{Det}(\not{D}+m)=\prod_l(i\lambda_l+m)\ . \tag{3.24}$$

A small change of A_μ leads Z to be changed to Z', which is given by

$$\begin{aligned}Z'&=\mathrm{Det}(\not{D}+\gamma^\mu\delta A_\mu+m)\\&=\mathrm{Det}(\not{D}+m)\exp\left[\mathrm{tr}\ln\left[1+\frac{1}{\not{D}+m}\gamma^\mu\delta A_\mu\right]\right]\\&=Z\left[1+\mathrm{tr}\left[\frac{1}{\not{D}+m}\gamma^\mu\delta A_\mu\right]\right]\ .\end{aligned} \tag{3.25}$$

Thus we have[12]

$$\begin{aligned}\delta\ln Z&=\mathrm{tr}\left[\frac{1}{\not{D}+m}\gamma^\mu\delta A_\mu\right]\\&=\mathrm{tr}\left[(\not{D}+m)\frac{1}{(\not{D}+m)(\not{D}-m)}\gamma^\mu\delta A_\mu\right]\\&=\mathrm{tr}\left[(\not{D}-m)\frac{1}{\not{D}^2-m^2}\gamma^\mu\delta A_\mu\right]\\&=\mathrm{tr}\left[(\not{D}-m)(-)\int_0^\infty ds\,e^{(\not{D}^2-m^2)s}\gamma^\mu\delta A_\mu\right]\\&=-\int_0^\infty ds\,e^{-m^2s}\frac{1}{(2\pi)^3}\\&\quad\times\int dk\{e^{-ikx}\mathrm{tr}[(\not{D}-m)e^{\not{D}^2s}\gamma^\mu\delta A_\mu]e^{kx}\}\\&=-\int_0^\infty ds\,e^{-m^2s}\frac{1}{(2\pi)^3}\\&\quad\times\int dk\,\mathrm{tr}[(-m+\not{D})\\&\qquad\times e^{-(k+A)^2s}(1+\tfrac{1}{2}s[\not{D}\not{D}]+\cdots)\gamma^\mu\delta A_\mu]\\&=\frac{1}{4\pi\hbar}\epsilon^{\mu\nu\rho}\delta A_\mu F_{\nu\rho}+O\left[\frac{1}{m}\right]\delta A_\mu\ .\end{aligned} \tag{3.26}$$

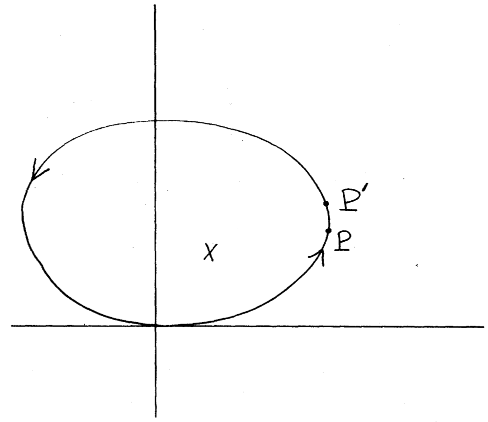

FIG. 1. The solid line shows the path along which the chiral transformation is defined. P and P' are the positions in the internal space before and after the transformation. The crosses show the singularities.

The induced current density is derived from the above equation:

$$\begin{aligned}j_\mu&=\frac{\delta}{\delta A_\mu}\ln Z\\&=\frac{e^2}{4\pi\hbar}\epsilon^{\mu\nu\rho}F^{\nu\rho}+O\left[\frac{1}{m}\right]\ .\end{aligned} \tag{3.27}$$

The first term in the above equation[13] is independent of mass m. The second term, on the other hand, depends on the mass m and vanishes in the large-mass limit.

In the previous section, it was shown that the asymmetry between the positive-energy solution and the negative-energy solution exists in the Dirac equation (2.11). Since the solution, Eq. (2.12a), is the eigenstate of γ_0, there is an asymmetry in the electric charge. This electric charge can be seen as the induced charge density j_0 in Eq. (3.27). Similarly, the induced current density j_x can be understood from the asymmetry of the solution of Dirac equations which are obtained by replacing (t,y) with (x,y) in Eq. (2.11).

