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Covariant generalization of the *Zitterbewegung* of the electron and its SO(4,2) and SO(3,2) internal algebras

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The internal geometry of the Dirac electron is studied in a proper-time formalism with γ^0-Hermitian operators. We solve the Heisenberg equations, separate external and internal coordinates, and identify the SO(3,2) internal algebra as the projection of an SO(3,3) geometry to the hyperplane (perpendicular to the center-of-mass momentum) where the *Zitterbewegung* takes place. We also give covariant intrinsic-spin and magnetic-moment operators. The system can be generalized to a larger system with the internal geometry SO(4,2) with the inclusion of dynamical variables γ^5 and $i\gamma^5\gamma^\mu$. The resultant internal algebras have higher-dimensional representations generalizing the Dirac electron to multifermion states.

I. INTRODUCTION

Barut and Bracken have recently reexamined the *Zitterbewegung* of the free Dirac electron.[1] One obtains the *Zitterbewegung* by solving the Heisenberg equations of motion for the position operator $x(t)$ using the Dirac Hamiltonian

$$H=m\gamma^0+\vec{p}\cdot\vec{\alpha} \quad (\hbar=c=1)$$

as the generator for time translations. One finds that the coordinate operator $x(t)$ contains a "center-of-mass" part

$$\vec{X}(t)=H^{-1}\vec{p}t+\vec{a} \quad (\vec{a}=\text{const vector})$$

which on momentum eigenstates moves with a uniform velocity, and an oscillatory part

$$\vec{\xi}(t)=\frac{i}{2}[\vec{\alpha}(0)-H^{-1}\vec{p}]H^{-1}e^{-2iHt}$$

called the *Zitterbewegung*. Then

$$\vec{x}(t)=\vec{X}(t)+\vec{\xi}(t)$$

is interpreted as the center of charge for the electron which oscillates rapidly about the center of mass $X(t)$. We shall call $\xi(t)$ the relative or the internal-position operator of the electron.

Barut and Bracken then pass to the rest frame of the center of mass and define an internal momentum proportional to the time rate of change of the internal position in this frame. The internal position and momentum generate then the ten-dimensional Lie algebra of SO(5), consisting of the three components of the internal-position, internal-momentum, spin (=internal-angular-momentum) operators, and the rest-frame Hamiltonian.

This description of the *Zitterbewegung* is not manifestly covariant since the internal position and momentum defined in some particular frame, which is the rest frame, are functions of coordinate time rather than proper time, and hence are three-vectors rather than four-vectors. The purpose of this work is to develop a Lorentz-covariant description of the *Zitterbewegung* and to investigate the Lie algebra associated with it.

II. PROPER-TIME FORMALISM

We expect that the relationship between ordinary time and energy should be similar to that between the two Lorentz-invariant quantities: proper time and (rest) mass. Just as energy is associated with eigenvalues of the partial derivative with respect to time, mass should be associated with eigenvalues with respect to proper time. Therefore, as our starting point, we introduce the auxiliary real Lorentz-invariant parameter s which is independent of spacetime and has the dimensions of a length ($c=1$). We make the (first quantized) Dirac wave function depend on this parameter, as well as on spacetime, and replace mass in the free-particle Lagrangian by the operator $-i\partial/\partial s$,

$$\mathscr{L}=\bar{\psi}(x,s)(i\partial_\mu\gamma^\mu+i\partial_s\cdot\mathbb{1})\psi(x,s) . \quad (1)$$

At this point we do not yet identify s with proper time but think of it as an independent "fifth" dimension. Along somewhat different but related lines DeVox and Hilgevoord[2] and others[3,4] introduce mass as a fifth spacelike coordinate in a de Sitter momentum space. In our approach this fifth momentum would be identified with the operator $-i\partial/\partial s$. However, we shall not find it necessary in this work to attach an additional metric structure to the five-dimensional space besides the Minkowski metric on four-dimensional spacetime. Nor is this a new extended formalism. It is, as we shall see, merely a device to obtain covariant equations for operators in the proper-time Heisenberg picture.

The Lagrangian (1) leads to the equation of motion

$$i\frac{\partial}{\partial s}\psi(x,s)=-i\partial_\mu\gamma^\mu\psi(x,s) . \quad (2)$$

We now introduce "mass eigenstates"

$$\psi_m(x,s)=e^{ims}\psi_m(x,0) \quad (3)$$

on which Eq. (2) reduces to the Dirac equation for

$$\psi_m(x,0)\equiv\psi(x) \quad \text{or} \quad i\partial_\mu\gamma^\mu\psi(x)=m\psi(x) .$$

Introducing the generator for translations in s,

$$\mathscr{H}=-i\partial_\mu\gamma^\mu=-p_\mu\gamma^\mu , \quad (4)$$

we find that (2) has the formal solution

$$\psi(x,s)=e^{-i\mathscr{H}s}\psi(x) \ . \tag{5a}$$

Taking the Dirac adjoint of (5a) we find

$$\bar{\psi}(x,s)=\bar{\psi}(x)e^{i\mathscr{H}s}=\psi^{\dagger}(x,s)\gamma^{0} \ , \tag{5b}$$

where the dagger stands for the Hermitian adjoint.