We present now a direct calculation of j_x by using the complete set of the eigenfunctions of

$$\sum_{\mu\neq x}\gamma^x\gamma^\mu(i\hbar\partial_\mu+eA_\mu)-m\gamma^x$$

for a constant electric field in the y direction. The spectrum has an asymmetry between the positive and negative values, just as the spectrum of Hamiltonian, with a constant magnetic field, has asymmetry, Eq. (2.12). This asymmetry causes the induced current, the first term in Eq. (3.27), to appear. The complete set ξ_n is defined by

$$\sum_{\mu\neq x}[\gamma^x\gamma^\mu(i\hbar\partial_\mu+eA_\mu)-m\gamma^x]\xi_n=i\mu_n\xi_n\ , \tag{3.28}$$

where we assume

$$\begin{aligned}&A_x=A_y=0\ ,\\&A_0=yE\ .\end{aligned} \tag{3.29}$$

Note that the operator is anti-Hermitian in Euclidean space since the following relations are satisfied:

$$(\gamma^\mu)^\dagger = -\gamma^\mu ,$$
$$(\gamma^x\gamma^\mu)^\dagger = \gamma^{\mu\dagger}\gamma^{x\dagger} = -\gamma^x\gamma^\mu . \tag{3.30}$$

Since Eq. (3.28) is almost equivalent to Eq. (2.11), it is easy to see that ξ_n is decomposed to

$$\xi_n = e^{ikt}\eta_n[y - y_0(k)] , \tag{3.31}$$

where

$$y_0(k) = \frac{\hbar k}{eE} . \tag{3.32}$$

The induced current can be written as

$$\begin{aligned} j_x &= e\hbar \left[\gamma^x \frac{1}{\not{D} - m} \right] \\ &= e\,\mathrm{tr}\left[\frac{1}{i\hbar\partial_x + \sum_{\mu\neq x} \gamma^x\gamma^\mu(i\hbar\partial_\mu + eA_\mu) - m\gamma^x} \right] \\ &= \frac{e}{2\pi}\int_{-\infty}^{\infty} d\lambda \frac{1}{2\pi}\int_{-\infty}^{\infty} dk \sum_n \frac{1}{\lambda + i\mu_n} \mathrm{Tr}[e^{i(\lambda/\hbar)x}\xi_n^\dagger e^{-i(\lambda/\hbar)x}\xi_n] \\ &= \frac{e}{2\pi}\int_{-\infty}^{\infty} d\lambda \frac{1}{2\pi}\int_{-\infty}^{\infty} dk \sum_n \frac{1}{\lambda + i\mu_n} \mathrm{Tr}\{\eta_n^\dagger[y - y_0(k)]\eta_n[y - y_0(k)]\} \\ &= \frac{e}{2\pi}\int_{-\infty}^{\infty} d\lambda \sum_n \frac{1}{\lambda + i\mu_n} \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{eE}{\hbar} dy_0 \mathrm{Tr}\{\eta_n^\dagger[y - y_0(k)]\eta_n[y - y_0(k)]\} \\ &= \frac{e}{2\pi}\int_{-\infty}^{\infty} d\lambda \sum_n \frac{1}{\lambda + i\mu_n} \frac{1}{2\pi} \frac{eE}{\hbar} 2 \\ &= \frac{e^2E}{2\pi^2\hbar}\int_0^{\infty} d\lambda \left[\frac{1}{\lambda + im} + \frac{1}{-\lambda + im} + \sum_{\mu_n \geq 0} \left[\frac{1}{\lambda + i\mu_n} + \frac{1}{-\lambda + i\mu_n} + \frac{1}{\lambda - i\mu_n} + \frac{1}{-\lambda - i\mu_n} \right] \right] \\ &= i\frac{e^2E}{2\pi\hbar} . \end{aligned} \tag{3.33}$$

In the above equations, we have used (1) the normalization condition of the function $\eta_n(y - y_0)$ and (2) the fact that the contributions from the paired solutions are canceled. Thus the result Eq. (3.27) is obtained by an intuitively clear method. It is obvious that the existence of asymmetry in the spectrum caused the induced current to appear.