For any operator A with no explicit dependence on s we make the transition to the (proper-time) "Heisenberg picture" by equating

$$\bar{\psi}(x,s)A\psi(x,s)=\bar{\psi}(x)A_H(s)\psi(x) \ ,$$

where the Heisenberg operator is defined by

$$A_H(s)=e^{i\mathscr{H}s}Ae^{-i\mathscr{H}s} \ . \tag{6}$$

In this picture states depend only on spacetime while operators depend on the parameter s. At this point we interpret s as a "proper time" in the Heisenberg picture. This interpretation is implemented by defining the expectation value of the Heisenberg operator $\hat{A}(s)$ (with subscript H dropped) as follows:

$$\langle A(s)\rangle=\int d^4x\ \bar{\psi}(x)A(s)\psi(x)\delta(n_\mu x^\mu-s) \ , \tag{7}$$

where n is a unit vector in the time direction of the frame in which $A(s)$ is measured. This definition is manifestly covariant since d^4x and the argument of the δ function are Lorentz invariant and $(\bar{\psi}(x)A(s)\psi(x))$ forms a bilinear covariant quantity. In this expression the laboratory frame, characterized by n, is in general independent of the frame with coordinates x^μ. If these two frames are chosen to coincide then

$$n=(1,0,0,0)$$

and (7) reduces to

$$\langle A(s)\rangle=\int d^3x\ \bar{\psi}(\vec{x},s)A(s)\psi(\vec{x},s) \ .$$

The proper time s parametrizes a timelike precession of spacelike hypersurfaces in spacetime.

Let us find the condition which must be satisfied by $\hat{A}$ in order that its expectation value, defined by (7), be real:

$$\begin{aligned}\langle A(s)\rangle^*&=\int dx\ \delta(n\cdot x-s)[\psi^\dagger(x)\gamma^0A(s)\psi(x)]^*\\&=\int dx\ \delta(n\cdot x-s)\psi^\dagger(x)A^\dagger(s)\gamma^{0\dagger}\psi(x)\\&=\int dx\ \delta(n\cdot x-s)\psi^\dagger(x)A^\dagger(s)\gamma^0\psi(x) \ ,\end{aligned}$$

where the asterisk denotes complex conjugation. Thus $\langle A(s)\rangle^*=\langle A(s)\rangle$ implies that

$$\gamma^0A(s)=A^\dagger(s)\gamma^0$$

or (8)

$$A^\dagger(s)=\gamma^0A(s)\gamma^0 \ .$$

We call an operator A, "γ^0-Hermitian," if it satisfies (8). Then its expectation value is real. Every γ^0-Hermitian operator A can be associated with a Hermitian operator A',

$$A'=\gamma^0A, \quad A'^\dagger=A' \ , \tag{9a}$$

and conversely any Hermitian operator A' is associated with the γ^0-Hermitian operator A,

$$A=\gamma^0A' \ . \tag{9b}$$

In particular, in the nonmanifestly covariant normalization of the Dirac wave function,

$$\int d^3x\ \psi^\dagger(x)\psi(x)=1 \ ,$$

$A'=1$, and hence $A=\gamma^0$ so that

$$\int d^3x\ \bar{\psi}(x)\gamma^0\psi(x)=1 \ .$$

Since $\gamma^0=n_\mu\gamma^\mu$ for $n=(1,0,0,0)$ the normalization can be put into the manifestly covariant form

$$1=\int d^4x\ \delta(n\cdot x-s)n_\mu\bar{\psi}\gamma^\mu\psi=\int_{\sigma(s)}d\sigma_\mu\bar{\psi}\gamma^\mu\psi \ ,$$

where $\sigma(s)$ is the spacelike hypersurface orthogonal to n at the proper time s and $d\sigma_\mu$ is an oriented area element on that surface. In general n_μ and $d\sigma_\mu$ in (10) can depend on x. The normalization (10) must hold at any proper time.

The Heisenberg operator $A(s)$ obeys the equation of motion

$$\frac{dA}{ds}=i[\mathscr{H},A(s)] \tag{10}$$

so that $\mathscr{H}$ generates proper-time translations in the Heisenberg picture. Several authors including Corben,[5] Szamosi,[6] Drechsler,[7] Ellis,[8] and Barut[9] have chosen this operator as the generator for proper-time translations of particles which obey the Dirac equation. We shall use this generator to find the proper-time development for the position operator of a free electron.

III. HEISENBERG EQUATIONS FOR THE POSITION OPERATOR AND THEIR SOLUTIONS

Applying the Heisenberg equation (10) to $x^\mu(s)$, we get

$$\dot{x}^\mu(s)=i[\mathscr{H},x^\mu(s)]=\gamma^\mu(s) \ ,$$

where the dot denotes differentiation with respect to proper time. A further differentiation gives

$$\begin{aligned}\ddot{x}_\mu(s)&=i[\mathscr{H},\dot{x}_\mu(s)]\\&=i[2\mathscr{H}\gamma_\mu(s)-\{\mathscr{H},\gamma_\mu(s)\}]\\&=2i\mathscr{H}\gamma_\mu(s)+2ip_\mu \ .\end{aligned}$$