IV. PHYSICAL IMPLICATIONS

In this section several implications of the anomaly-induced interaction, which was obtained in Sec. III, are discussed. The quantized Hall effect, a light emission due to a density fluctuation, a parity-violating light reflection, and transmission, is studied.

Before we go on into detailed discussions about phenomenological applications, we describe two independent problems. The dimensional reduction from the 3 + 1 to the 2 + 1 dimension is one. The derivation of the effective Dirac equation and Lagrangian for the equation of motion for the correlated two levels is the other.

The dimensional reduction is made in a situation where there is an energy gap in the z direction. The 2×2 γ_μ ($\mu = 0$, 1,2) matrices in the 2 + 1 dimension are derived from the 4×4 γ_μ matrices in the 3 + 1 dimension.

The derivation of the effective Dirac equation and Lagrangian which is satisfied by nearby two levels is similar to that of the effective Lagrangian for bound states in high-energy physics. A state in one level is regarded as a bound state. Since the state has a spatial structure, there may be some physical cutoff in the system described by the effective Lagrangian. However, the result which is independent of the cutoff should be physically accepted. The electromagnetic coupling in the effective Lagrangian is also uniquely determined, only from the charge.

Some phenomena which we will discuss have been seen experimentally, but the others have not been seen so far. We hope that they will be observed.

We study the dimensional reduction in the following Lagrangian:

$$\mathcal{L}=\bar{\psi}\gamma^0(i\hbar\partial_0+A_0+B_0^{\text{ex}})\psi+\bar{\psi}\gamma^j(i\hbar\partial_j+A_j+B_j^{\text{ex}})\psi+g\bar{\psi}\psi\rho+\mathcal{L}' , \tag{4.1}$$

where a fermion ψ behaves in a two-dimensional plane (xy plane) due to a boundary condition. The electromagnetic field A_μ is not confined in a plane, but moves the full (3 + 1)-dimensional space. A scalar ϕ may or may not be confined in the plane. $\mathcal{L}'$ depends on A_μ and ϕ.

We assume that the confining condition is given through the external fields $B_0^{\text{ex}}(z)$ and $B_3^{\text{ex}}(z)$. It is assumed that B_0^{ex} and B_3^{ex} depend on only the variable z. This is an example to obtain a planar structure and may be general enough. If the wave equation in the z direction

$$\{\gamma_0\gamma_3[i\hbar\partial_3+B_3^{\text{ex}}(z)]+B_0^{\text{ex}}(z)\}u_i=\lambda_i u_i \tag{4.2}$$

has discrete eigenvalue, an eigenfunction which corresponds to this discrete value has a finite support. The other direction, the x and y direction, does not have an energy gap and is the same; thus the (2 + 1)-dimensional phenomena, if the energy scale in the z direction is less than a gap energy, is expected. We expand ψ as

$$\psi_\xi(x,y,z,t)=\sum u_\xi^{(i)\delta}(z)\tilde{\psi}_\delta^{(i)}(x,y,t) ,$$
$$\psi_\xi^\dagger(x,y,z,t)=\sum\tilde{\psi}_\delta^{\dagger(i)}(x,y,t)u_\xi^{\dagger(i)\delta}(z) , \tag{4.3}$$
$$\xi=1,\ldots,4,\quad \delta=1,2 ,$$

where $u_\xi^{i\delta}(z)$ ($\delta=1,2$) are the two independent solutions of Eq. (4.2) with the same eigenvalue λ_i. From the symmetry of the equation, namely the operator on the left-hand side commutes with γ_1 and γ_2,

$$e^{i\gamma_1\theta}e^{i\gamma_2\phi}u \tag{4.4}$$

is the solution of the equation if u is the solution. Matrices $a_{1(2)}$, which are defined by