Following Barut and Bracken[1] we define the operator

$$\eta^\mu(s)=\gamma^\mu(s)+\mathscr{H}^{-1}p^\mu \quad (p^2\neq0) \ , \tag{11}$$

which obeys the equation

$$\dot{\eta}^\mu(s)=2i\mathscr{H}\eta^\mu(s)$$

with the solution

$$\begin{aligned}\eta^\mu(s)&=e^{2i\mathscr{H}s}\eta^\mu(0)\\&=\eta^\mu(0)e^{-2i\mathscr{H}s}\end{aligned}$$

since η^μ anticommutes with $\mathscr{H}$. The velocity operator then has the solution

$$\gamma^\mu(s)=[\gamma^\mu(0)+\mathscr{H}^{-1}p^\mu]e^{-2i\mathscr{H}s}-\mathscr{H}^{-1}p^\mu \ , \tag{12}$$

and integrating, we can write

$$x^\mu(s)=X^\mu(s)+Q^\mu(s) \ , \tag{13}$$

where

$$X^\mu(s)=a^\mu+s(p\cdot\gamma)^{-1}p^\mu \quad [a^\mu=x^\mu(0)-Q^\mu(0)] \tag{13a}$$

is the center-of-mass position operator, a^μ being a constant operator which depends on initial conditions, and

$$Q^\mu(s)=\frac{i}{2}[\gamma^\mu(0)+\mathscr{H}^{-1}p^\mu]\mathscr{H}^{-1}e^{-2i\mathscr{H}s} \tag{13b}$$

is the internal-position operator.

On mass eigenstates the center-of-mass position operator becomes

$$X^\mu(s)=a^\mu+s\frac{p^\mu}{m} \ .$$

This is precisely the behavior that one would expect for the center of mass of a free relativistic particle and we see that s should be interpreted as the proper time of the electron's center of mass, exactly as in the classical model of the Dirac electron.[10]

IV. THE INTERNAL ALGEBRA

Let us now focus our attention on the internal operators associated with the *Zitterbewegung*. We have found an internal position $Q^\mu(s)$ and the internal velocity operator $\dot{Q}^\mu(s)$ which turns out to be equal to $\eta^\mu(s)$, introduced before

$$\dot{Q}^\mu(s)=\eta^\mu(s) \ .$$

The algebra generated by these operators does not close since the commutator of Q^μ with η^ν introduces the operator $(p^2)^{-1}\mathscr{H}$; then the commutator of Q^μ with $\mathscr{H}/p^2$ brings in $(p^2)^{-1}\eta^\mu$ and so on, so that continually higher powers of $(p^2)^{-1}$ must be introduced into the algebra. However, if we impose the condition that the *Zitterbewegung* operators act on the Hilbert space spanned by positive- and negative-frequency solutions of the Dirac equation with fixed mass m, then

$$p^2=m^2, \quad (p^2)^{-1}=1/m^2$$

become constants. With this restriction let us introduce the internal-momentum operator:

$$P^\mu=m\dot{Q}^\mu=m[\gamma^\mu(0)+\mathscr{H}^{-1}p^\mu]e^{-2is\mathscr{H}} \ . \tag{14}$$

Then we obtain the following algebra generated by the internal-position and -momentum operators:

$$[Q^\mu,Q^\nu]=-\frac{i}{m^2}\widetilde{S}^{\mu\nu} \ , \tag{15a}$$

$$[P^\mu,P^\nu]=-4im^2\widetilde{S}^{\mu\nu} \ , \tag{15b}$$

$$[Q^\mu,P^\nu]=-i\widetilde{g}^{\mu\nu}\mathscr{H}/m \ , \tag{15c}$$

$$[Q^\mu,\widetilde{S}^{\nu\lambda}]=i(\widetilde{g}^{\mu\nu}Q^\lambda-\widetilde{g}^{\mu\lambda}Q^\nu) \ , \tag{15d}$$

$$[P^\mu,\widetilde{S}^{\nu\lambda}]=i(\widetilde{g}^{\mu\nu}P^\lambda-\widetilde{g}^{\mu\lambda}P^\nu) \ , \tag{15e}$$

$$[Q^\mu,\mathscr{H}/m]=\frac{i}{m^2}P^\mu \ , \tag{15f}$$

$$[P^\mu,\mathscr{H}/m]=-4im^2Q^\mu \ , \tag{15g}$$

$$[\widetilde{S}^{\mu\nu},\mathscr{H}/m]=0 \ , \tag{15h}$$

$$[\widetilde{S}^{\alpha\beta},\widetilde{S}^{\mu\nu}]=-i(\widetilde{g}^{\alpha\mu}\widetilde{S}^{\beta\nu}+\widetilde{g}^{\beta\nu}\widetilde{S}^{\alpha\mu}-\widetilde{g}^{\alpha\nu}\widetilde{S}^{\beta\mu}-\widetilde{g}^{\beta\mu}\widetilde{S}^{\alpha\nu}) \ , \tag{15i}$$

where we have introduced

$$\widetilde{g}^{\mu\nu}\equiv g^{\mu\nu}-\frac{p^\mu p^\nu}{m^2} \ , \tag{16}$$

$g^{\mu\nu}$ being the Minkowski metric of signature $(+,-,-,-)$, and

$$\widetilde{S}^{\mu\nu}=S^{\mu\nu}-\frac{p^\mu p_\alpha}{m^2}S^{\alpha\nu}-\frac{p^\nu p_\alpha}{m^2}S^{\mu\alpha} \ ,$$
$$S_{\mu\nu}=\tfrac{1}{2}\sigma_{\mu\nu}=\frac{i}{4}[\gamma_\mu,\gamma_\nu] \ . \tag{17}$$

It is easily verified that all of the operators in this algebra are γ^0-Hermitian.