$$a_i^{\delta\delta'}=\int dz\, u^{\dagger\delta}(z)\gamma^0\gamma^i u\delta'(z) , \tag{4.5}$$

are Hermitian and nondiagonal in the following standard representation of γ_μ matrices:

$$\gamma_0=\begin{bmatrix}1&0\\0&1\end{bmatrix},\quad \gamma_i=\begin{bmatrix}0&\sigma_0\\-\sigma_i&0\end{bmatrix},$$

where

$$\sigma^1=\begin{bmatrix}0&1\\1&0\end{bmatrix},\quad \sigma^2=\begin{bmatrix}0&-i\\i&0\end{bmatrix},\quad \sigma^3=\begin{bmatrix}1&0\\0&-1\end{bmatrix}.$$

After we substitute Eq. (4.3) into the action, the terms which express the discrete level states in the action

$$\int dt\,dx\,dy\,dz\,\mathcal{L}$$

reduce to

$$\sum\int dt\,dx\,dy[\tilde{\psi}_i^\dagger(x,y,t)[(i\hbar\partial_0+e\tilde{A}_0)+a_1(i\hbar\partial_1+e\tilde{A}_1)+a_2(i\hbar\partial_2+e\tilde{A}_2)+\lambda_i+\tilde{A}_3+y\tilde{\phi}]\tilde{\psi}_i+\int dz\,\mathcal{L}'] , \tag{4.6}$$

where

$$\tilde{A}_0=\int dz\, A_0u^\dagger(z)u(z) ,$$
$$a_1\tilde{A}_1=\int dz\, A_1u^\dagger(z)\gamma_0\gamma_1u(z) ,$$
$$a_2\tilde{A}_2=\int dz\, A_2u^\dagger(z)\gamma_0\gamma_2u(z) , \tag{4.7}$$
$$A_3=\int dz\, A_3u^\dagger(z)\gamma_0\gamma_3u(z) ,$$
$$\tilde{\phi}=\int dz\,\phi u^\dagger(z)\gamma_o u(z) .$$

If A_μ and ϕ are smooth functions of z, much smoother than that of $u(z)$, $\tilde{A}_\mu$ and $\tilde{\phi}$ becomes A_μ and ϕ at the constant z variable z_0, where z_0 is the position of a plane. The a_i can be written as

$$a_1=ci\sigma^3\sigma^1+ei\sigma^3\sigma^2 ,$$
$$a_2=di\sigma^3\sigma^2+fi\sigma^3\sigma^1 , \tag{4.8}$$

where σ_i are the Pauli matrices.

Whether coefficients c, d, e, and f are of the simple Dirac equation depends on the function $u(z)$, hence on the boundary condition.

We define

$$\bar{\bar{\psi}}=\tilde{\psi}^\dagger\sigma^3 ,$$
$$\gamma^0=\sigma^3,\quad \gamma^1=i\sigma^1,\quad \gamma^2=i\sigma^2 . \tag{4.9}$$

Then Eq. (4.6) becomes

$$\int dt\,dx\,dy\left[\bar{\bar{\psi}}[\gamma^0(i\hbar\partial_0+e\tilde{A}_0)+(\gamma^1+e\gamma^2)(i\hbar\partial_1+e\tilde{A}_1)+(d\gamma^2+f\gamma^1)(i\hbar\partial_2+e\tilde{A}_2)+\lambda_1\gamma^0+\gamma^0e\tilde{A}_3]\tilde{\psi}+g'\bar{\bar{\psi}}_i\tilde{\psi}_i\phi+\int dz\,\mathcal{L}'\right] . \tag{4.10}$$

This is the action in the (2 + 1)-dimensional space. We can apply the results of the previous section, if the energy in the z direction is much smaller than the gap energy.

The consistency condition Eq. (3.20) now becomes

$$\int dx\,dy(\partial_1\tilde{A}_2-\partial_2\tilde{A}_1)=\frac{1}{2}\frac{h}{e}\times\text{integer} . \tag{4.11}$$

If the z dependence of A_i is negligible, then $\tilde{A}_i$ in the above equation is replaced with A_i.