Acting on plane waves of the form $u(p)e^{-ipx}$, the Lie algebra (15) is characterized by the fixed four-vector p, i.e., the center-of-mass momentum of the particle. In order to transform from the internal algebra characterized by p to the algebra characterized by

$$p'^\mu=\Lambda^\mu{}_\nu p^\nu \ ,$$

where $(\Lambda^\mu{}_\nu)$ is the matrix of a Lorentz transformation, we introduce the corresponding spinor transformation

$$U(\Lambda)\in \mathrm{SL}(2,C)\otimes \mathrm{SL}(2,C)^* \ ,$$

such that

$$U(\Lambda)^{-1}\gamma^\mu U(\Lambda)=\Lambda^\mu{}_\nu\gamma^\nu \ ,$$
$$U(\Lambda)\gamma^\mu U(\Lambda)^{-1}=(\Lambda^{-1})^\mu{}_\nu\gamma^\nu$$

and obtain the following transformation law between algebras:

$$Q^\mu(p')=\Lambda^\mu{}_\nu U(\Lambda)Q^\nu(p)U(\Lambda)^{-1} \ , \tag{18a}$$

$$P^\mu(p')=\Lambda^\mu{}_\nu U(\Lambda)P^\nu(p)U(\Lambda)^{-1} \ , \tag{18b}$$

$$\mathscr{H}(p')=U(\Lambda)\mathscr{H}(p)U^{-1}(\Lambda) \ , \tag{18c}$$

$$\widetilde{S}^{\mu\nu}(p')=\Lambda^\mu{}_\alpha\Lambda^\nu{}_\beta U(\Lambda)\widetilde{S}^{\alpha\beta}(p)U^{-1}(\Lambda) \ . \tag{18d}$$

The algebra (15) is therefore manifestly covariant.

The vector and tensor operators of (15), acting on a state of momentum p, are confined to the hyperplane orthogonal to p:

$$p_\mu Q^\mu=p_\mu P^\mu=p_\mu\widetilde{S}^{\mu\nu}=0 \ . \tag{19}$$

The tensor

$$\widetilde{g}_{\mu\nu}=g_{\mu\nu}-p_\mu p_\nu/m^2$$

acts like a metric on this hypersurface, mapping pairs of vectors into scalars with precisely the same effects as $g_{\mu\nu}$, and can in fact be written as the composition of the projection operator $g^{\mu}{}_{\nu}$ onto this plane with the Lorentz metric:

$$\tilde{g}_{\mu\nu}=g_{\alpha\beta}\tilde{g}^{\alpha}{}_{\mu}\tilde{g}^{\beta}{}_{\nu}\ ,$$

where

$$\tilde{g}^{\alpha}{}_{\mu}=\delta^{\alpha}{}_{\mu}-p^{\alpha}p_{\mu}/m^2\ .$$

The restriction (19) implies that only ten of fifteen generators in (15) are linearly independent. In order to identify the algebra generated by these ten elements we write (15) in the standard dimensionless form

$$[\theta^{ab},\theta^{cd}]=-i(\tilde{g}^{ac}\theta^{bd}+\tilde{g}^{bd}\theta^{ac}-\tilde{g}^{ad}\theta^{bc}-\tilde{g}^{bc}\theta^{ad})\ , \tag{20}$$

where

$$\tilde{g}^{ab}=\begin{cases}\tilde{g}^{\mu\nu}, & a,b=\mu,\nu=0,1,2,3\\ \delta^{ab}, & a=4,5\end{cases} \tag{21}$$

and the antisymmetric matrix θ is given by

$$\begin{aligned}&\theta^{\mu\nu}=\tilde{S}^{\mu\nu},\quad \mu,\nu=0,1,2,3\\ &\theta^{\mu4}=mQ^{\mu}\ ,\\ &\theta^{\mu5}=\frac{1}{2m}P^{\mu}\ ,\\ &\theta^{45}=\frac{1}{2m}\mathscr{H}\ .\end{aligned} \tag{22}$$

Expression (20) looks like the standard form for the Lie algebra of a (pseudo-) orthogonal group in six dimensions. The fifteen generators of this algebra can be viewed as the result of projecting the operators of the so(3,3) algebra

$$[\hat{\theta}^{ab},\hat{\theta}^{cd}]=-i(g^{ac}\hat{\theta}^{bd}+g^{bd}\hat{\theta}^{ac}-g^{ad}\hat{\theta}^{bc}-g^{bc}\hat{\theta}^{ad}) \tag{23}$$

with

$$g^{ab}=\begin{cases}g^{\mu\nu}, & a,b=\mu,\nu=0,1,2,3\\ \delta^{ab}, & a=4,5\end{cases} \tag{24}$$

onto the hyperplane orthogonal to p. Using the projection operator

$$\begin{aligned}\Pi^{a}{}_{b}&=\tilde{g}^{\mu}{}_{\nu},\quad a,b=\mu,\nu=0,1,2,3\\ &=\delta^{a}_{b},\quad a=4,5\end{aligned}$$

and writing

$$\theta^{ab}=\pi^{a}{}_{e}\pi^{b}{}_{f}\hat{\theta}^{ef},\quad \tilde{g}^{ab}=\pi^{a}{}_{e}\pi^{b}{}_{f}g^{ef}$$

we can obtain (20) from (23). Indeed the tensor $\tilde{S}^{\mu\nu}$, defined in (17), can be obtained from the ordinary spin tensor $S^{\mu\nu}$ by this projection.

To identify the algebra (20) we diagonalize the 4×4 nondiagonal part $g^{\mu\nu}$ of g^{ab}. It has already been mentioned that $g_{\mu\nu}$ and therefore $g^{\mu\nu}$ annihilate vectors along the direction of p, $g^{\mu\nu}p_{\nu}=0$, so that the four covariant components of p, $(E,-\vec{p})$, form an eigenvector of $g^{\mu\nu}$ with eigenvalue zero. Three other eigenvectors of $g^{\mu\nu}$ can be readily found by constructing a triad of orthogonal spacelike vectors all orthogonal to p^{ν}. The corresponding eigenvalues are -1, -1, $-(E^2+\vec{p}^{\,2})/m^2$. Therefore the diagonal form of $\tilde{g}^{ab}$ is

$$\tilde{g}'^{ab}=\mathrm{diag}\left[0,-1,-1,-\frac{E^2+\vec{p}^{\,2}}{m^2},+1,+1\right]\ . \tag{25}$$

Corresponding to the similarity transformation which diagonalizes $\tilde{g}$:

$$\tilde{g}'^{ab}=S^{-1ae}\tilde{g}_{ef}S^{fb}\ ,$$

where the matrix S is chosen to be orthogonal, there exists an automorphism of the Lie algebra

$$\theta'^{db}=S^{-1ae}\theta_{ef}S^{fb}\ , \tag{26}$$

which brings the commutation relations into the form

$$[\theta'^{ab},\theta'^{cd}]=-i(\tilde{g}'^{ac}\theta'^{bd}+\tilde{g}'^{bd}\theta'^{ac}-\tilde{g}'^{ad}\theta'^{bc}-\tilde{g}'^{bc}\theta'^{ad}) \tag{27}$$

with the diagonal metric given in (25). Moreover, the automorphism (26) eliminates five of the fifteen generators in the algebra

$$\tilde{S}'^{0j}=Q'^{0}=P'^{0}=0\ . \tag{28}$$

However, the automorphism has the effect of dilating Q, P, and $\tilde{S}$ by a factor of $[(E^2+\vec{p}^{\,2})/m^2]^{1/2}$ along the direction of $\vec{p}$ in the plane orthogonal to $p=(E,\vec{p})$. We therefore choose to augment (26) by rescaling these operators so that the internal-position and -momentum vectors and the tensor $\tilde{S}$ are left invariant by the resulting Lie-algebra automorphism, which amounts merely to reexpressing these operators in a coordinate system intrinsic to the hyperplane in which the *Zitterbewegung* takes place. Then the space part of the metric (25) becomes diag $(-1,-1,-1)$.

For simplicity let us choose for the moment an orthonormal frame (e_0,e_1,e_2,e_3) in which the space component of the center-of-mass momentum points in the three direction

$$p=Ee_0+|\vec{p}|e_3\ .$$

Then the automorphism (26) leaves Q^1, Q^2, P^1, P^2, and $\tilde{S}^{12}$ unchanged and leads to

$$Q'^{3}=\frac{|\vec{p}|Q^0+EQ^3}{(E^2+\vec{p}^{\,2})^{1/2}}$$

with similar expressions for P'^{3} and $\tilde{S}'j^3$. The spacelike unit vector pointing in the direction of $\vec{p}$ along the hyperplane orthogonal to p is

$$e_{3'}=\frac{|\vec{p}|e_0+Ee_3}{m}\ . \tag{29}$$

Rescaling Q'^{2}, P'^{3}, and $\tilde{S}'^{j3}$,

$$Q^{3'}=\frac{mQ'^{3}}{(E^2+\vec{p}^{\,2})^{1/2}}=\frac{m(|\vec{p}|Q^0+EQ^3)}{E^2+\vec{p}^{\,2}}\ , \tag{30a}$$

$$P^{3'}=\frac{mP'^{3}}{(E^2+\vec{p}^{\,2})^{1/2}}=\frac{m(|\vec{p}|P^0+EP^3)}{E^2+\vec{p}^{\,2}}\ , \tag{30b}$$

$$\tilde{S}^{j3'} = \frac{m\tilde{S}'^{j3}}{(E^2+\vec{p}^{\,2})^{1/2}} = \frac{m(|\vec{p}|\tilde{S}^{03}+E\tilde{S}^{j3})}{E^2+\vec{p}^{\,2}} , \quad (30c)$$

we find that

$$Q^{3'}e_{3'} = Q^0 e_0 + Q^3 e_3 , \quad (31a)$$

$$P^{3'}e_{3'} = P^0 e_0 + P^3 e_3 , \quad (31b)$$

$$\tilde{S}^{j3'}e_j \times e_{3'} = \tilde{S}^{j0}e_j \times e_0 + \tilde{S}^{j3}e_j \times e_3 . \quad (31c)$$

Thus the redundant 0 and 3 coordinates have been replaced by the single 3′ coordinate which lies in the hyperplane of the *Zitterbewegung*. From the commutation of $Q^{3'}$ with $P^{3'}$ we also find that

$$\tilde{g}^{3'3'} = g^{3'3'} = -1 .$$

Now if $\vec{p}$ points in an arbitrary direction we can rotate the expression (31) and write the operators of the algebra (20) in terms of the orthonormal frame $(e_{1'}, e_{2'}, e_{3'})$ which is intrinsic to the hyperplane orthogonal to $p=(E,\vec{p})$, and obtain the algebra (32)

$$[\theta^{ab},\theta^{cd}] = -i(g^{ac}\theta^{bd} + g^{bd}\theta^{ac} - g^{ad}\theta^{bc} - g^{bc}\theta^{ad}) , \quad (32)$$

where

$$\theta^{jk} = \tilde{S}^{jk}, \quad j,k = 1',2',3'$$
$$\theta^{4j} = mQ^j ,$$
$$\theta^{5j} = \frac{1}{2m}P^j , \quad (33)$$
$$\theta^{45} = \frac{1}{2m}\mathcal{H} ,$$

and

$$g^{ab} = \mathrm{diag}(-1,-1,-1,+1,+1) . \quad (34)$$

This Lie algebra of γ^0-Hermitian operators is so(3,2). The algebra so(3,2) is the noncompact form of so(5) proposed in Ref. 1 to be the dynamical algebra of the Dirac electron in an arbitrary frame. It turned out that so(3,2) worked for this purpose while so(4,1) did not. It is interesting that when the *Zitterbewegung* is formulated in proper time with γ^0-Hermitian operators, the algebra so(3,2) comes out automatically.