Next we investigate the effect of additional magnetic field on the system of planar electrons. A strong magnetic field in an orthogonal direction to the planar electrons leads electrons to split into many levels, as was discussed in Sec. II. The field operators $\psi_N(X)$ are used for expressing the state in N level at the center position X.

For our study the following Hamiltonian is assumed:[14]

$$H=\sum_{N,X} E_N \psi_N^\dagger(x)\psi_N(x) + \sum_{N,X;N',X'} C(N,X;N',X')\psi_N^\dagger(X)\psi_{N'}(X') . \quad (4.12)$$

The second term in the above equation shows the effect of electrons scattering with scatterers. The functions $C(N,X;N',X)$ may be decreasing with respect to $|X-X'|$. Thus it would be reasonable to write

$$\sum_{X'} C(N,X;N',X')\psi_{N'}(X') = C_0(N,N')\psi_{N'}(X)+C_1^i(N,N')\frac{\partial}{\partial x_i}\psi_{N'}(x) + C_2^{i_1 i_2}(N,N')\frac{\partial^2}{\partial x_{i_1}\partial x_{i_2}}\psi_{N'}(x)+\cdots , \quad (4.13)$$

where

$$C_0(N,N')=\sum_{X'} C(N,X;N',X') ,$$
$$C_1^i(N,N')=\sum_{X'} C(N,X;N',X')(X-X')^i ,$$
$$C_2^{i_1 i_2}(N,N')=\sum_{X'} C(N,X;N',X')(X-X')^{i_1}(X-X)^{i_2} ,$$
$$\ldots . \quad (4.14)$$

The contributions of $C_2, C_3, \ldots$ to a large-scale property of the higher system may be less important than that of C_1, because the higher derivative of ψ in Eq. (4.13) is smaller than the first derivative of ψ if the wave length of ψ is large. A simple dimensional analysis also suggests that C_1 is most important in that region. The induced current which depends on C_2 is proportional to the spatial derivative of the electric field if the C_1-dependent current is proportional to the electric field from the dimensional analysis.

Localization of the electrons[15] may suggest that $C_l^{i_1 \cdots i_l}$ ($l \geq 1$) vanish. If $C_l^{i_1 \cdots i_l}$ vanish exactly, there is no induced current. However, the nonzero value of C_1, even though the value is very small, leads the nonvanishing induced current to exist. The induced current has a quantized value. A perturbative treatment of C_2 does not change the result.

The wide separation of energy levels, which may occur under the strong magnetic field, leads us to study the mixing of only two nearest-neighbor levels. We do not need to consider the mixing of two levels with others, especially when the Fermi energy is located in their gap region. Hereafter we concentrate on this case.

The Hamiltonian density is given by

$$H=(\psi_1^\dagger,\psi_2^\dagger)\begin{bmatrix} E_1 & \delta_1 \\ \delta_1^* & E_2 \end{bmatrix}\begin{bmatrix}\psi_1\\ \psi_2\end{bmatrix} + (\psi_1^\dagger,\psi_2^\dagger)\begin{bmatrix} 0 & C^1 i\hbar\frac{\partial}{\partial x} \\ C^{1*} i\hbar\frac{\partial}{\partial x} & 0 \end{bmatrix}\begin{bmatrix}\psi_1\\ \psi_2\end{bmatrix} + (\psi_1^\dagger,\psi_2^\dagger)\begin{bmatrix} 0 & C^2 i\hbar\frac{\partial}{\partial y} \\ C^{2*} i\hbar\frac{\partial}{\partial y} & 0 \end{bmatrix}\begin{bmatrix}\psi_1\\ \psi_2\end{bmatrix} . \quad (4.15)$$

In writing the above equation, parity conservation is assumed and the diagonal parts in the least two terms of the right-hand side are set to be zero.