V. PROPERTIES AND INTERPRETATION OF OPERATORS IN THE INTERNAL ALGEBRA

Expression (17) for $\tilde{S}^{\mu\nu}$ and the commutator (15a) are remarkably similar in form to Eqs. (2.13) and (2.14a) obtained by Aldinger *et al.*[11] in their work on the relativistic rotator, and to Eq. (14) in the classical theory of the electron.[10] Following them we shall call $\tilde{S}^{\mu\nu}$ the intrinsic-spin tensor.

In order to interpret $\tilde{S}^{\mu\nu}$ let us consider the expression for the total angular momentum of a (free) spin-$\frac{1}{2}$ particle:

$$J^{\mu\nu} = L^{\mu\nu} + S^{\mu\nu} , \quad (35)$$

where

$$L^{\mu\nu} = x^\mu p^\nu - x^\nu p^\mu \quad (36)$$

is the orbital angular momentum and $S^{\mu\nu}$ [defined in (17)] is the spin angular momentum. Both $L^{\mu\nu}$ and $S^{\mu\nu}$ vary in proper time, since they do not commute with $\mathcal{H}$, while their sum $J^{\mu\nu}$ is constant. However, by substituting expression (13) for the position operator into (36) and (35) we can rewrite the total angular momentum as the sum of two constant tensors:

$$J^{\mu\nu} = \mathcal{L}^{\mu\nu} + \tilde{S}^{\mu\nu} , \quad (37)$$

where

$$\mathcal{L}^{\mu\nu} = X^\mu p^\nu - X^\nu p^\mu \quad (38)$$

is the orbital angular momentum of the center of mass, and

$$\tilde{S}^{\mu\nu} = S^{\mu\nu} + Q^\mu p^\nu - Q^\nu p^\mu \quad (39)$$

is the intrinsic spin, being the sum of the orbital angular momentum which the internal coordinate $Q^\mu(s)$ has by virtue of the electron's center-of-mass motion plus $S^{\mu\nu}$. From the solution of Heisenberg's equations of motion for the spin operator

$$S^{\mu\nu}(s) = S^{\mu\nu}(0) - \frac{p^\mu p_\alpha}{m^2}S^{\alpha\nu}(0) - \frac{p^\nu p_\alpha}{m^2}S^{\mu\alpha}(0) + \left[\frac{p^\mu p_\alpha}{m^2}S^{\alpha\nu}(0) + \frac{p^\nu p_\alpha}{m^2}S^{\mu\alpha}(0)\right]e^{-2is\mathcal{H}} , \quad (40)$$

it is easily verified that (39) is equivalent to (17).

In analogy with expression (38) we can write the intrinsic spin of the electron in terms of its internal position and momentum as follows:

$$\tilde{S}^{\mu\nu} = (Q^\mu P^\nu - Q^\nu P^\mu)\left[-\frac{\mathcal{H}}{2m}\right] . \quad (41)$$

The presence of the term $-\mathcal{H}/2m$ in (41), in contrast to the absence of a corresponding term in (38), exemplifies the difference between the internal and center-of-mass motion of the electron.

When $\tilde{S}$ is written in terms of the coordinate system characterized by the basis $(e_{1'}, e_{2'}, e_{3'})$, intrinsic to the hyperplane of the *Zitterbewegung*, it becomes clear that $\tilde{S}$ should indeed be regarded as an internal angular momentum, since it generates rotations of the internal position and momentum vectors. Using the formula

$$e^{-A}Be^{A} = B + \frac{1}{1!}[B,A] + \frac{1}{2!}[[B,A],A] + \cdots ,$$

where A and B are Lie algebra elements, we find that

$$e^{-i\theta\tilde{S}^{jk}}Q^j e^{+i\theta\tilde{S}^{jk}} = Q^j\cos\theta + Q^k\sin\theta , \quad (42a)$$

$$e^{-i\theta\tilde{S}^{jk}}P^j e^{i\theta\tilde{S}^{jk}} = P^j\cos\theta + P^k\sin\theta , \quad (42b)$$

for $j,k = 1',2',3'$, and $j \neq k$.

While $e^{i\theta\tilde{S}^{jk}}$ generates rotations, the operator

$$e^{i(\pi/2)\mathcal{H}/m} = i\frac{\mathcal{H}}{m}$$

generates *parity* transformations on the internal space of the *Zitterbewegung,*

$$e^{-i(\pi/2)\mathscr{H}/m}Q^j e^{i(\pi/2)\mathscr{H}/m} = -Q^j \ , \qquad (43a)$$

$$e^{-i(\pi/2)\mathscr{H}/m}P^j e^{i(\pi/2)\mathscr{H}/m} = -P^j \ , \qquad (43b)$$

$$e^{-i(\pi/2)\mathscr{H}/m}\tilde{S}^{jk} e^{i(\pi/2)\mathscr{H}/m} = \tilde{S}^{jk} \ , \qquad (43c)$$

$$e^{-i(\pi/2)\mathscr{H}/m}\mathscr{H} e^{i(\pi/2)\mathscr{H}/m} = \mathscr{H} \ , \qquad (43d)$$

$$j,k = 1',2',3' \ .$$

The commutators (15f) and (15g) indicate that the internal position and momentum of the electron exhibit the harmonic-oscillator dynamics found also in Ref. 1:

$$\dot{Q}^\mu(s) = i[\mathscr{H},Q^\mu(s)] = \frac{1}{m}P^\mu(s) \ , \qquad (44a)$$

$$\dot{P}^\mu(s) = i[\mathscr{H},P^\mu(s)] = -4m^3Q^\mu(s) \ , \qquad (44b)$$

implying that

$$\ddot{Q}^\mu(s) + 4m^2Q^\mu(s) = 0 \ , \qquad (45a)$$

$$\ddot{P}^\mu(s) + 4m^2P^\mu(s) = 0 \ . \qquad (45b)$$

The harmonic oscillator has a frequency of $2m$.

VI. EQUATION OF MOTION OF γ^5 AND THE INTERNAL ALGEBRA so(4,2)

The Lie algebra so(3,2) was generated by the internal position and velocity of the *Zitterbewegung*. There exists a larger algebra associated with the Dirac electron which can be obtained by considering, in addition to the dynamical variables described above, the motion of the pseudoscalar operator

$$\gamma^5 = \gamma^0\gamma^1\gamma^2\gamma^3 \qquad (46a)$$

with

$$(\gamma^5)^2 = -1 \ . \qquad (46b)$$

Using Heisenberg's equation of motion we find that the proper-time derivative of γ^5 is

$$\dot{\gamma}^5 = i[\mathscr{H},\gamma^5] = 2i\mathscr{H}\gamma^5$$
$$= 2p_\alpha\gamma^{5\alpha} \ , \qquad (47)$$

where the axial-vector operator

$$\gamma^{5\alpha} \equiv i\gamma^5\gamma^\alpha \qquad (48)$$

is also defined to be γ^0-Hermitian. The solution of (47) is

$$\gamma^5(s) = e^{2i\mathscr{H}s}\gamma^5(0) = \gamma^5(0)e^{-2is\mathscr{H}} \ .$$

Commuting $\gamma^5(s)$ with the other operators of (15) and completing the algebra we obtain

$$\left[\gamma^5(s),\frac{\mathscr{H}}{m}\right] = 2i\frac{p_\alpha\gamma^{5\alpha}(s)}{m} \ , \qquad (49a)$$

$$[\gamma^5(s),\tilde{S}^{\mu\nu}] = 0 \ , \qquad (49b)$$

$$[\gamma^5(s),P^\mu(s)] = -2iP^{5\mu} \ , \qquad (49c)$$

$$[\gamma^5(s),Q^\mu(s)] = 0 \ , \qquad (49d)$$

$$\left[\frac{p_\alpha\gamma^{5\alpha}(s)}{m},\frac{\mathscr{H}}{m}\right] = -2i\gamma^5(s) \ , \qquad (49e)$$

$$\left[\frac{p_\alpha\gamma^{5\alpha}(s)}{m},\tilde{S}^{\mu\nu}\right] = 0 \ , \qquad (49f)$$

$$\left[\frac{p_\alpha\gamma^{5\alpha}(s)}{m},Q^\mu(s)\right] = \frac{2i}{m^2}P^{5\mu} \ , \qquad (49g)$$

$$\left[\frac{p_\alpha\gamma^{5\alpha}(s)}{m},P^\mu(s)\right] = 0 \ , \qquad (49h)$$

$$\left[\frac{p_\alpha\gamma^{5\alpha}(s)}{m},\gamma^5(s)\right] = -2i\frac{\mathscr{H}}{m} \ , \qquad (49i)$$

$$\left[\frac{p_\alpha\gamma^{5\alpha}(s)}{m},P^{5\mu}\right] = 4im^2Q^\mu(s) \ , \qquad (49j)$$

$$[P^{5\mu},\gamma^5(s)] = 2iP^\mu(s) \ , \qquad (49k)$$

$$[P^{5\mu},\mathscr{H}] = 0 \ , \qquad (49l)$$

$$[P^{5\mu},P^\nu(s)] = 2im^2\tilde{g}^{\mu\nu}\gamma^5(s) \ , \qquad (49m)$$

$$[P^{5\mu}(s),Q^\nu(s)] = -i\tilde{g}^{\mu\nu}\frac{p_\alpha\gamma^{5\alpha}}{m} \ , \qquad (49n)$$

$$[P^{5\mu},P^{5\nu}] = 4im^2\tilde{S}^{\mu\nu} \ , \qquad (49o)$$

$$[P^{5\mu},\tilde{S}^{\alpha\beta}] = i(\tilde{g}^{\mu\nu}P^{5\beta} - \tilde{g}^{\mu\beta}P^{5\alpha}) \ , \qquad (49p)$$

where

$$P^{5\mu} \equiv i\gamma^5(0)P^\mu(0)$$
$$= m\left[\gamma^{5\mu}(0) - \frac{p_\alpha\gamma^{5\alpha}(0)}{m^2}p^\mu\right] \qquad (50)$$

is constant and

$$p_\alpha\gamma^{5\alpha}(s) = p_\alpha\gamma^{5\alpha}(0)e^{-2is\mathscr{H}} \ . \qquad (51)$$