The Lagrangian density by which the previous Hamiltonian is derived is

$$\mathcal{L}=\bar\psi[\gamma_0(i\hbar\partial_0+e_0)-d_i^j\gamma^i(i\hbar\partial_j+f_j)-m]\psi , \quad (4.16)$$

where

$$\bar\psi=(\psi_1^\dagger,\psi_2^\dagger)\gamma_0,\quad \psi=\begin{bmatrix}\psi_1\\ \psi_2\end{bmatrix} ,$$
$$(d)=\begin{bmatrix}\mathrm{Im}C^1 & -ke C^1\\ \mathrm{Im}C^2 & -keC^2\end{bmatrix} , \quad (4.17)$$
$$\gamma^0=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix} ,\quad \gamma_1=i\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} ,$$
$$\gamma_2=\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix} ,$$

and the numerical constants e_0, m, f_i are linear combinations of E_1, E_2, δ_1.

The coupling of ψ with the external electromagnetic field is determined by the usual minimal coupling. Thus we have

$$\mathcal{L}=\bar\psi[\gamma_0(i\hbar\partial_0+eA_0+e_0)-d_i^j\gamma^i(i\hbar\partial_j+f_j+eA_j)-m]\psi . \quad (4.18)$$

The above Lagrangian density is the same as the ordinary Dirac equation except the numerical coefficient d_i^j. As we will see this factor does not change the anomaly-induced current.

The states ψ_1 and ψ_2 are actually generated by the external magnetic field. The vector potential in Eq. (4.18) should not include the above magnetic field, in order to avoid double counting. Only the additional term, by which a small perturbation of the system is produced, is put into Eq. (4.18). Then the system's response against the small change of the external fields is obtained. We understand the vector potential in Eq. (4.18) to be this small term.

The induced current is obtained for the system described by the Lagrangian density, Eq. (4.18). We have the following induced current:

$$j_\mu = \frac{e^2}{4\pi\hbar}\epsilon_{\mu\nu\rho}F^{\nu\rho} + O(\epsilon) , \tag{4.19}$$

where

$$\epsilon = \det(d) . \tag{4.20}$$

Thus in the vanishing d limit, which corresponds to the electron's localization, the Hall current does not vanish.

Since the time derivative of the particle's position

$$\langle \mathbf{r}_1 \rangle \equiv \left\langle \int d\mathbf{x}\, \psi_1^\dagger(t,\mathbf{x})\mathbf{x}\psi_1(t,x) \right\rangle$$

is proportional to d^{ij}, the particles do not move long distances if d^{ij} are infinitesimal. They are localized. The nonzero Hall current in the localized region is understood to be carried by phases of the fields but not by particles. In this sense, the Hall current may be similar to the Josephson current.

Now we study the physical applications.

A. Quantized Hall effect (Ref. 16)

The external electric field is added to the parallel direction (y) in addition to the external magnetic field to the orthogonal direction (Z). The anomalous term in the current in the vanishing d limit becomes

$$j_i = \frac{e^2}{2\pi\hbar}F^{0l} \quad (l \neq i) . \tag{4.21}$$

For the external electric field in the y direction,

$$F^{0l} \equiv F^{0x} = 0 . \tag{4.22}$$

Then the current in the x direction is given by

$$J_x = \frac{e^2}{h}\int dy\, F^{0y} = \frac{e^2}{h}V_y . \tag{4.23}$$

Thus at each gap region, the current equation (4.23) is added. The total current and conductivity become

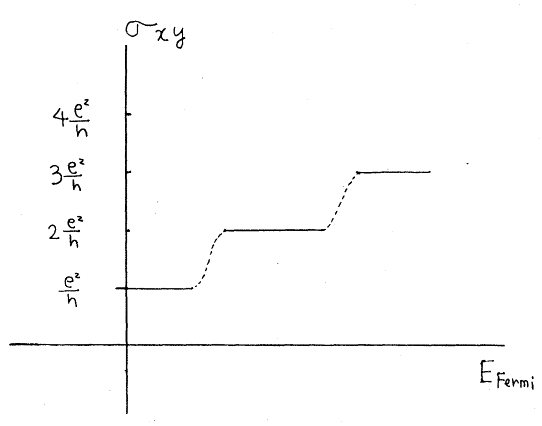

FIG. 2. The behavior of the Hall conductivity σ_{xy} is drawn. The value is the $(e^2/n)\times$integer in the gap region. In the transition region, one unit is added to σ_{xy}.