Since

$$p_\mu P^{5\mu} = 0$$

this axial vector can be written in terms of the coordinate system intrinsic to the *Zitterbewegung:*

$$P^{5\mu}e_\mu = P^{51'}e_{1'} + P^{52'}e_{2'} + P^{53'}e_{3'} \ .$$

In this coordinate system we introduce the dimensionless and rescaled operators:

$$\theta^{jk} = \tilde{S}^{jk}, \quad j,k = 1',2',3'$$
$$\theta^{4j} = mQ^j \ ,$$
$$\theta^{5j} = \frac{1}{2m}P^j \ , \qquad (52)$$
$$\theta^{45} = \mathscr{H}/2m \ ,$$
$$\theta^{46} = p_\alpha\gamma^{5\alpha}, \quad \theta^{56} = \tfrac{1}{2}\gamma^5$$

and obtain

$$[\theta^{ab},\theta^{cd}]=-i(g^{ac}\theta^{bd}+g^{bd}\theta^{ac}-g^{ad}\theta^{bc}-g^{bc}\theta^{ad}) , \tag{53}$$

where g is diagonal and

$$g^{jj}=g^{66}=-1,\ \ g^{44}=g^{55}=+1 . \tag{54}$$

The algebra obtained is therefore so(4,2).

For completeness, let us also find the proper-time evolution of $\gamma^{5\mu}$. Heisenberg's equation of motion yields

$$\dot{\gamma}^{5\mu}=i[\mathscr{H},\gamma^{5\mu}]=-2p^{\mu}\gamma^{5} . \tag{55}$$

Differentiating again and using the fact that p^{μ} is constant and γ^5 is given by (47), we obtain

$$\begin{aligned}\ddot{\gamma}^{5\mu}&=-4p^{\mu}p_{\alpha}\gamma^{5\alpha}\\ &=2i\mathscr{H}\dot{\gamma}^{5\mu} .\end{aligned} \tag{56}$$

Solving (56) for $\gamma^{5\mu}(s)$

$$\begin{aligned}\dot{\gamma}^{5\mu}(s)&=e^{2is\mathscr{H}}\dot{\gamma}^{5\mu}(0)=\dot{\gamma}^{5\mu}(0)e^{-2is\mathscr{H}}\\ &=-2p^{\mu}\gamma^{5}(0)e^{-2is\mathscr{H}} ,\end{aligned} \tag{57}$$

and integrating again we find

$$\gamma^{5\mu}(s)=a^{5\mu}+\frac{p^{\mu}p_{\alpha}}{m^2}\gamma^{5\alpha}(0)e^{-2is\mathscr{H}} , \tag{58}$$

where the constant $a^{5\mu}$ is determined by initial conditions to be

$$a^{5\mu}=\gamma^{5\mu}(0)-\frac{p^{\mu}p_{\alpha}}{m^2}\gamma^{5\alpha}(0)\equiv\eta^{5\mu} . \tag{59}$$

Therefore we have obtained

$$\gamma^{5\mu}(s)=\eta^{5\mu}+\frac{p^{\mu}p_{\alpha}}{m^2}\gamma^{5\alpha}(s) . \tag{60}$$

It is interesting to note that while the vector operator $\gamma^{\mu}(s)$ is constant in the direction of the center-of-mass momentum p^{μ} and oscillates in the directions orthogonal to p^{μ}, the axial-vector operator $\gamma^{5\mu}(s)$ oscillates along the p^{μ} directions and is constant in the orthogonal directions.

We see from solutions (48) and (58) that the motions of $\gamma^5(s)$ and $\gamma^{5\mu}(s)$ are decoupled from the other dynamical variables in the case of the Dirac electron. This would not be the case if we had generalized the Dirac equation to include a γ^5 term which is allowed by Lorentz invariance alone,[12] i.e., to

$$(\gamma^{\mu}p_{\mu}-m+\lambda\gamma^{5})\psi=0 .$$

The internal Lie algebra o(4,2) given by (53) defines a more general quantum system with internal dynamics of which the Dirac electron and further the neutrino[13] are special projections. The general Lie algebra o(4,2) has many other representations and realizations. If we go further to infinite-dimensional representations, we arrive at systems like an H atom, or a hadron, where now the internal dynamics refer to a "*Zitterbewegung*" which we know well, namely, the motion of the electron around the proton in the H atom, for example.[14] We emphasize that in all these cases the internal or dynamical algebra o(4,2) is quite distinct from the external algebra o(4,2) of the motion of the system as a whole in space-time, although some connections exist.[15]

VII. COVARIANT-MAGNETIC-MOMENT OPERATOR

Generalizing the three-vector magnetic-moment operator derived earlier[16] to a four-vector, we define a covariant magnetic-moment operator

$$\mu^{\sigma\lambda}=\frac{e}{2}(\dot{x}^{\sigma}x^{\lambda}-x^{\sigma}\dot{x}^{\lambda}) ,$$

and using Eqs. (12) and (13) and separating the even and odd parts of the product[16] we obtain after some calculations

$$\mu^{\sigma\lambda}=\frac{e}{2}(\mathscr{L}^{\sigma\lambda}+2\widetilde{S}^{\sigma\lambda})(p\cdot\gamma)^{-1}$$

showing again the g factor 2 and the role of $S^{\sigma\lambda}$ as the intrinsic spin of the particle. The first term is the orbital magnetic moment of the center of mass.

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