$$J_x = N\frac{e^2}{h}V_y , \qquad \text{conductivity} = N\frac{e^2}{h} , \tag{4.24}$$

and behave as shown in Fig. 2. These behaviors have been observed in recent experiments.[17]

B. Light emission by a density fluctuation

We replace both of the external electromagnetic fields or one of them with the nonzero density fluctuation or the nonzero momentum fluctuation of the system which is produced by the external condition. These external variables are represented in the theory by adding

$$\mu(x)\psi^\dagger\psi - \mu_i(x)\psi^\dagger\gamma_0\gamma^i\psi \tag{4.25}$$

to the Lagrangian. The Lagrange multiplier $\mu(x)$ represents the density fluctuation and $\mu_i(x)$ represents the momentum density fluctuation. The $\mu(x)$ plays the same role as $A_0(x)$ and $\mu_i(x)$ plays the same role as $A_i(x)$. The equivalent phenomenon, known as the Hall effect, is expected to occur.

We study light emission by the induced current. The current density is given by

$$J_i(x,y) = N\frac{e^2}{h}\epsilon_{0ij}\partial_j\mu(x,y) . \tag{4.26}$$

The charge density is given by

$$\rho(x,y) = N\frac{e^2}{h}(\partial_1\mu_2 - \partial_2\mu_1) . \tag{4.27}$$

These currents and charges become sources of the Maxwell equation.

C. Parity-violating light scattering

Integrating Eq. (3.26) with respect to A_μ, we have

$$\frac{e^2}{h}\int dt\, dx\, dy\, \epsilon_{\mu\nu\rho}A^\mu F^{\nu\rho} \tag{4.28}$$

as the action. This can cause parity-violating light scattering. A simple way to see the effect is to see a reflection and a transmission of the light along the z direction and to study a rotation of a polarization plane.

We study the maxwell equation using the vector potential in the $A_0=0$ gauge. We study the following equation, which is free except on the surface $z=0$

$$\begin{aligned} \left[\Delta - \frac{1}{c^2}\partial_t{}^2\right]A_1 &= \frac{e^2}{h}\delta(z)i\partial_t A_2 , \\ \left[\Delta - \frac{1}{c^2}\partial_t{}^2\right]A_2 &= -\frac{e^2}{h}\delta(z)i\partial_t A_1 , \\ \left[\Delta - \frac{1}{c^2}\partial_t{}^2\right]A_3 &= 0 . \end{aligned} \tag{4.29}$$

We solve the equation with a boundary condition where a plane wave with a polarization $\mathbf{e}^2$ is coming to the surface, namely

$$\mathbf{A}=\mathbf{e}^{(1)}a_1e^{i(kz-wt)}+\mathbf{e}^{(i)}b_ie^{i(-kz-wt)} \tag{4.30}$$

for $z>0$, and

$$\mathbf{A}=\mathbf{e}^{(j)}c_je^{i(kz-wt)}$$

for $z\leq 0$, where the linear polarization vector $\mathbf{e}^{(i)}$ is

$$\begin{aligned}\mathbf{e}^{(1)}&=(1,0,0)\ ,\\ \mathbf{e}^{(2)}&=(0,1,0)\ .\end{aligned} \tag{4.31}$$

By substituting Eq. (4.26) in Eq. (4.25), we have

$$\begin{aligned} b_1&=\frac{1}{2}\left[\frac{e^2}{h}\right]^2\frac{1}{2-\frac{1}{2}\left[N\frac{e^2}{h}\right]^2}a_1\ ,\\ b_2=c_2&=-\frac{e^2}{h}i\frac{1}{2-\frac{1}{2}\left[N\frac{e^2}{h}\right]^2}a_1\ ,\\ c_1&=\left[1+\frac{1}{2}\left[\frac{e^2}{h}\right]^2\frac{1}{2-\frac{1}{2}\left[N\frac{e^2}{h}\right]^2}\right]a_1\ .\end{aligned} \tag{4.32}$$

The effect of the parity violation is seen in the coefficients b_2 and c_2. It may be interesting if this effect can be seen by an experiment.

It is amazing to see the parity-violating phenomenon in the order of the electromagnetic interaction strength. However, a similar phenomenon has been known a long time.

One is an occurrence of permanent electric dipole moment in the hydrogen atom[18] in the nonrelativistic approximation. The first-order energy perturbation due to the external electric field vanishes generally for parity-conserving unpeturbed Hamiltonian systems, since the perturbed Hamiltonian

$$H'=-eEz \tag{4.33}$$

is odd against parity transformation. However, if opposite parity states are degenerate, the nonzero electric dipole moment can appear. Due to the degeneracy of the Schrödinger equation with the Coulomb potential, the electric dipole moment of the $n=2$, $l_z=0$ hydrogen becomes nonzero. The symmetry-breaking term exists due to the degeneracy of the opposite-parity states.

V. SUMMARY

The importance of the zero-energy solution of the massless Dirac equation has been discussed. The zero-energy solution was shown to exist, to cause the asymmetry of the spectrum of the massive theory, and to break the symmetry of the Lagrangian.

The axial anomaly in even dimensions is due to the existence of the singularity in the partition function and the free energy which is produced by the vanishing eigenvalue of the Dirac equation. The same solution is interpreted as that of the zero energy of massless Dirac theory and leads the spectrum to have the asymmetry in the massive Dirac theory in odd dimensions. Due to the asymmetry, there appears the induced current and the induced effective Lagrangian which violates parity invariance as well as time-reversal invariance.

The simplest, but macroscopically most important, theory, QED, has been discussed. Inclusion of other interactions such as quantum chromodynamics, the Glashow-Weinberg-Salam unified electroweak theory, or the Georgi-Glashow grand unified theory is possible. Since the structure of the fermion zero-eigenvalue solution is sensitive to the structure of the Lagrangian, a classification based on topological argument is needed. In fact, the several relations[14] between the zero eigenvalues and the topologically invariant quantity is well known. Although the relation is that of classical theory, it can be applied in the quantized theory too.

A remarkable property is that the anomaly-induced interaction is irrelevant to the ordinary symmetry breaking. The term Eq. (3.26) is induced in the theory described by the Eq. (3.1) as the mass-independent term.

The effective coupling in the unified model is reduced by the large mass of the vector meson. The anomaly-induced term, on the contrary, is not reduced by the large mass. The induced interaction is determined from the structure of the Lagrangian.

The induced action, which is obtained by integrating Eq. (3.26),

$$\frac{e^2}{2h}\int dt\,dx\,dy\,\epsilon_{\mu\nu\rho}A^\mu\partial^\nu A^\rho\ , \tag{5.1}$$

breaks parity invariance in the strength of QED. The implication of this term, other than the one discussed here, might exist in a wide area.

The existence of the degeneracy is very critical in our argument. A Dirac-type equation was used in describing the electrons of low energy. The relativistic effect is not the important factor, but the degeneracy which is expressed by the Dirac equation is the important factor. We may use the Schrödinger equation if we treat the degeneracy correctly.

The explanation of the quantized Hall effect seems to us a little complicated in the ordinary method of solid state physics,[16] contrary to our simple explanation. This may be related to the treatment of the degeneracy.

As a summary, the anomaly-induced interaction Eq. (5.1), which violates parity, and the consistency condition Eq. (3.20), were shown to exist in the system of planar fermions. They are produced by the zero-eigenvalue solution. The existence of this kind of solution in the uniform magnetic field was shown explicitly. The physical implications were also investigated.

ACKNOWLEDGMENT

The author would like to thank Dr. H. Takayama for instruction about the quantized Hall effect.

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